Categorification via geometry (5 hr minicourse)

Higher Structures in China, II Changchun, China 8/11

Lecture 2: Categorifying the regular representation of the symmetric group

Let me remind you what we're done and what we're planning to do.

Yesterday: $X = \mathrm{Flag}(\mathbb{C}^n)$.

$$S_n \curvearrowright H_T^{2d}(X) \cong \mathbb{C}[S_n]$$

Today:
- Find an abelian category $\mathcal{C}$ with $K(\mathcal{C}) \otimes \mathbb{C} \cong H_T^{2d}(X)$
- Construct an action $B_n \curvearrowright D(\mathcal{C})$ categorifying the S_n action.
- Find an autoequivalence of $D(\mathcal{C})$ that swaps the two actions (categorifying $w \leftrightarrow w^{-1}$)

Let me start by telling you what our category will be. The definition will be entirely algebraic, though we will eventually need to tie it to geometry in order to get the Grothendieck group.

Let $\mathfrak{g} = \mathfrak{sl}_n$, $Z(\mathfrak{g})$ = center of $U(\mathfrak{g})$.

Let $Z(\mathfrak{g}) \to \mathbb{C}$ be as in the trivial rep, and let $U(\mathfrak{g})_0 = U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} \mathbb{C}$.

So reps of $U(\mathfrak{g})_0$ are reps of $U(\mathfrak{g})$ on which the center acts the way it does on the trivial rep. There are lots of these (just not many finite-dim'l ones)!

We'll be interested not in all $U(\mathfrak{g})_0$-modules, but in a particular subcategory.

Let $B = \{(0 \backslash \star)\} \subseteq SL_n$

$\mathfrak{b} = \mathrm{Lie}(\mathfrak{b}) \subseteq \mathfrak{g}$

$U(\mathfrak{b}) \subseteq U(\mathfrak{g})$.

Let $\mathcal{O}_0$ = f.g. $U(\mathfrak{g})_0$-modules on which $U(\mathfrak{b})$ acts locally finitely

($\forall v \in V$, $U(\mathfrak{b}) \cdot v$ is finite-dimensional)

This thing is called "category $\mathcal{O}$", and was introduced by BBG.

Two lies: ① This is just a block; we ought to consider all central characters

② Really we should allow $Z(\mathfrak{g})$ to act locally finitely and force $U(\mathfrak{h})$ to act semisimply, though the resulting categories (that is, the $\mathcal{O}$-blocks) are equivalent.

Let me now give you an overview of how we will identify the Grothendieck group of $\mathcal{O}_0$ with the top equivariant cohomology group of the flag variety.

$$\mathcal{O}_0 \xrightarrow{\text{"localization"}} \text{D-modules on } X \overset{gr}{\rightsquigarrow} \text{coherent sheaves on } M = T^*X \overset{\text{support}}{\rightsquigarrow} \text{cycles on } M$$

$$K(\mathcal{O}_0) \otimes \mathbb{C} \xrightarrow{\quad cc \quad} H^{2d}_T(M)$$

- First (I'll explain why $U(\mathfrak{g})_0$-modules can be thought of as $\mathcal{D}$-modules on X

- Next I'll explain how to get a coherent sheaf on M from a $\mathcal{D}$-module on X. This involves some choices, but the support of the sheaf (computed with multiplicity) is independent of these choices.

- It turns out that the support cycle is always T-equivariant and Lagrangian, so we get a class in $H^{2d}_T(M)$.

- Finally, this assignment descends to the Grothendieck group, and we get an isomorphism!

Let's begin with $\mathcal{D}$-modules.

Let $\mathcal{D}$ be the sheaf of diff'l operators on X (locally generated by functions and vector fields)

The group SL_n acts on X, so every element of the Lie algebra $\mathfrak{sl}_n$ gives a vector field on X. This induces a map from the universal enveloping algebra to the ring of global sections of $\mathcal{D}$.

$\mathfrak{sl}_n \curvearrowright X$ induces $\mathfrak{g} \longrightarrow \Gamma(TX)$

induces $U(\mathfrak{g}) \longrightarrow \Gamma(\mathcal{D})$

Thm (BB): $\mathcal{O}U(\mathfrak{g}) \longrightarrow \Gamma(\mathcal{D})$ (2) The functors $\mathcal{D}\text{-mod} \xrightarrow{\Gamma} U_0\text{-mod}$ and $U_0\text{-mod} \xrightarrow{Loc} \mathcal{D}\text{-mod}$, $L \longmapsto \mathcal{D} \otimes_{U_0} L$ are inverse equivalences of abelian categories.

$\mathcal{O}U(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g})_0$

We can ask where objects of $\mathcal{O}_0$ go under this equivalence, and there's a very nice answer.

On any D-module, $U(\mathfrak{g})_0 = \Gamma(D)$ acts,
so $U(\mathfrak{g})$ acts,
so $U(\mathfrak{b})$ acts,
so $\mathfrak{b}$ acts.

Def: A D-module is B-equivariant if the $\mathfrak{b}$-action integrates to a B-action.

Prop:
$$\begin{array}{ccc} U(\mathfrak{g})_0\text{-mod} & \xrightarrow{\sim} & D\text{-mod} \\ \cup & & \cup \\ \mathcal{O}_0 & \xrightarrow{\sim} & \text{fg B-eq D-mod} \end{array}$$

The $U(\mathfrak{b})$ local finiteness condition exactly translates to the B-equivariance condition.

Next I want to tell you how to go from a D-module to a coherent sheaf on M.

Fact: D is filtered by order, and $\operatorname{gr} D \cong p_* \mathcal{F}un_M$ (where $M \xrightarrow{p} X$)

Ex: $\Gamma(\operatorname{gr} D) = \operatorname{gr} \Gamma(D) = \operatorname{gr} U(\mathfrak{g})_0$

$\operatorname{gr} U(\mathfrak{g}) = \operatorname{Sym} \mathfrak{g} = \mathbb{C}[\mathfrak{g}^*]$

$\operatorname{gr} U(\mathfrak{g})_0 = \mathbb{C}[\text{nilpotent cone}] = \mathbb{C}[\operatorname{Spec} \mathbb{C}[M]]$

$= \mathbb{C}[M] = \Gamma(p_* \mathcal{F}un_M)$

$U(\mathfrak{g})_0$ is a quotient of $U(\mathfrak{g})$, so we have to get here some subspace of $\mathfrak{g}^*$

$\mathcal{P}$ affine $\Rightarrow$ $\left\{\begin{array}{l}\text{sheaves on } X \\ \text{of } \mathcal{P}_0\text{-Fun}_n\text{-modules}\end{array}\right\} \simeq \left\{\begin{array}{l}\text{sheaves on } M \\ \text{of Fun}_n\text{-modules}\end{array}\right\}$

f.g. filtered $\mathcal{D}$-modules $\xrightarrow{\text{gr}}$ coherent sheaves on M

Of course, this coherent sheaf depends on the choice of filtration, but amazingly enough its support cycle does not.

Thm (Ginzburg): 1. $CC(\mathcal{I}) = \text{Supp}(\text{gr}\,\mathcal{I})$ does not depend on the choice of filtration of $\mathcal{I}$

2. If $0 \to \mathcal{I}' \to \mathcal{I} \to \mathcal{I}'' \to 0$, then $CC(\mathcal{I}) = CC(\mathcal{I}') + CC(\mathcal{I}'')$.

Get $CC: K(U(\mathfrak{g})_0\text{-mod}) \to$ cycles on M (coisotropic)

This isn't quite the domain or the range we want. We want $K(\mathcal{O}_0)$ on the left, and we want to produce a T-invariant d-cycle, so that it defines a class in $H_T^{2d}(M)$.

Facts: 1. If $\mathcal{I}$ is B-eq, then $CC(\mathcal{I})$ is contained in the union of the conormal bundles to the B-orbits on X.

This would be true for any group action.

2. These conormal bundles are Lagrangian and B-invt, $\therefore$ also T-invt.

$\therefore$ Since there are finitely many B-orbits, $CC(\mathcal{I})$ is a sum of T-invt Lagrangians.

Get $CC: K(\mathcal{O}_0) \longrightarrow H_T^{2d}(M)$

Thm: This is an isomorphism.

Now I need to tell you how to categorify the convolution action. We do this via tensor products with bimodules.

Def: Let HC_0 be the category of f.g. $U(\mathfrak{g})_0$-bimodules that are locally finite for the adjoint $U(\mathfrak{g})_0$-action.

$$HC_0 \xrightarrow[\sim]{Loc} \text{f.g. } G\text{-eq } D\boxtimes D^{op}\text{-modules on } X\times X \xrightarrow{CC} \text{cycles on } M\times M$$

This is a tensor category, if you like, a 2-category!

What kind of cycles do we get? They will be coisotropic and contained in the union of the conormal bundles to the G-orbits on $X\times X$. It turns out that we have already seen these conormal bundles!

Fact: $\bigcup_{\substack{\text{conormal bundles}\\ \text{to orbits of}\\ G\circlearrowright X\times X}} = M\times_N M = Z = \bigcup_{i=1}^{n!} Z_i$

So $CC : K(HC_0) \longrightarrow \mathbb{C}\{[Z_1],\dots,[Z_{n!}]\} = B$

This is the convolution algebra from yesterday's lecture.

This says that the 2-category HC_0 acts on the category $\mathcal{O}_0$.

Thm 1. If $P\in HC_0$, then $P\otimes\_ : \mathcal{O}_0 \to \mathcal{O}_0$

2. CC is an algebra homomorphism

3. For all $L\in\mathcal{O}_0$, $\underset{\substack{\rotatebox{90}{$\in$}\\ H^{2d}_T(M)}}{CC(P\otimes L)} = \underset{\substack{\rotatebox{90}{$\in$}\\ B}}{CC(P)} * \underset{\substack{\rotatebox{90}{$\in$}\\ H^{2d}_T(M)}}{CC(L)}$

The ring structure on the left is tensor product.

This says that tensor product categorifies convolution.

Now the last remarkable fact is that when we lift our convolution operators to bimodules, we get an action of the braid group on the derived category.

Thm: $\forall w \in S_n$, $\exists P_w \in HC_0$ such that

i) $CC(P_w) \in B$
$\quad\;\; \| \qquad\quad \wr\|$
$\quad\;\; w \;\;\in\;\; \mathbb{C}[S_n]$

ii) $F_w : P_w \overset{L}{\otimes} _ : D(\mathcal{O}_0) \longrightarrow D(\mathcal{O}_0)$ is invertible

iii) If $\ell(w) + \ell(w') = \ell(ww')$, then $F_w \circ F_{w'} = F_{ww'}$.

In particular, if w is a simple transposition, this does not say that F_w is its own inverse. This is exactly what we need for a braid group action (rather than a symmetric group action) on our category.

I promised one more piece of structure, namely an autoequivalence that categorifies the map sending w to w^{-1}. This comes from Koszul duality.

Let $\mathcal{C}$ be an abelian category with simples $S_1, \ldots, S_r$ and projective covers $P_1, \ldots, P_r$.

Let $A := \mathrm{Hom}(\oplus P_i, \oplus P_i)$. Then

$\mathrm{Hom}(\oplus P_i, -) : \mathcal{C} \xrightarrow{\sim} \text{f.g. right } A\text{-mod}$

There's another ring we can build out of $\mathcal{C}$, this one graded

Let $E = Ext^{\cdot}_{\mathcal{C}}(\oplus S_i, \oplus S_i) = RHom_{D(\mathcal{C})}(\oplus S_i, \oplus S_i)$

Let $\mathcal{C}^! = $ f.g. right $E^!$-mod; get $A^!$ and $E^!$

Def: If $E^! \cong A$, $A^! \cong E$, then $\mathcal{C}$ & $\mathcal{C}^!$ are Koszul dual.

There is a more intrinsic definition of Koszulity, but this will do for now.

Let $\mathcal{C}_{gr}$ = f.g. right graded $E^!$-mod

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Define $i : D(\mathcal{C}_{gr}) \longrightarrow D(\mathcal{C}^!_{gr})$

$Q \longmapsto RHom(\oplus S_i, -)$

Thm (BGS): 1. i is an equivalence.

2. If $\mathcal{C} = \mathcal{O}_0$, then $A \cong E$, so $\mathcal{O}_0$ is self-dual.

I didn't quite deliver what I promised, because i is a derived autoequivalence of the graded lift of $\mathcal{O}_0$, rather than of $\mathcal{O}_0$ itself, and i does not respect the grading. But it's pretty good.

If there's still time, I'll say a few words about how the grading fits into the picture.

$K(\mathcal{O}_0^{gr}) \otimes \mathbb{C} \cong H_n = \mathbb{C}[q, q^{-1}]$-algebra st.
$q \mapsto -1$ or $q \mapsto 1$ gives $\mathbb{C}[S_n]$

$B_n \subset \mathcal{O}_0$ lifts to $B_n \subset \mathcal{O}_0^{gr}$

inducing $B_n \subset H_n$

and $S_n \subset \mathbb{C}[S_n]$ when $q \mapsto \pm 1$

$i : K(\mathcal{O}_0^{gr}) \longrightarrow K(\mathcal{O}_0^{gr})$
$q \longmapsto -q$

so it induces

$$\begin{array}{ccc} H_n/\langle q-1\rangle & \xrightarrow{\sim} & H_n/\langle q+1\rangle \\ \| & & \| \\ \mathbb{C}[S_n] & \longrightarrow & \mathbb{C}[S_n] \\ w & \longmapsto & w^{-1} \end{array}$$

H_n is called the Hecke algebra.