Categorification via geometry (3 hr minicourse)

Higher Structures in China, II Changchun, China 8/11

Lecture 3: Geometric category $\mathcal{O}$ and symplectic duality

The point of this lecture is that everything in my previous two lectures had to do with the geometry of the cotangent bundle to the flag variety, and if we start with a similar space, we should be able to tell a similar story.

What is a "similar space"?

Def: A conical symplectic resolution is:

- a smooth complex variety M
- an algebraic 2-form ω
- an action of $S \cong \mathbb{C}^\times$

satisfying

- $$\begin{array}{c} M \\ \downarrow \pi \\ N = \operatorname{Spec} \mathbb{C}[M] \end{array}$$ is a resolution of singularities (M has lots of functions)

- $S \circlearrowright \mathbb{C}[M]$ with non-negative weights, and $\mathbb{C}[M]^S = \mathbb{C}$

- $s^*\omega = s^m \omega$ for some $m > 0$

This is the "conical" part.

Examples: 1. X a projective variety,
$M = T^* X$
$s \cdot (x, \alpha) = (x, s^{-1}\alpha)$
$m = 1$.

Conj: The only examples for which $\begin{smallmatrix} M \\ \downarrow \\ N \end{smallmatrix}$ is a resolution are when $X = G/P$ for some reductive G and parabolic $P \subseteq G$, eg $SL_n/B \cong \mathrm{Flag}(\mathbb{C}^n)$.

This conjecture has been made by many people. It makes sense, because M has lots of functions iff X has lots of vector fields.

2. $M = \mathrm{Hilb}^n \mathbb{C}^2$ — S acts via action on $\mathbb{C}^2$
$\downarrow$ — $m = 2$
$N = \mathrm{Sym}^n \mathbb{C}^2 = \mathbb{C}^{2n}/S_n$

3. $\Gamma \subset SL_2 \mathbb{C}$ finite
$M = \widetilde{\mathbb{C}^2/\Gamma}$ — S acts via action on $\mathbb{C}^2$
$\downarrow$ — $m = 2$
$N = \mathbb{C}^2/\Gamma$

4. $G \circlearrowright V$ linear — S contracts T^*V
$G \circlearrowright T^*V \xrightarrow{\Phi} \mathfrak{g}^*$ — $m = 2$
$\alpha \in \mathrm{Hom}(G, \mathbb{C}^\times)$
$M = T^*V /\!\!/_\alpha G = \Phi^{-1}(0) /\!\!/_\alpha G$
$\downarrow$
$N = \Phi^{-1}(0) /\!\!/_0 G$ — eg hyperkaehler varieties (G abelian) quiver varieties

In fact, 2 and 3 are special cases of 4, as is 1 if $G = SL_n$ (with a modified S-action).

Okay, now you know what kind of spaces we want to look at. What do we want to do with them?

Def: A quantization of (M, ω, S) is an S-eq sheaf of $\mathbb{C}[[\hbar]]$-algebras W on M such that

- $W \underset{loc}{\cong} \mathrm{Fun}_M[[\hbar]]$
- $f * g = fg + \hbar\{f,g\} + O(\hbar^2)$ (thus $W/\hbar W = \mathrm{Fun}_M$)

Note that the Poisson bracket lowers degree by m, so $\hbar$ has to have weight m.

Ex: $\mathbb{C}^2$ has a unique quantization W with $\Gamma(W) = \mathbb{C}[[\hbar]]\langle x, y\rangle / \langle xy - yx - \hbar\rangle$

Thm (BK): Quantizations are classified by $H^2(M; \mathbb{C})$.

At this point you may be wondering what this has to do with the previous two lectures. The point is that D-modules on X may be interpreted as modules over a quantization of T^*X.

Let $\tilde{W} = W[\hbar^{-1/m}]$, and let $\tilde{W}$-mod be the category of sheaves of modules $\tilde{P}$ over $\tilde{W}$ st $\exists$ a W-module $P \subset \tilde{P}$ with
- $\tilde{W} \otimes_W P = \tilde{P}$,
- $P/\hbar P$ is a coherent sheaf.

Note that the choice of P is not part of the data!

Prop: Let X be a projective variety, D the sheaf of diff'l operators on X. $\exists!$ quantization W of T^*X and map $p^{-1}D \to W$ st

$$\mu : \text{f.g. } D\text{-mod} \xrightarrow{\sim} \widetilde{W}\text{-mod}$$
$$L \longmapsto \widetilde{W} \otimes_{p^{-1}D} p^{-1}L$$

microlocalization

$$\Gamma(L) = \Gamma(\mu(L))^S$$

filtration $\longleftrightarrow$ choice of lattice

$$\operatorname{gr} L = \mu(L)(0) / \hbar\, \mu(L)(0)$$

(Interestingly enough, the parameter in $H^2(T^*X;\mathbb{C})$ corresponding to this quantization is not 0, but rather $\frac{1}{2}$ the Euler class of the canonical bundle of X.)

So now we've come to the main point: we have a category of sheaves on M that generalizes the notion of D-modules on a projective variety.

(Note: All of this works without the condition that $M \to N$ is a resolution of singularities. That will be important later.)

Let $A = \Gamma(\widetilde{W})^S$.

This is the analogue of global differential operators.

Ex: 1. If $M = \mathbb{C}^2$,

$$A = \left(\mathbb{C}((\hbar^{1/2}))\langle x, y\rangle / \langle xy - yx - \hbar\rangle\right)^S \cong \Gamma(D_{\mathbb{C}})$$

$$\hbar^{-1/2} x \longmapsto z$$
$$\hbar^{-1/2} y \longmapsto \partial$$

Sle that is filtered by grees in $\hbar$, d its associated rated is morphic to e ring of rations on M.

We saw yesterday that gr of this guy is the ring of functions on M.

2. If $M = T^*(G/B)$, A is a central quotient of $U(\mathfrak{g})$.

3. If $M = \mathrm{Hilb}^n \mathbb{C}^2$, A is a central quotient of the spherical rational Cherednik algebra.

I say this only to point out that it's an algebra which is interesting enough to have a name...

4. If $M = T^*V /\!\!/_{\alpha} G$, A is a central quotient of $\Gamma(\mathcal{D}_V)^G$

When G is a torus, there's a whole book written about this ring, by Musson & van den Bergh.

Once again, we have sections and localization functors.

$$\tilde{W}\text{-mod} \underset{\tilde{W} \otimes_A -}{\overset{\Gamma(-)^S}{\rightleftarrows}} \text{fg } A\text{-mod}$$

Prop (BLPW): For "many" quantizations of M, these functors are inverse equivalences.

I won't try to make this precise now, but basically it says that if you have any parameter in $H^2(M;\mathbb{C})$, you can obtain a new parameter where this works by adding a large enough multiple of the Euler class of any ample line bundle.

This is the analogue of the Beilinson-Bernstein theorem. From now on we'll assume that we've chosen a quantization for which this holds.

Of course, we don't want to think about all W-modules or all A-modules, we want to think about the analogue of category $\mathcal{O}$. For this we need another piece of structure, namely a second action of $\mathbb{C}^*$.

Let $T = \mathbb{C}^* \circlearrowright (M, \omega, S, W)$ st

- M^T is finite
- T preserves ω
- T commutes with S
- $A = \bigoplus_{n \in \mathbb{Z}} A_n$ (T-wt spaces)

and $\exists\, \xi \in A$ st $[\xi, a] = na \;\forall\, a \in A_n$.

A choice of such a ξ is sometimes called a noncommutative moment map, because its image in $\operatorname{gr} A \cong \mathbb{C}[M]$ is a classical moment map.

Let $A^+ = \bigoplus_{n \geq 0} A_n$.

Ex: If $M = T^*(G/B)$, $A = U(\mathfrak{g})_0$, $A^+ \supsetneq$ image of $U(\mathfrak{b})$.

Def: $\mathcal{O}$ = f.g. A-modules on which A^+ acts locally finitely.

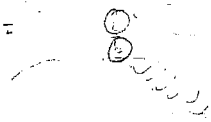
It's not too hard to check that in the Lie theory case, we get the same category as before. (A priori the new one is smaller.)

Where does this category go under localization?

Let $M^+ = \{ m \in M \mid \lim_{t \to 0} t \cdot m \text{ exists} \} = \bigcup_{p \in M^T} \{ m \in M \mid \lim_{t \to 0} t \cdot m = p \}$

Lagrangian

Ex: $M = T^*\mathbb{P}^1$, $M^+ =$ 

$M = \widetilde{\mathbb{C}^2/\mathbb{Z}_3}$, $M^+ =$ 

$M = T^*\mathrm{Flag}(\mathbb{C}^n)$, $M^+ = \bigcup$ conormal bundles to B-orbits

Thm (BLPW): 1. fg A-mod $\xrightarrow[\text{Loc}]{\sim}$ W-mod

$\cup\!\!\mid$ $\qquad\qquad\qquad\qquad$ $\cup\!\!\mid$

$\mathcal{O}$ $\xrightarrow{\sim}$ modules that admit a T-inv't lattice and are supported on M^+

2. $K(\mathcal{O})\otimes\mathbb{C} \xrightarrow[CC]{\sim} H_T^{\dim M}(M)$

Supports are always coisotropic, so this tells us that they are Lagrangian and T-invariant.

Now I just want to try to make all of the obvious statements that generalize the results from my previous two lectures. I'm supposed to have a convolution action on the cohomology of M (ordinary or equivariant), and it is supposed to be categorified by an action on $\mathcal{O}$ given by tensor products with bimodules.

Let $Z = M \times_N M = \{(m, m') \mid \pi(m) = \pi(m')\}$

Fact: Every component $Z_1, \ldots, Z_r \subseteq Z$ has the same dimension as M.

Let $B = \mathbb{C}\{[Z_1], \ldots, [Z_r]\}$. B has an algebra structure, and $B \circlearrowright H^*(M)$, $H_T^{\dim M}(M)$.

All of this B just like the first lecture. What is not necessarily true is that B is the group algebra of some group, and $H^*(M)$ is the regular representation. The following theorem is the analogue of those statements.

Thm (Ginzburg): 1. B is semisimple. In particular, $H_T^{\dim M}(M) \cong \operatorname{gr} H_T^{\dim M}(M) \cong H^*(M)$ as a B-module.

2. Every irrep of B shows up in $H^*(M)$.

Now let's categorify as before.

Let HC be the tensor-category of A-bimodules that are locally finite for the adjoint action

$$HC \xrightarrow{Loc} \widetilde{W} \boxtimes \widetilde{W}^{op}\text{-mod} \xrightarrow{CC} \text{cycles on } Z \subset M \times M$$

$$K(HC) \otimes \mathbb{C} \longrightarrow B$$

As before, the localization to $M \times M$ of a Harish-Chandra bimodule is supported on Z, thus we get a map from the Grothendieck group to B.

Thm (BLPW): 1. HC acts on $\mathcal{O}$ by tensor product

2. CC is an algebra homomorphism

3. If $P \in HC$, $L \in \mathcal{O}$, $CC(P \otimes L) = CC(P) \cdot CC(L)$.

Conj: $CC : K(HC) \otimes \mathbb{C} \longrightarrow B$ is surjective.

We've generalized all the stuff about group actions on categories, but I still haven't said anything about Koszul duality. That's the most interesting part!

Conj (BLPW): 1. $\mathcal{O}$ is Koszul.
2. $\exists\, (M', \omega', S', W', T')$ st $\mathcal{O}'$ is dual to $\mathcal{O}$.

Ex: $T^*(G/B)$ dual to $T^*(G^\vee/B^\vee)$

$T^*\mathbb{P}^n$ dual to $\widetilde{\mathbb{C}^2/\mathbb{Z}_{n+1}}$

Both of these generalize the self-duality of $T^*\mathbb{P}^1$.

3. The actions $B \circlearrowright K(\mathcal{O}) \cong K(\mathcal{O}') \circlearrowleft B'$ satisfy the double commutator property.

That is, the action of B' includes all the endomorphisms that commute with the action of B, and vice-versa. This is well-known for the left and right actions on the regular representation of a group.

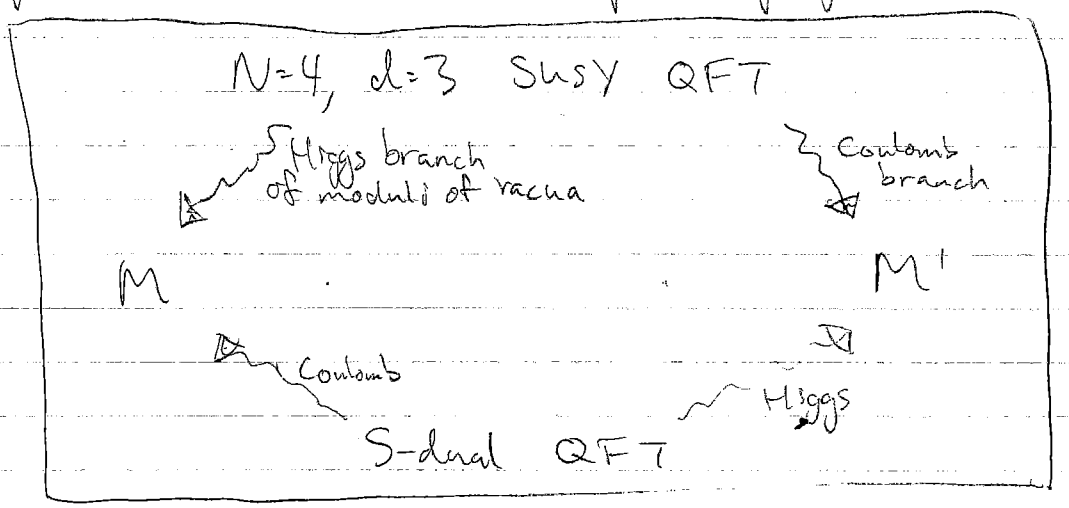
We call this package of conjectures (along with many others that I haven't explained) "symplectic duality". There are two large families of examples for which we can prove everything.

Thm (BLPW): These conjectures hold for
- various spaces related to $T^*(G/B)$
- hypertoric varieties $T^*V /\!\!/ G$, G abelian

In the first case, the work was to show that the categories we get are certain much-studied categories in representation theory that were already known to come in dual pairs (parabolic-singular duality).

In the second case, the categories are completely new! This picture categorifies something known as Gale duality of matroids.

Finally, let me note that the same examples of pairs of spaces seem to come up in physics!



Crazy observation: All known examples of symplectic dual pairs arise as Higgs & Coulomb branches of a QFT, or as Higgs branches of two S-dual theories!

The connection is still completely mysterious.