Band Schemes and Moduli Spaces of Matroids

Joint work with Oliver Lorscheid and Tong Jin

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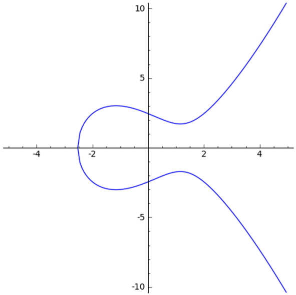
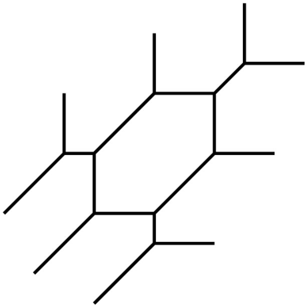
Analogy #1: Tropical geometry

Let K be a complete and algebraically closed non-archimedean field.

Given a closed subscheme X of $\mathbb{G}_m^n$, one associates to X a *tropical variety* $\mathrm{Trop}(X) \subseteq \mathbb{R}^n$. It is a balanced weighted polyhedral complex.





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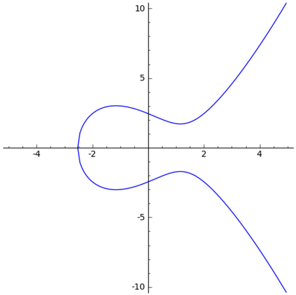

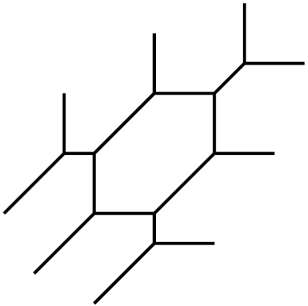

Analogy #2: Tits' dream (don't Google this)

Analogy: Weyl groups $\leftrightarrow$ "Algebraic groups over $\mathbb{F}_1$"

Example: Weyl group of GL_n is S_n, which should be "$\mathrm{GL}_n(\mathbb{F}_1)$".

Heuristic: $\#\mathrm{GL}_n(\mathbb{F}_q) = (q-1)^n q^{\binom{n}{2}} [n]_q!$, where

$$[n]_q! = (1 + q + \cdots + q^{n-1})(1 + q + \cdots + q^{n-2}) \cdots (1+q) \cdot 1.$$

If T is the diagonal torus in GL_n, then

$$\lim_{q \to 1} \frac{\#\mathrm{GL}_n(\mathbb{F}_q)}{\#T(\mathbb{F}_q)} = n! = \#S_n.$$

Similar phenomenon happens for other reductive groups.

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Analogy #3: Grassmannian and matroids

Following the philosophy of Tits, we have $\#\mathrm{Gr}(r,n)(\mathbb{F}_q) = \binom{n}{r}_q$, and

$$\lim_{q\to 1} \#\mathrm{Gr}(r,n)(\mathbb{F}_q) = \binom{n}{r}.$$

So perhaps the combinatorial analogue of the Grassmannian is the collection of all r-element subsets of $[n] := \{1,\ldots,n\}$.

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Goal

Our goal in this talk is to illuminate all of these analogies – especially the last one – through the unifying lens of *band schemes*.

Bands

Bands are like commutative rings, but where we don't assume that $+$ comes from a binary operation.

A *pointed monoid* is a commutative monoid B with an identity element 1 and an absorbing element 0.

The *ambient semiring* of a pointed group B is the group semiring

$$B^+ = \mathbb{N}[B - \{0\}].$$

An *ideal of B^+* is a subset that contains 0 and is closed under addition and under multiplication by elements of B^+.

A *band* is a pointed monoid B, together with an element $-1 \in B$ and an ideal N_B of B^+, called the *null set of B*, such that for every $a \in B$ we have $a + b \in N_B$ iff $b = (-1) \cdot a$.

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Tracts and homomorphisms

A band B is called a *tract* if every nonzero element of B has a multiplicative inverse.

A *homomorphism* of bands is a multiplicative map $f\colon B_1 \to B_2$ with $f(0) = 0$, $f(1) = 1$, and $f(-1) = -1$ such that the induced map $B_1^+ \to B_2^+$ sends every element of N_{B_1} to an element of N_{B_2}.

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Examples of bands

1. Every ring R (meaning commutative ring with 1) is naturally a band, where $N_R = \{\sum a_i \mid \sum a_i = 0 \in R\}$.
2. The *initial band* is $\mathbb{F}_1^{\pm} = \{0, 1, -1\}$ with the usual multiplication and null set $\{0, 1-1, 1+1-1-1, \ldots\}$.
3. The *Krasner hyperfield* is $\mathbb{K} = \{0, 1\}$ with the usual multiplication and null set $\{0, 1+1, 1+1+1, 1+1+1+1, \ldots\}$.
4. The *tropical hyperfield* is $\mathbb{T} = \mathbb{R}_{\geq 0}$ with the usual multiplication and null set consisting of 0 and all $\sum a_i$ such that the maximum among $a_1, \ldots, a_n$ occurs at least twice.

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More examples

1. The *sign hyperfield* is $\mathbb{S} = \{0, 1, -1\}$ with the usual multiplication and null set consisting of 0 and all $\sum a_i$ such that at least one a_i is positive and at least one is negative.
2. The *triangle hyperfield* is $\mathbb{T}_1 = \mathbb{R}_{\geqslant 0}$ with the usual multiplication and null set consisting of 0 and all $\sum a_i$ such that the a_i form the side lengths of a polygon.

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Examples of band homomorphisms

1. If R is a ring, a homomorphism $R \to \mathbb{K}$ is the same thing as a *prime ideal* of R.
2. If K is a field, a homomorphism $K \to \mathbb{S}$ is the same thing as an *ordering* on K.
3. If K is a field, a homomorphism $K \to \mathbb{T}$ is the same thing as a *non-archimedean absolute value* on K.
4. If R is a ring, a homomorphism $R \to \mathbb{T}$ is the same thing as a prime ideal $\mathfrak{p}$ of R and a non-archimedean absolute value on the fraction field of $R/\mathfrak{p}$.

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Properties of the category of bands

The category of bands is complete and cocomplete, and in particular admits products and tensor products.

For example:

1. The product of $\mathbb{F}_2$ and $\mathbb{F}_3$ is $\mathbb{F}_1^{\pm}$.
2. The tensor product of $\mathbb{F}_2$ and $\mathbb{F}_3$ is $\mathbb{K}$.

We also have free objects, for example $B[x_1, \ldots, x_n]$.
The pointed monoid of $B[x_1, \ldots, x_n]$ consists of all *monomials*

$$\{b \cdot \textstyle\prod_{i=1}^{n} x_i^{m_i}\}$$

and the null set consists of all formal sums of monomials in which, for each fixed monomial, the sum of the coefficients of belongs to N_B.

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Prime ideals

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A *prime m-ideal* in a band B is the kernel of a monoid homomorphism $B \to \mathbb{K}$.

We let $\operatorname{Spec} B$ denote the set of prime m-ideals of B, with the Zariski topology generated by $U_h = \{\mathfrak{p} \mid h \notin \mathfrak{p}\}$ for $h \in B$.

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Band schemes

The *structure sheaf* $\mathcal{O}_X$ on $X = \operatorname{Spec} B$ is characterized by $\mathcal{O}_X(U_h) = B[h^{-1}]$.

An *affine band scheme* is a pair consisting of $X = \operatorname{Spec} B$ and the sheaf of bands $\mathcal{O}_X$.

A *band space* is a topological space X together with a sheaf of bands $\mathcal{O}_X$. Band spaces form a category in the "standard" way.

A *band scheme* is a band space which is locally isomorphic to an affine band scheme. A morphism of band schemes is a morphism of the associated band spaces.

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Functors between scheme theories

There is a functor from band schemes to schemes which locally takes a band B to the ring generated by B^+ modulo the ideal generated by N_B.

There is no functor from schemes to band schemes. However, we can associate a band scheme $\mathfrak{X}$ to a separated scheme X together with an open affine covering $\mathcal{U}$.

We call such a band scheme $\mathfrak{X}$ a *model* for X.

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The Tits space of a band scheme

If X is a band scheme and C is a band, we can define $X(C)$ as in usual algebraic geometry to be $\mathrm{Mor}(\mathrm{Spec}(C), X)$.

There is a natural topology on the Krasner hyperfield $\mathbb{K}$ whose open subsets are $\emptyset$, $\{1\}$, and $\mathbb{K}$. This induces a topology on the point set $X(\mathbb{K})$.

For example, if $X = \mathrm{Spec}\, B$ is affine, then $X(\mathbb{K})$ is naturally identified with the set of prime k-ideals of B, and the topology on $X(\mathbb{K})$ is the Zariski topology.

We define the *Tits space* of X to be the set of closed points of $X(\mathbb{K})$.

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Every toric variety X over a field K admits a canonical band scheme model $\mathfrak{X}$ over $\mathbb{F}_1^{\pm}$.

If Δ is a rational polyhedral fan in $\mathbb{R}^n$, σ is a cone in Δ, and $A_\sigma = \sigma^\vee \cap \mathbb{Z}^n$ is the set of lattice points of the dual cone, we set

$$U_\sigma = \operatorname{Spec} \mathbb{F}_1^{\pm}[t^\lambda]_{\lambda \in A_\sigma}$$

and $X(\Delta) = \operatorname{colim}_{\sigma \in \Delta} U_\sigma$.

The points of $X(\Delta)$ correspond bijectively to the cones $\sigma \in \Delta$, with points of the open affine subset U_σ of $X(\Delta)$ corresponding to the faces of σ.

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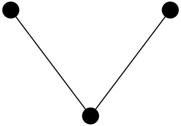
Examples



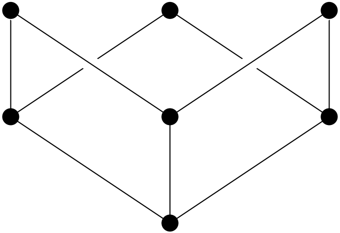


Figure: The projective line $\mathbb{P}^1$ and the projective plane $\mathbb{P}^2$

Analytification

Let K be a field equipped with a valuation, i.e., a band homomorphism $v : K \to \mathbb{T}$.

Let $X = \operatorname{Spec} R$ be an affine K-scheme of finite type, which we can consider as a band scheme by considering the K-algebra R as a band.

Then $X(\mathbb{T}) = \operatorname{Hom}_K(R, \mathbb{T})$ is canonically homeomorphic to the Berkovich analytification of X.

With more care, one can show that $X(\mathbb{T})$ is canonically homeomorphic to the Berkovich analytification of X for any scheme (not necessarily affine) of finite type over K.

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Choosing generators $a_1, \ldots, a_n$ for R as a K-algebra yields a presentation $R = K[a_1, \ldots, a_n]/I$ for some ideal I.

Let C be the band consisting of the pointed submonoid of R generated by K and $a_1, \ldots, a_n$, with null set given by those formal sums $\sum c_i$ with $\sum c_i \in I$.

Let Y be the K-band scheme Spec C.

Theorem (Lorscheid)

$Y(\mathbb{T})$ is canonically homeomorphic to the tropicalization X^{trop} of X with respect to the embedding into $\mathbb{A}^n_k$ given by $a_1, \ldots, a_n$.

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For example, when $G = \mathrm{SL}_2$, we let B be the band whose underlying pointed monoid is generated by indeterminates a, b, c, d and whose null set is generated by $ad - bc - 1$.

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Inspired by Hannah Markwig's talk this morning, suppose F is a tract. Here is a (tentative) definition of the *Grothendieck–Witt band* $\mathrm{GW}(F)$:

1. The pointed monoid of $\mathrm{GW}(F)$ is $F^\times/(F^\times)^2$.
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When $F = K$ is a field, this coincides with the usual Grothendieck–Witt ring of K.

When $F = \mathbb{F}_1^\pm$, $\mathrm{GW}(\mathbb{F}_1^\pm)$ is the pointed monoid $\{0, \langle 1\rangle, \langle -1\rangle\}$, with null set generated by $\langle 1\rangle - \langle 1\rangle$.

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Matroids

Recall that a *matroid* M of rank r on $[n]$ is a non-empty collection $\mathcal{B}$ of r-element subsets of E, called the *bases* of M, such that for all $B, B' \in \mathcal{B}$ and $x \in B - B'$, there exists $y \in B' - B$ such that $B - x + y$ and $B' - y + x$ belong to $\mathcal{B}$.

Let $\mathcal{G}(r, n)$ be the projective band scheme over $\mathbb{F}_1^{\pm}$ defined by the *Plücker relations*.

For example, the "homogeneous null set" of $\mathcal{G}(2, 4)$ is defined by the relation $x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23}$.

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Properties of the band scheme Grassmannian

1. If K is a field, $\mathcal{G}(r,n)(K)$ is the set of r-dimensional subspaces of K^n.
2. For the Krasner hyperfield $\mathbb{K}$, $\mathcal{G}(r,n)(\mathbb{K})$ is the set of rank r matroids on $[n]$.
3. The Tits space of $\mathcal{G}(r,n)$ consists of the rank r matroids on $[n]$ with a unique basis, which can be identified with the set of r-element subsets of $[n]$. In particular, the Tits space has $\binom{n}{r}$ elements.

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For any tract F, we can define the set $\mathcal{G}(r,n)(F)$ of *F-matroids* of rank r on $[n]$.

Nathan Bowler and I gave cryptomorphic descriptions of F-matroids in terms of *F-circuits* and *dual pairs* of F-circuits and F-cocircuits.

Given a tract homomorphism $F \to F'$ and an F-matroid M, we can define a push-forward F'-matroid M'.

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Thin Schubert cells, realization spaces, and the foundation

Given a matroid M and a tract F:

1. The *thin Schubert cell* $\mathrm{Gr}_M(F)$ is the set of F-matroids with underlying matroid M.
2. The *realization space* $\underline{\mathrm{Gr}}_M(F)$ is the set of $(F^\times)^n$-orbits of $\mathrm{Gr}_M(F)$.

Note for the experts: we frequently use just the 3-term Plücker relations when defining realization spaces.

For every matroid M, $\mathrm{Gr}_M(F)$ and $\underline{\mathrm{Gr}}_M(F)$ can naturally be identified with the F-points of affine band schemes Gr_M and $\underline{\mathrm{Gr}}_M$, respectively, which are defined over $\mathbb{F}_1^\pm$.

The unique tract F_M with the property that $\underline{\mathrm{Gr}}_M(F) = \mathrm{Hom}(F_M, F)$ for every tract F is called the *foundation* of M.

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Thin Schubert cells, realization spaces, and the foundation

Given a matroid M and a tract F:

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Example: $U_{2,4}$

Let $M = U_{2,4}$, whose bases are all 2-element subsets of $\{1,2,3,4\}$.

If K is a field, then up to the action of $(K^\times)^4$, every 2-dimensional subspace W of K^4 can be written as the row space of

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & t \end{pmatrix}$$

for a unique $t \in K - \{0,1\}$.

The foundation F_M has underlying pointed monoid generated by x, x^{-1}, y, y^{-1} and null set generated by $x + y - 1$.
The realization space $\underline{\mathrm{Gr}}_M$ is the affine band scheme Spec F_M.

The associated scheme over $\mathbb{Z}$ (the "classical" realization space of M) is $\mathbb{P}^1 - \{0,1,\infty\} = \operatorname{Spec} \mathbb{Z}[x, x^{-1}, y, y^{-1}]/(x + y - 1)$.

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Valuated matroids and the Dressian

A *valuated matroid* of rank r on $[n]$ is a $\mathbb{T}$-matroid.

For example, a valuated matroid with underlying matroid $U_{2,4}$ is a point $[p_{12} : p_{13} : p_{14} : p_{23} : p_{24} : p_{34}] \in \mathbb{P}^5(\mathbb{T})$ with $p_{ij} > 0$ such that the maximum of $p_{12}p_{34}, p_{13}p_{24}, p_{14}p_{23}$ is achieved at least twice.

Valuated matroids are canonically in bijection with tropical linear spaces, i.e., tropical varieties of degree 1.

The (logarithm of the) tropical realization space $\underline{\mathrm{Gr}}_M(\mathbb{T})$ is called the *Dressian* of M.

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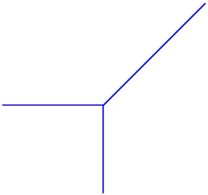
Example

The Dressian of $U_{2,4}$ is homeomorphic to the tropical line defined by $x + y + 1$:



Representability

A matroid M is *representable* over a tract F if the set of F-matroids with underlying matroid M is non-empty.

Equivalently, M is representable over F iff there is a homomorphism from F_M to F.

A matroid M is representable over F_1 and F_2 iff M is representable over $F_1 \times F_2$. Since $\mathbb{F}_2 \times \mathbb{F}_3 \cong \mathbb{F}_1^{\pm}$, the initial object in the category of tracts, we immediately obtain Tutte's theorem that a matroid M is representable over $\mathbb{F}_2$ and $\mathbb{F}_3$ iff M is representable over every field.

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Properties of the foundation

1. The foundation of M is isomorphic to the foundation of the dual matroid M^*.
2. The foundation of $M_1 \oplus M_2$ is isomorphic to the tensor product $F_{M_1} \otimes F_{M_2}$.
3. If N is an *embedded minor* of M, i.e., $N = M \backslash I / J$ for disjoint sets $I, J \subseteq [n]$ with J independent and I co-independent, there is a canonical morphism $F_N \to F_M$. In particular, if M is representable over a tract F then so is N.

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Canonical presentation for the foundation

There are special elements in the foundation of a matroid M called *cross-ratios*. These are the (finitely many) elements $x \in F_M$ for which there exists a $y \in F_M$ with $x + y - 1$ in the null set of F_M.

There is a natural action of S_3 on the set of such pairs (x, y), and the orbits of this action are canonically in bijection with embedded $U_{2,4}$-minors of M.

Theorem (B–Lorscheid)

1. *The cross-ratios of M generate the foundation of M.*
2. *All additive relations in F_M are inherited from embedded $U_{2,4}$ minors.*
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The fact that cross-ratios generate the foundation immediately implies the well-known fact that a matroid with no $U_{2,4}$-minor must be binary.

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Cross-ratios also give a canonical fan structure on the Dressian of M, which lives in the tropicalization of T, and this gives a canonical way to compactify $\underline{\mathrm{Gr}}_M$, by taking the closure in the associated toric variety.

The canonical presentation can also be used to prove:

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If M has no minor isomorphic to $U_{2,5}$ or $U_{3,5}$, then $F_M \cong F_1 \otimes \cdots \otimes F_s$, where each F_i is one of five specific tracts.

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Exotic bijections between realization spaces

The previous theorem can be used to prove new results about matroid representations over **fields**. For example:

Theorem (B–Lorscheid)

Suppose q, q_1, q_2 are prime powers with $3 \nmid q$ such that $q - 2 = (q_1 - 2)(q_2 - 2)$. Then for every ternary matroid M,

$$\underline{\mathrm{Gr}}_M(\mathbb{F}_q) \cong \underline{\mathrm{Gr}}_M(\mathbb{F}_{q_1}) \times \underline{\mathrm{Gr}}_M(\mathbb{F}_{q_2}).$$

For example, if M is ternary then

$$\underline{\mathrm{Gr}}_M(\mathbb{F}_{17}) \cong \underline{\mathrm{Gr}}_M(\mathbb{F}_5) \times \underline{\mathrm{Gr}}_M(\mathbb{F}_7).$$

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Application to Lorentzian polynomials

As explained in June Huh's second lecture, if M is any matroid the space $\underline{L}_M$ of Lorentzian polynomials with support M, modulo rescaling of the variables, is canonically homeomorphic to $\underline{\mathrm{Gr}}_M(\mathbb{T}_1)$, where $\mathbb{T}_1$ is the triangle hyperfield.

Using this and our classification theorem, one can determine all possible homeomorphism types of $\underline{L}_M$ for matroids without $U_{2,5}$ or $U_{3,5}$ minors:

Theorem (B.–Huh–Kummer–Lorscheid)

Let M be a matroid that does not have a $U_{2,5}$ or $U_{3,5}$ minor. Then $\underline{\mathrm{Gr}}_M(\mathbb{T}_1)$, and hence $\underline{L}_M$, is homeomorphic to a (finite) product of half-open intervals and discs with three points removed from the boundary.

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A more general result

Theorem (B.–Huh–Kummer–Lorscheid)

If M is any matroid, then $\underline{L}_M$ is homeomorphic to the inverse limit of a finite directed system of topological spaces, each of which is either a half-open interval, a disc with three points removed from the boundary, or a five-dimensional ball with a copy of the Petersen graph removed from the boundary.

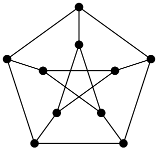

Figure: The Petersen graph

Thanks

Thank you!