# Quantum Schubert Calculus and Rigid Local Systems

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# Plan

To a unitary rigid local system (of arbitrary rank $\ell$) on $\mathbb{P}^1 - \{p_1, \ldots, p_s\}$ with eigenvalues of local monodromies $n$th roots of unity, we associate a divisor class of enumerative significance (a degeneracy locus) on the moduli stack of rank $n$ parabolic bundles on $\mathbb{P}^1$. The construction passes through strange duality.

This imposes several restrictions on the unitary local system. For example it answers a question of Nicholas Katz on the non-existence of rigid local systems of rank $> 1$ with finite global monodromy when $n$ is a prime number.

# Rigid local systems

A rank $\ell$ local system on $\mathbb{P}^1_{\mathbb{C}} - S$, $S = \{p_1, \ldots, p_s\}$, is a locally constant sheaf of rank $\ell$ vector spaces on $\mathbb{P}^1_{\mathbb{C}} - S$. Examples of such local systems arise from local solutions to $\ell$th order differential equations (with singularities of the coefficients at points of $S$)

A rank $\ell = 2$ hypergeometric example.

$$z(1-z)w'' + (\frac{1}{2} - z)w' + \frac{1}{4}w = 0$$

Gives a rank 2 local system on $\mathbb{P}^1 - \{0, 1, \infty\}$.

# Rigid local systems of rank $\ell$

Rank $\ell$ local sytems correspond to representations

$$\rho : \pi_1(\mathbb{P}^1_{\mathbb{C}} - S, b) \to \mathrm{GL}(\ell, \mathbb{C}).$$

The image of $\rho$ is called the (global) monodromy group of the local system. The conjugacy classes of $\rho(\text{loop around } p_i) = A_i$ are the local monodromies of the local system.

Equivalently, in matrix form we may consider $s$-tuples $(A_1, \ldots, A_s)$ of matrices $A_i \in \mathrm{GL}(\ell, \mathbb{C})$ with product $A_1 A_2 \cdots A_s = I$. Such tuples are taken up to conjugacy, i.e.,
$(A_1, \ldots, A_s) \sim (CA_1C^{-1}, CA_2C^{-1}, \ldots, CA_sC^{-1})$.

# Irreducibility and rigidity

- The local system is irreducible if $\rho$ is irreducible.
- The local system is rigid, if any other local system with the same local monodromies $B_i$ (i.e, $B_i$ is conjugate to $A_i$) is conjugate to it. That is, there is a single $C$ such that $CB_iC^{-1} = A_i$.

The rigidity condition on an irreducible local system can be captured by a numerical equation invoving multiplicities and Jordan blocks:

$$\sum \dim Z(A_i) = (s-2)\ell^2 + 2$$

We will restrict to having all eigenvalues of local monodromies $n$th roots of unity for a fixed $n$.

# A rank 2 hypergeometric example from Schwarz's list

The rank 2 example from before

$$z(1-z)w'' + (\frac{1}{2} - z)w' + \frac{1}{4}w = 0$$

which corresponds to the matrix equation ($i = \sqrt{-1}$)

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = I.$$

The global monodromy is a finite group (dihedral). It is hence also unitary. All three matrices are conjugate, with eigenvalues $i$ and $-i$. Here $\ell = 2$ and $n = 4$.

# Katz's work

The concept of rigid local systems has origins in the work of Riemann on hypergeometric functions.

Katz has shown that rigid local systems are produced by an inductive, algorithmic procedure (middle convolution and tensoring), and have Hodge-theoretic realizations and have geometric origin.

Henceforth "rigid local systems" will refer to "irreducible rigid local systems", with eigenvalues of local monodromies $n$th roots of unity.

# Questions

- Classify rigid local systems of rank $\ell$ whose local monodromies have eigenvalues that are $n$th roots of unity, and *whose global monodromy is finite.*

For the hypergeometric systems of rank 2 this problem was studied by Riemann and Schwarz. More recently this problem was solved for all hyergeometric local systems by Beuckers and Heckman in 1980's.

- Classify unitary rigid local systems (image in $U(\ell)$) with eigenvalues of local monodromy $n$th roots of unity.

A rigid local system has finite monodromy iff it and all of its Galois conjugates are unitary. Note that if $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ then $\sigma(A_1)\cdots\sigma(A_s) = I$, so we get a "Galois conjugate" local system. The action on local monodromies is easy to compute (really only cyclotomic extensions come into play).

# Quantum Schubert Calculus and divisor classes on the moduli stacks of parabolic bundles.

Some definitions.

1. $Gr(r, n)$ is the Grassmann variety of rank $r$ subspaces of $\mathbb{C}^n$. It has dimension $r(n-r)$.
2. A complete flag on $\mathbb{C}^n$ is a flag $F_\bullet$ of subspaces.

   $$F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n = \mathbb{C}^n.$$

   The space of such flags is denoted by $Fl(n)$.
3. Any $r$ element subset $I = \{i_1 < i_2 < \cdots < i_r\}$ of $[n] = \{1, \ldots, n\}$ determines a Schubert variety $\Omega_I(F_\bullet) = \{V \mid \forall k, \operatorname{rk} V \cap F_{i_k} \geq k\}$.
4. Let $\sigma_I \in H^{2|\sigma_I|}(Gr(r, n))$ be cohomology class of $\Omega_I(F_\bullet)$. Here $|\sigma_I| = \sum_a (n - r + a - i_a)$.

# Gromov-Witten numbers

Let $I^1, \ldots, I^s$ be subsets of $[n]$, and $d$ a positive integer such that $\sum |\sigma_{I^j}| = nd + \dim(Gr(r,n))$.

Then we define $\langle \sigma_{I^1}, \ldots, \sigma_{I^s} \rangle_d$ to be the number of maps $f : \mathbb{P}^1 \to Gr(r,n)$ of degree $d$ so that $f(p_j) \in \Omega_{I^j}(F^j_\bullet)$ for generic choices of the flags $F^j_\bullet$.

The codimension condition shows that the expected dimension of this set of maps is zero. These GW numbers are encoded in the small quantum cohomology ring and are computed by quantum Schubert calculus.

Recall also that a map $f : \mathbb{P}^1 \to Gr(r,n)$ is also the data of a rank $r$ subbundle of the trivial bundle $\mathcal{O}^{\oplus n}$ on $\mathbb{P}^1$ of degree $-d$. So the problem is about counting rank $r$ subbundles of a rank $n$ bundle with special properties.

# Divisors in the moduli of parabolic bundles on $\mathbb{P}^1$

Assume now that

$$\sum |\sigma_{I^j}| = nd + \dim Gr(r,n) + 1.$$

Then the set of maps $f : \mathbb{P}^1 \to Gr(r,n)$ of degree $d$ so that $f(p_i) \in \Omega_{I^j}(F^j_\bullet))$ for a generic choice of the flags $F^j_\bullet$ is empty. But we can look at a locus in the space of flags $Fl(n)^s$ such that there are such $f$. This is an irreducible divisor (possibly zero). Need to define this properly as a cycle theoretic push-forward.

Note that $Fl(n)^s$ parametrises parabolic bundles on $\mathbb{P}^1$ such that the underlying bundle is trivial. We can extend this to a divisor on the moduli stack of rank $n$ parabolic bundles on $\mathbb{P}^1$.

# Moduli of rank $n$ parabolic bundles on $\mathbb{P}^1$

Let $\mathcal{P}\mathrm{ar}_n$ be stack parametrizing rank $n$ vector bundles $E$ with trivialized determinant on $\mathbb{P}^1$ with complete filtrations at the fibers $p_1, \ldots, p_s$. It has a large open subset that is $[Fl(n)^s/SL(n)]$.

A line bundle on this stack is product of local factors $L_{\lambda_i}$ at the points $p_i$ and a determinant of cohomology factors raised to a level $\ell$: i.e., looks like

$$L_{\lambda^1} \otimes L_{\lambda^2} \otimes \cdots \otimes L_{\lambda^s} \otimes \det(H^*(E))^{\ell}$$

Here $\lambda^i = \sum c_b^i \omega_b$ is a sum of fundamental dominant integral weights, note that $c_b^i \geq 0$ and $\ell \geq 0$ if the line bundle is effective.

The divisor class from the earlier page can also be expressed in this way: All the numbers $c_b^i$ and $\ell$ are GW invariants. For example $c_b^1 = \langle \sigma_{I'^1}, \ldots, \sigma_{I^s} \rangle_d$.

Here $I'^1 = (I^1 - \{b\}) \cup \{b+1\}$ if this is a $r$ element subset, otherwise $c_b^i = 0$. Therefore $c_b^i$ and $c_{b+1}^i$ cannot both be non-zero.

The $\ell$ for this divisor class is also a GW number:

$$\ell = \langle \sigma_{I^1}, \ldots, \sigma_{I^s}, \sigma_J \rangle_{d+1}$$

where $J = \{1, n-r+1, \ldots, n-1\}$.

# From rigids to divisors

Let $\mathcal{A} = (A^1, \ldots, A^s)$ correspond to a rank $\ell$ unitary rigid (irreducible) local system (with local monodromy $n$th roots of unity). Assume that the eigenvalues of $A_i$ are $\exp(2\pi\sqrt{-1}\mu^i_j/n)$ with integers $0 \leq \mu^i_j < n$, so that the $\mu^i = (\mu^i_1, \ldots, \mu^i_\ell)$ are Young diagrams that fit in an $\ell \times n$ box.

Let $\mathcal{P}\mathrm{ar}_\ell$ be the moduli stack of rank $\ell$ parabolic bundles on $\mathbb{P}^1$ with parabolic structures at points of $S$.

Note that $\mathcal{A}$ gives a tuple of representations of $GL(\ell)$ at "level" $n$, thus a line bundle $\mathcal{L}$ on $\mathcal{P}\mathrm{ar}_\ell$ at level $n$. This corresponds to the determinant of cohomology line bundle (raised to $n$) tensored with local factors at the parabolic points that come from the Borel-Weil theorem that correspond to $\mu^i$.

### Theorem

*There exists an unitary local system with this local monodromy data iff* $h^0(\mathcal{P}ar_\ell, \mathcal{L}) \neq 0$

This is the theorem of Mehta and Seshadri combined with quantum saturation. In fact an irreducible rep exists if for the general parabolic bundle, all subbundles satisfy a semistability condition, which is an algebro-geometric notion what is equivalent to $h^0(\mathcal{P}\mathrm{ar}_\ell, \mathcal{L}^N) \neq 0$ for some $N > 0$.
Also,

### Theorem

$h^0(\mathcal{P}ar_\ell, \mathcal{L}) = 1$ *if and only if there is only one such unitary local system (possibly reducible).*

This uses a generalization of a conjecture of Fulton.

# Strange duality, or level rank duality

Consider the vector of transpose Young diagrams $\vec{\lambda} = (\lambda^1, \dots, \lambda^s)$ with $\lambda^i = (\mu^i)^T$.

- This transpose gives a line bundle $\mathcal{L}^T$ on $\mathcal{P}\mathrm{ar}_n$ at level $\ell$ which is also effective (has space of sections dual to $H^0(\mathcal{P}\mathrm{ar}_n, \mathcal{L})$). This gives an effective divisor on $Par_n$ from the rigid local system.
- This effectiveness is explained by $h^0(\mathcal{P}\mathrm{ar}_\ell, \mathcal{L}) =$ structure constant in cohomology of Grassmannians $Gr(n, n+\ell)$ and we use
$$Gr(n, n+\ell) = \mathrm{Gr}(\ell, n+\ell)$$
to get
$$h^0(\mathcal{P}\mathrm{ar}_\ell, \mathcal{L}) = h^0(\mathcal{P}\mathrm{ar}_n, \mathcal{L}^T).$$

The rigidity property on the $\ell$ side transforms into a geometric property of the divisor on $\mathcal{P}\mathrm{ar}_n$ on the other side:

Let $D$ be the divisor on the $\mathcal{P}\mathrm{ar}_n$ side (of $\mathcal{L}^T$). It has exactly one dimensional space of global sections even after scaling, and is irreducible: Because reducibility of the divisor on the $Par_n$ side means reducibility of the representation on the other.

We can consider the non-semistable locus of $\mathcal{O}(D)$. It is characterized by parabolic bundles which have a subbundle which maximally contradicts semistability (Harder-Narasimhan). The numerical type of the bundle, defines an enumerative property. (It should be essentially the subset where some extra Schubert intersection happens.) This means that the $D$ is of enumerative type.

# In the Schwarz example

The degeneracy locus is the set of all rank 4 parabolic bundles of degree 0 with $s = 3$ such that there is a rank 2 subbundle such whose fibers at the points $p_1$, $p_2$, $p_3$ contain the first elements of the flags and are contained in the third elements of the flags (codim 3 conditions). $I^1 = I^2 = I^3 = \{1, 3\}$.

# Katz's question

The property that $c^i_b$ and $c^i_{b+1}$ cannot both be non-zero translates into the following: The eigenvalues of a unitary rigid local system with eigenvalues (at all points) $n$th roots of unity canot differ by a factor of $\exp(2\pi\sqrt{-1}/n)$.
For a local system with finite monodromy: The eigenvalues of a finite global monodromy rigid local system with eigenvalues (at all points) $n$th roots of unity canot differ by a primitive $n$th root of unity. Hence if $n$ is prime there are none (answers a question of Katz).
We also see that allowing ourselves to vary $\ell$, there are only finitely many unitary rigids with eigenvalues of local monodromy $n$th roots of unity.
What if we drop the unitarity condition on the rigid, but require that the Hodge filtration has a fixed length? Are there only finitely many such rigids?

# KZ equations

The form of $\ell$ as rank of a conformal blocks suggests that the rank $\ell$ unitary local system is the same as the corresponding KZ local system for $SL_n$ conformal blocks (the added point is moving and corresponds to the vector representation). Therefor all unitary rigid local systems occur as solutions to KZ/WZW equations.

# KZ

There is a larger context: Determining the effective $\mathbb{Q}$ cone of $Par_n$, equivalently the problem of existence of rank $n$ unitary local sytems with given monodromy. There is a polyhedral cone controlling the solutions:

1. The face equations are given by GW numbers as shown by Agnihotri-Woodward and B.
2. The extremal rays have been studied more recently: These are of two kinds. There are ones corresponding to rigid local systems on the strange dual side (F-rays), and there is a second kind.
3. Any situation with a $\langle \sigma_{I^1}, \dots, \sigma_{I^s} \rangle_d = 1$ produces a bunch of F-rays by overdetermining the number of conditions by one. We get all unitary rigids such that there is some $A_i$ with an eigenvalue of multiplicity one.