ANNALS OF MATHEMATICS
Vol. 46, No. 3, July, 1945

RELATIONS BETWEEN HOMOLOGY AND HOMOTOPY GROUPS OF SPACES*

BY SAMUEL EILENBERG AND SAUNDERS MACLANE

(Received February 16, 1945)

CONTENTS

INTRODUCTION

A. This paper is a continuation of an investigation, started by H. Hopf [5][6], studying the influence of the fundamental group $\pi_1(X)$ on the homology structure of the space X.

We shall consider an *arcwise connected topological space*[1] X and the following groups derived from X:

$\pi_n(X)$—the n^{th} homotopy group of X constructed relative to some point $x_0 \in X$ as base point. In particular, the 1^{st} homotopy group $\pi_1(X)$ is the fundamental group of X, see [7_1].

$H^n(X, G)$—the n^{th} homology group of X with coefficient group G. Both G and $H^n(X, G)$ are discrete abelian groups. If $G = I$ is the additive group of integers, we write $H^n(X)$ instead of $H^n(X, I)$.

$\Sigma^n(X)$—the *spherical* subgroup of $H^n(X)$; this is the image of the group $\pi_n(X)$ under the natural homomorphism $\nu_n : \pi_n(X) \to H^n(X)$.

$H_n(X, G)$—the n^{th} cohomology group of X with G as coefficient group. Both G and $H_n(X, G)$ are topological abelian groups.[2]

The homology and cohomology groups of X are defined (Ch. II) using singular simplexes in X, with ordered vertices, as recently introduced by one of the authors [1].

---

* Presented to the American Mathematical Society, April 23, 1943. Most of the results were published without proof in a preliminary report [3]. The numbers in brackets refer to the bibliography at the end of the paper.

(Added in proof) After this paper had been submitted for publication, a paper by H. Hopf, *Über die Bettischen Gruppen, die zur einer beliebigen Gruppe gehören*, Comment. Math. Helv. 17 (1944), pp. 39–79, came to the authors' attention. Although the methods employed are quite different, the two papers overlap considerably.

[1] A topological space is a set with a family of subsets called "open sets" subject to the following axioms: The union of any number of open sets is open, the intersection of two open sets is open: the empty set and the whole space are open.

[2] A topological group is one which carries a topology with respect to which the group operations are continuous. No separation axioms are assumed.

B. We list the known results dealing with the relations between the homology and homotopy groups.

1°) The group $H^1(X)$ is isomorphic with the factor group of $\pi_1(X)$ by its commutator subgroup.

2°) If $\pi_n(X) = 0$ for $0 < n < r$ then $H^r(X) \cong \pi_r(X)$; Hurewicz [7_2].

3°) A space X is called *aspherical* if $\pi_i(X) = 0$ for $i > 1$. In an arcwise connected aspherical space the fundamental group $\pi_1(X)$ determines all the homology and cohomology groups of X; Hurewicz [7_4]. The algebraic mechanism of this determination was unknown.

4°) The group $\pi_1(X)$ determines the group $H^2(X)/\Sigma^2(X)$. This was proved by Hopf [5] only in the case when X is a connected polyhedron. The word "determines" is used in the following sense. Given any group $\Pi = F/R$ represented as a factor group of a (non abelian) free group F by an invariant subgroup R, consider the group

$$h^2(\Pi) = R \cap [F, F]/[F, R] \tag{1}$$

where $\cap$ stands for set theoretic intersection and $[A, B]$ is the subgroup generated by all elements of the form $aba^{-1}b^{-1}$, for $a \in A$, $b \in B$. It was shown by Hopf that $h^2(\Pi)$ depends only on Π and not on the representation $\Pi = F/R$, and that

$$H^2(X)/\Sigma^2(X) \cong h^2[\pi_1(X)].$$

5°) If $\pi_n(X) = 0$ for $1 < n < r$ then the group $\pi_1(X)$ determines the group $H^r(X)/\Sigma^r(X)$. This was proved by Hopf [6] only in the case when X is a connected polyhedron. The proof gives no algebraic procedure for the determination.

The theorems of this paper include and generalize all these results. Moreover, we succeed in getting a complete algebraic formulation of the group constructions needed for the various "determinations."

C. The task of handling a variety of group constructions, which, judging from the complexity of formula (1), is likely to become quite involved, is simplified by the following device developed in Chapter I. Given a (non abelian) group Π we construct an abstract complex $K(\Pi)$. The homology and cohomology groups of this complex, denoted by $H^n(\Pi, G)$ and $H_n(\Pi, G)$, turn out to be precisely the groups needed for the various descriptions. The cohomology groups $H_n(\Pi, G)$ can be described directly (without the complex $K(\Pi)$) as follows.

A function f of n variables from the group Π with values in the topological abelian group G will be called an n-cochain. The coboundary δf of f is the $(n + 1)$-cochain defined by

$$\begin{aligned}(\delta f)(x_1, \cdots, x_{n+1}) &= f(x_2, \cdots, x_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i f(x_1, \cdots, x_i x_{i+1}, \cdots, x_{n+1}) + (-1)^{n+1} f(x_1, \cdots x_n).\end{aligned} \tag{2}$$

The cochains form an additive group $C_n(\Pi, G)$.[3] The cocycles (i.e., cochains f with $\delta f = 0$) form a subgroup $Z_n(\Pi, G)$. Since $\delta\delta f = 0$ it follows that the coboundaries (i.e., cochains f of the form $f = \delta g$ for $g \in C_{n-1}(\Pi, G)$) form a subgroup $B_n(\Pi, G)$ of $Z_n(\Pi, G)$. The cohomology groups of Π are then defined as the factor groups[4]

$$H_n(\Pi, G) = Z_n(\Pi, G)/B_n(\Pi, G).$$

For $n = 0, 1, 2$ the cohomology groups $H_n(\Pi, G)$ furnish well known invariants

$$H_0(\Pi, G) \cong G$$

$$H_1(\Pi, G) \cong \text{Hom}\ (\Pi, G)$$

$$H_2(\Pi, G) \cong \text{Extcent}\ (G, \Pi)$$

where Hom (Π, G) stands for the group of all homomorphisms $\varphi: \Pi \to G$, and Extcent (G, Π) for the group of all central group extensions of the group G by the group Π.[5]

The homology groups $H^n(\Pi)$ with integral coefficients have the following values for $n = 0, 1, 2.$

$$H^0(\Pi) \cong I$$

$$H^1(\Pi) \cong \Pi/[\Pi, \Pi]$$

$$H^2(\Pi) \cong \text{CharExtcent}\ (P, \Pi),$$

where P denotes the group of real numbers reduced mod 1, while Char G is the group of all characters of the group G, i.e., Char G = Hom (G, P).

The development of the algebraic ideas of this paper was purposely limited to the needs of the topological applications. Consequently, we have entirely omitted the discussion of the algebraically important case when the group Π acts as a group of operators on the coefficient group G. We will return to this subject in another paper.[6]

D. Proposition 3°) can now be formulated as follows:

THEOREM I. *If X is arcwise connected and aspherical then the homology and*

---

[3] $C_n(\pi, G)$ is topologized as follows. Given an n-tuple $(x_1, \cdots, x_n)$ and given an open set U in G, consider the set of n-cochains f such that $f(x_1, \cdots, x_n) \in U$ as a basic open set in $C_n(\pi, G)$. Arbitrary open sets in $C_n(\pi, G)$ can be obtained from the basic ones using finite intersections and arbitrary unions. It is easy to verify that the homomorphism δ is continuous with respect to this topology.

[4] Since no separation axioms in topological groups are postulated, the factor group Z_q/B_q is topological even if B_q is not a closed sub-group of Z_q.

[5] For more details see §4 below. This is the second application of the group of group extensions to problems in topology. In a previous paper [2] the authors have studied the group of abelian extensions in connection with the problem of classifying and computing the homology and cohomology groups for various coefficient groups. Both groups Hom (Π, G) and Extcent (G, Π) carry a topology; see [2], p. 762 and p. 770.

[6] See Bull. Amer. Math. Soc. 50 (1944), p. 53.

cohomology groups of X are determined by the fundamental group $\pi_1(X)$. More precisely,

$$H^n(X, G) \cong H^n(\pi_1(X), G),$$
$$H_n(X, G) \cong H_n(\pi_1(X), G).$$

In order to formulate our generalization of Hopf's propositions 4°) and 5°) we need the following groups derived from the spherical subgroup $\Sigma^n(X)$:
$\Sigma^n(X, G)$—the subgroup of $H^n(X, G)$ consisting of all elements of the form $\Sigma g_i z_i$ where $g_i \in G$, $z_i \in \Sigma^n(X)$.
$\Lambda_n(X, G)$—the subgroup of $H_n(X, G)$ consisting of those cohomology classes that annihilate every element of $\Sigma^n(X)$, when the Kronecker index is the multiplication.

THEOREM II. *If X is arcwise connected and*

$$\pi_n(X) = 0 \quad \text{for} \quad 1 < n < r$$

then

$$H^n(X, G) \cong H^n(\pi_1(X), G) \quad \text{for } n < r$$
$$H_n(X, G) \cong H_n(\pi_1(X), G) \quad \text{for } n < r$$
$$H^r(X, G)/\Sigma^r(X, G) \cong H^r(\pi_1(X), G)$$
$$\Lambda_r(X, G) \cong H_r(\pi_1(X), G).$$

Both Theorems I and II are derived in Chapter II.

E. It was shown by Hopf [5] that the group $\pi_1(X)$ not only determines the group H^2/Σ^2 but also has a bearing upon the products (i.e., the cup and cap products) in X. In Chapter III we define products in the complex $K(\Pi)$ and show that the isomorphisms constructed in proving Theorems I and II preserve the products.

F. In Chapter IV we consider the case when $\pi_n(X) = 0$ for $n < q$ and study the influence of the group $\pi_q(X)$ upon the homology structure of X. We obtain a theorem similar to the preceding Theorem II. The group constructions are also derived from a suitably defined abstract complex; however, we do not have the algebraic interpretations as in the preceding case.

G. Theorem II implies that

$$H^2(X)/\Sigma^2(X) \cong H^2(\pi_1(X))$$

for every arcwise connected space X. On the other hand Hopf [5] has shown that

$$H^2(X)/\Sigma^2(X) \cong h^2(\pi_1(X))$$

for every connected polyhedron, with the group h^2 defined as above. Comparing these two results we obtain an isomorphism

$$h^2(\Pi) \cong H^2(\Pi) \cong \text{CharExtcent}\,(P, \Pi) \tag{3}$$

for all groups Π that are fundamental groups of connected polyhedra; i.e., for groups that can be described by means of a finite number of generators and relations. The fact is that the isomorphism (3) can be established purely algebraically for all discrete groups Π. The proof is not given in this paper.

CHAPTER I

CONSTRUCTIONS ON GROUPS

1. The complex $K(\Pi)$

For any discrete (multiplicative) group Π we define an abstract complex $K(\Pi)$ as follows. An n-dimensional cell σ^n ($n \geqq 0$) in $K(\Pi)$ is to be an ordered array

$$\sigma^n = [x_0, \cdots, x_n] \tag{1.1}$$

of $n + 1$ elements of Π, subject to the equivalence relation

$$[x_0, \cdots, x_n] = [xx_0, \cdots, xx_n] \tag{1.2}$$

for all $x \in \Pi$. Consequently there is only one 0-cell, namely $[x]$.

The boundary relation is

$$\partial[x_0, \cdots, x_n] = \sum_{i=0}^{n} (-1)^i [x_0, \cdots, \hat{x}_i, \cdots, x_n] \tag{1.3}$$

where $\hat{x}_i$ on the right means that the argument x_i is to be omitted.

After collecting terms, (1.3) can be rewritten as a sum over all the $(n-1)$-cells σ^{n-1} of $K(\Pi)$ with integral coefficients $[\sigma^n; \sigma^{n-1}]$, in the form

$$\partial\sigma^n = \sum_{\sigma^{n-1}} [\sigma^n; \sigma^{n-1}]\sigma^{n-1}$$

which defines the incidence numbers $[\sigma^n; \sigma^{n-1}]$. Because of possible repetitions on the right side of (1.3) the incidence numbers may be integers others than 0, 1 or -1.

We verify at once that

$$\partial\partial = 0;$$

and consequently the cells σ^n with the incidence numbers $[\sigma^n; \sigma^{n-1}]$ define a closure finite abstract complex $K(\Pi)$; see [1, §1].

In this complex we may consider finite chains over a discrete abelian coefficient group G; they lead to discrete homology groups

$$H^n(\Pi, G) = H^n(K(\Pi), G).$$

Similarly we may consider (infinite) cochains in $K(\Pi)$ with coefficients in a topological abelian group G; they lead to topologized cohomology groups[7]

$$H_n(\Pi, G) = H_n(K(\Pi), G).$$

[7] It is easy to see that the topology of $C_n(\Pi, G)$ as defined in [3] is the usual topology of the group of cochains in an abstract complex; see [1, §3].

In general the homology and cohomology groups of the complex $K(\Pi)$ will be referred to as the homology and cohomology groups of the group Π over a suitable abelian group G as coefficient group. Similarly we shall understand the groups $C^n(\Pi, G)$, $Z^n(\Pi, G)$, and $B^n(\Pi, G)$ of the chains, cycles, and bounding cycles of Π to be the appropriate groups associated with the complex $K(\Pi)$. The topological groups $C_n(\Pi, G)$, $Z_n(\Pi, G)$, and $B_n(\Pi, G)$ of the cochains, cocycles, and coboundaries are interpreted in analogous fashion.

Since an n-dimensional cochain over a group G is a function which with each n-cell σ^n associates an element of G, a cochain of $C_n(\Pi, G)$ can be interpreted as a function F of $n + 1$ variables on Π, with values in G, subject to the conditions

$$F(xx_0, \cdots, xx_n) = F(x_0, \cdots, x_n) \tag{1.4}$$

for all $x \in \Pi$. The coboundary is then given by the formula

$$(\delta F)(x_0, \cdots, x_{n+1}) = \sum_{i=0}^{n+1} (-1)^i F(x_0, \cdots, \hat{x}_i, \cdots, x_{n+1}). \tag{1.5}$$

In the following sections two alternative definitions of the complex $K(\Pi)$ will be given. For purposes of identification the method adopted in the present section will be designated as the *homogenous* method.

In the complex $K(\Pi)$ we have the well known duality between homology and cohomology, expressed as an isomorphism (see [8, p. 129])

$$H^n(\Pi, G) \cong \text{Char } H_n(\Pi, \text{Char } G) \tag{1.6}$$

for any discrete abelian group G. This formula gives the homology groups of Π once the cohomology groups of Π with compact coefficients are known.

2. Matric definition of $K(\Pi)$

For the purpose of the applications of the complex $K(\Pi)$ it is convenient to have an alternative definition based on certain matrices.

We shall consider $(n + 1) \times (n + 1)$ matrices

$$\Delta = \| d_{ij} \|, i, j = 0, 1, \cdots, n,$$

with elements d_{ij} in Π such that

$$d_{ij}d_{jk} = d_{ik}, \quad i, j, k = 0, 1, \cdots, n. \tag{2.1}$$

This condition implies that

$$d_{ii} = 1, \quad d_{ij} = d_{ji}^{-1}, \quad d_{ik} = d_{0i}^{-1}d_{0k}.$$

With each such matrix Δ we will associate an n-cell of the complex $K(\Pi)$ according to the correspondence

$$\Delta \to [d_{00}, d_{01}, \cdots, d_{0n}]. \tag{2.2}$$

Conversely, given an n-cell $[x_0, \cdots, x_n]$ of $K(\Pi)$ we construct a matrix $\Delta = \| x_i^{-1}x_j \|$. This matrix satisfies condition (2.1). We verify at once that this procedure gives a 1-1-correspondence between the matrices Δ described above and the n-cells of $K(\Pi)$. Consequently we may regard each n-cell of $K(\Pi)$ as a matrix and vice-versa. In order to translate the boundary relation (1.3) into the matric terminology we notice that if

$$[x_0, \cdots, x_n] \to \Delta$$

then

$$[x_0, \cdots, \hat{x}_i, \cdots, x_n] \to \Delta^{(i)},$$

where $\Delta^{(i)}$ is the matrix obtained from Δ by erasing the i^{th} row and the i^{th} column. Consequently the boundary relation becomes

$$\partial\Delta = \sum_{i=0}^{n} (-1)^i \Delta^{(i)} \tag{2.3}$$

3. The non-homogenous description of $K(\Pi)$

For the algebraic applications of the complex $K(\Pi)$ it is convenient to have a third, "non-homogenous", description of $K(\Pi)$.

We shall consider n-tuples $(x_1, \cdots, x_n)$ of elements $x_i \in \Pi$. With each n-tuple associate an n-cell of $K(\Pi)$ as follows:

$$(x_1, \cdots, x_n) \to [1, x_1, x_1x_2, \cdots, x_1 \cdots x_n];$$

conversely with each n-cell we can associate an n-tuple by the rule

$$[x_0, \cdots, x_n] \to (x_0^{-1}x_1, x_1^{-1}x_2, \cdots, x_{n-1}^{-1}x_n). \tag{3.1}$$

This gives a 1-1 correspondence between the n-cells of $K(\Pi)$ and the n-tuples $(x_1, \cdots, x_n)$, so that each n-cell of $K(\Pi)$ may be regarded as an n-tuple $(x_1, \cdots, x_n)$ and vice versa. The 0-cell of $K(\Pi)$ becomes then the (vacuous) 0-tuple ( ).

In order to translate the boundary relation (1.3) we notice that

$$\partial[1, x_1, x_1x_2, \cdots, x_1 \cdots x_n] = [x_1, x_1x_2, \cdots, x_1 \cdots x_n] + \sum_{i=1}^{n} (-1)^i [1, x_1, x_1x_2, \cdots, x_1 \cdot \hat{\cdot} \cdot x_i, \cdots, x_1 \cdots x_n],$$

and that

$$[x_1, x_1x_2, \cdots, x_1 \cdots x_n] = [1, x_2x_3, \cdots, x_2 \cdots x_n] \to (x_2, \cdots, x_n),$$

$$[1, x_1, x_1x_2, \cdots, x_1 \cdot \hat{\cdot} \cdot x_i, \cdots, x_1 \cdots x_n] \to (x_1, \cdots, x_ix_{i+1}, \cdots, x_n) \text{ for } i < n,$$

$$[1, x_1, x_1x_2, \cdots, x_1 \cdots x_{n-1}] \to (x_1, \cdots, x_{n-1}),$$

so that

$$\partial(x_1, \cdots, x_n) = (x_2, \cdots, x_n) + \sum_{i=1}^{n-1} (-1)^i (x_1, \cdots, x_ix_{i+1}, \cdots, x_n) + (-1)^n (x_1, \cdots, x_{n-1}). \tag{3.2}$$

This is the boundary relation in the non-homogenous definition of the complex $K(\Pi)$.

Using this definition, an n-cochain of Π over a coefficient group G is a function f of n variables on Π to G, while the coboundary δf is defined by

$$\begin{aligned}(\delta f)(x_1, \cdots, x_{n+1}) &= f(x_2, \cdots, x_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i f(x_1, \cdots, x_i x_{i+1}, \cdots x_{n+1}) + (-1)^{n+1} f(x_1, \cdots, x_n).\end{aligned} \tag{3.3}$$

Each non-homogeneous cochain can be translated into a homogeneous one and vice-versa, by means of the formulae

$$\begin{aligned} F(x_0, \cdots, x_n) &= f(x_0^{-1}x_1, x_1^{-1}x_2, \cdots, x_{n-1}^{-1}x_n) \\ f(x_1, \cdots, x_n) &= F(1, x_1, x_1x_2, \cdots, x_1 \cdots x_n). \end{aligned} \tag{3.4}$$

We shall also exhibit directly the connection between the matric and the non-homogeneous description of $K(\Pi)$. Given an $(n + 1) \times (n + 1)$ matrix

$$\Delta = \| d_{ij} \|$$

satisfying condition (2.1), we have by (2.2)

$$\Delta \to [d_{00}, d_{01}, \cdots, d_{0n}]$$

and by (3.1)

$$[d_{00}, d_{01}, \cdots, d_{0n}] \to (d_{00}^{-1}d_{01}, d_{01}^{-1}d_{02}, \cdots, d_{0,n-1}^{-1}d_{0n}).$$

But

$$d_{0,i-1}^{-1}d_{0i} = d_{i-1,0}d_{0i} = d_{i-1,i}$$

and consequently the formula

$$\Delta \to (d_{01}, d_{12}, \cdots, d_{n-1,n}) \tag{3.5}$$

gives the desired connection.

4. The cases $n = 0, 1, 2$

Using the non-homogenous description of the complex $K(\Pi)$ we shall now give a group theoretic interpretation of the cohomology groups $H_n(\Pi, G)$ for $n = 0, 1, 2$.

$n = 0$. Since $K(\Pi)$ contains only one 0-cell, written as the 0-tuple ( ), each 0-cochain is a constant in G so that $C_0(\Pi, G) = G$. The coboundary of a 0-cochain f is

$$(\delta f)(x) = f(\ ) + (-1)^1 f(\ ) = 0,$$

so that $\delta f = 0$. Hence $Z_0(\Pi, G) = G$. Since there are no cells of dimension -1 we have $B_0(\Pi, G) = 0$ and therefore

$$H_0(\Pi, G) = G. \tag{4.1}$$

From (1.6) we get for the 0^{th} homology group

$$H^0(\Pi, G) \cong \text{Char } H_0(\Pi, \text{Char } G) \cong \text{Char Char } G \cong G$$

and therefore

$$(4.2) \qquad H^0(\Pi, G) \cong G.$$

$n = 1$. A 1-cochain $f \in C_1(\Pi, G)$ is a function of 1 variable on Π to G. The coboundary of f is

$$(\delta f)(x_1, x_2) = f(x_2) - f(x_1x_2) + f(x_1)$$

so that f is a cocycle if and only if f satisfies the identity

$$f(x_1x_2) = f(x_1) + f(x_2);$$

i.e., if and only if f is a homomorphic mapping of Π into G. Consequently $Z_1(\Pi, G)$ is the group $\text{Hom}(\Pi, G)$ of all the homomorphic mappings of Π into G. Since every 0-cochain is a cocycle, we have $B_1(\Pi, G) = 0$ so that

$$(4.3) \qquad H_1(\Pi, G) = \text{Hom } (\Pi, G).$$

In particular if $G = P$ is the additive group of reals reduced mod 1 we get

$$(4.4) \qquad H_1(\Pi, P) = \text{Char } \Pi.$$

From the duality formula (1.6) we get

$$(4.5) \qquad H^1(\Pi, G) \cong \text{Char Hom } (\Pi, \text{Char } G)$$

or, in terms of the tensor product (see [2, p. 788])

$$(4.6) \qquad H^1(\Pi, G) \cong \Pi \circ G.$$

For integral coefficients we find from (4.4) and (1.6) that

$$H^1(\Pi) \cong \text{Char } H_1(\Pi, P) \cong \text{Char Char } \Pi$$

and therefore

$$(4.7) \qquad H^1(\Pi) \cong \Pi/[\Pi, \Pi]$$

where $[\Pi, \Pi]$ is the commutator subgroup of Π.

$n = 2$. A 2-cochain $f \in C_2(\Pi, G)$ is a function of 2 variables on Π to G. The coboundary of f is

$$\delta f(x_1, x_2, x_3) = f(x_1, x_2) - f(x_1x_2, x_3) + f(x_1, x_2x_3) - f(x_1, x_2),$$

so that f is a cocycle if and only if f satisfies the identity

$$(4.8) \qquad f(x_1, x_2) + f(x_1, x_2x_3) = f(x_1x_2, x_3) + f(x_1, x_2).$$

This identity means that f is a central "factor set" of Π in G. The group of all such factor sets may be written as Factcent (Π, G). A 2-cochain f is a coboundary if it can be written as

$$(x_1, x_2) = h(x_1) - h(x_1x_2) + h(x_2)$$

for some function h on Π to G. Factor sets of this special form are known as "transformation sets"; the group of all such may be written as Trans (Π, G). The second cohomology group may thus be written as

$$(4.9) \qquad H_2(\Pi, G) = \text{Factcent } (\Pi, G)/\text{Trans } (\Pi, G).$$

This group occurs in the theory of group extensions. We call the group E a *central group extension* of G by Π if the center of E contains G and if $E/G = \Pi$. Given any factor set f, we may define a corresponding group extension E_f as the set of all pairs (g, x), with $g \in G$, $x \in \Pi$, and with the multiplication rule

$$(g_1, x_1)\cdot(g_2, x_2) = (g_1 + g_2 + f(x_1x_2), x_1x_2).$$

One observes that this multiplication is associative if and only if f satisfies the condition (4.8) above. The group E_f constructed from f is indeed an extension of G by Π, for the correspondence $(g, x) \to x$ is a nomomorphism of E_f onto Π. The kernel of this homomorphism is the set G_0 of all pairs $(g, 1)$. This set is isomorphic to the group G under the correspondence $G \leftrightarrow (g - f(1, 1), 1)$. Furthermore, G_0 lies in the center of E_f.

Every central group extension E of G by Π may be represented in the form E_f, for a suitable factor set f. The factor set f is not unique; two factor sets f and h determine "equivalent" groups extensions if and only if $f - h$ is a transformation set. Therefore, the cosets of Factcent (Π, G)/Trans (Π, G) are in one-to-one correspondence with the central group extensions of G by Π; this factor group consequently is known as the group Extcent (Π, G) of such group extensions. We may thus write (4.9) as

$$(4.10) \qquad H_2(\Pi, G) \cong \text{Extcent } (\Pi, G)$$

5. Examples

In this section we shall compute the cohomology groups $H_n(\Pi, G)$ for various groups Π. Except in one trivial case the computation will be carried out by constructing a space X whose fundamental group $\pi_1(X)$ is isomorphic with Π and for which suitable higher homotopy groups vanish, and then applying to this space the main theorem of this paper as stated in the introduction.

A) Π *is the null group*. The complex $K(\Pi)$ has only one n-cell σ^n for each $n \geqq 0$. Formula (1.3) shows that

$$\partial\sigma^n = \sigma^{n-1} \qquad \text{for } n \text{ even},$$

$$\partial\sigma^n = 0 \qquad \text{for } n \text{ odd}.$$

Consequently all the homology and cohomology groups of Π are null groups.

B) Π *is a free* (*non abelian*) *group*. Let X be a connected graph with Π as fundamental group. All the homotopy and all the cohomology groups of X are null for the dimensions $n > 1$. Consequently, by Theorem I

$$H_n(\Pi, G) = 0 \qquad \text{for } n > 1,$$

while from (4.3) we have $H_1(\Pi, G) = \text{Hom}\,(\Pi, G)$, so that $H_1(\Pi, G)$ is the direct sum of G with itself m times, where m is the number of free generators of Π.

C) Π *is a free abelian group with* r *generators.* Let X be the r-dimensional torus; i.e., the cartesian product of r circumferences. We have

$$\pi_1(X) \cong \Pi, \qquad \pi_n(X) = 0 \text{ for } n > 1.$$

Since $\Sigma^n(X) = 0$ for $n > 1$ we have $\Lambda_n(X, G) = H_n(X, G)$ for $n > 1$. Since

$$H_n(X, G) = 0 \qquad \text{for } n > r$$

$$H_n(X, G) = G^{\binom{r}{n}} \qquad \text{for } n \leqq r,$$

where G^j stands for the direct product $G \times G \times \cdots \times G$ j times, we have

$$H_n(\Pi, G) = 0 \qquad \text{for } n > r$$

$$H_n(\Pi, G) = G^{\binom{r}{n}} \qquad \text{for } n \leqq r.$$

D) Π *is a cyclic group of order* m. For each $n > 0$ there is a rotation λ of the $(2n + 1)$-sphere S^{2n+1} such that 1°) λ has the period m; 2°) λ preserves the orientation of S^{2n+1}; 3°) none of the rotations $\lambda, \lambda^2, \cdots, \lambda^{m-1}$ has a fixed point. By identifying the points $x, \lambda(x), \cdots, \lambda^{n-1}(x)$ we therefore get an orientable manifold X, often called a generalized lens space, whose universal covering space is S^{2n+1}. Consequently we have

$$\pi_1(X) = \Pi, \qquad \pi_i(X) = 0 \text{ for } 1 < i < 2n + 1.$$

Hence the application of the Theorem II gives

$$H_{2n}(\Pi, G) \cong H_{2n}(X, G)$$

$$H_{2n+1}(\Pi, G) \cong \Lambda_{2n+1}(X, G).$$

From the Poincaré duality relation we get $H_{2n}(X, G) \cong H^1(X, G)$. Since $H^1(X)$ is cyclic of order m we get $H^1(X, G) = G/mG$. Consequently,

$$H_{2n}(\Pi, G) \cong G/mG.$$

Since X is an orientable mainfold of dimension $2n + 1$ we have

$$H^{2n+1}(X) \cong I, \qquad H_{2n+1}(X, G) \cong G.$$

The sphere S^{2n+1} is an m-fold covering of X, so the spherical homology classes are the ones that are divisible by m:

$$\Sigma^{2n+1}(X) = mI.$$

From this it follows that a cohomology class f^{2n+1} will annihilate $\Sigma^{2n+1}(X)$ if and only if $mf^{2n+1} = 0$. Consequently $\Lambda_{2n+1}(X, G) \cong {}_mG$ and

$$H_{2n+1}(\Pi, G) \cong {}_mG,$$

where ${}_mG$ is the subgroup the elements $g \in G$ such that $mg = 0$.

Using the cartesian product of a suitable number of circumferences and generalized lens spaces we could in a similar fashion compute the cohomology group of any abelian group Π with a finite number of generators.

CHAPTER II

THE MAIN THEOREMS[8]

6. The singular complex $S(X)$

We briefly review the basic concepts of the singular homology theory as developed in [1].

Let $s = < p_0 , \cdots , p_n >$ be a simplex, in some euclidean space, taken with a definite order of vertices $p_0 < p_1 < \cdots < p_n$. A continuous mapping

$$T:s \rightarrow X$$

will be called a singular n-simplex in the topological space X. If s_1 is another simplex with ordered vertices of the same dimension as s, then there is a unique linear mapping $B:s_1 \rightarrow s$ preserving the order of the vertices. The singular simplexes

$$T:s \rightarrow X \qquad TB:s_1 \rightarrow X$$

will be called equivalent (notation: $T \equiv TB$). The equivalence classes obtained this way are the cells of the abstract complex $S(X)$. We shall use the same symbol to denote an individual singular simplex and its equivalence class.

We denote by $s^{(i)}$ the face of s opposite to the i^{th} vertex and by

$$T^{(i)}:s^{(i)} \rightarrow X$$

the partial mapping $T^{(i)} = T \mid s^{(i)}$. The boundary in $S(X)$ is defined by

$$\partial T = \sum_{i=0}^{n} (-1)^i T^{(i)}.$$

The homology and cohomology groups of the complex $S(X)$ will be called the singular homology and cohomology groups of the space X, and will be written as $H^n(X, G)$ and $H_n(X, G)$.

Let $x_0 \in X$ be a fixed point of X chosen as base point. We denote by $S_1(X)$ the subcomplex of $S(X)$ obtained by considering only those singular simplexes $T:s \rightarrow X$ which map all the vertices of s into x_0. Since $S_1(X)$ is a closed subcomplex of $S(X)$ the identity mapping $\eta_1(T) = T$ is a chain transformation (see [1, §4])

$$\eta_1:S_1(X) \rightarrow S(X).$$

It was shown in [1, §31] that if the space X is arcwise connected then the chain transformation η_1 is a chain equivalence; i.e., that there is chain transformation

[8] This chapter and the following ones lean rather heavily on the methods and the results of [1].

$$\bar{\eta}_1 : S(X) \to S_1(X)$$

such that the chain transformations

$$\bar{\eta}_1 \eta_1 : S_1(X) \to S_1(X) \qquad \eta_1 \bar{\eta}_1 : S(X) \to S(X)$$

are both chain homotopic to the identity chain transformation of the complex $S_1(X)$ into itself and of $S(X)$ into itself, respectively. In particular, the homology and cohomology groups of the complexes $S(X)$ and $S_1(X)$ are isomorphic.

We shall always choose the base point $x_0 \epsilon X$ used in the definition of $S_1(X)$ as the base point for the definition of the fundamental group $\pi_1(X)$ and of the higher homotopy groups $\pi_n(X)$. It is clear that the complex $S_1(X)$ is more closely connected with the fundamental group $\pi_1(X)$ than is the larger complex $S(X)$. In fact, every singular 1-simplex in $S_1(X)$ uniquely determines an element of $\pi_1(X)$. Consequently in studying the influence of the group $\pi_1(X)$ upon the homology structure of X we shall use the complex $S_1(X)$ almost exclusively.

We shall also have occasion to consider the subcomplexes $S_m(X)$ of $S(X)$ consisting of those $T: s \to X$ which map all the faces of s of dimension less than m into the point x_0. The identity mapping furnishes the chain transformation

$$\eta_m : S_m(X) \to S(X).$$

Clearly $S(X) = S_0(X)$.

A cycle z^n in the complex $S_m(X)$ $(m \leqq n)$ will be called *spherical* if it is homologous in $S_m(X)$ to a cycle in the subcomplex $S_n(X)$. In other words z^n is spherical if $z^n \sim 0$ in $S_m(X)$ mod $S_n(X)$. In each of the homology groups $H^n(S_m(X), G)$ (or $H^n(X, G)$ for $m = 0$) we thus get a subgroup $\Sigma^n(S_m(X), G)$ (or $\Sigma^n(X, G)$ for $m = 0$) of the spherical homology classes. It was proved in [1, §33] that the subgroup $\Sigma^n(X)$ of the integral homology group $H^n(X)$ is the image of the homotopy group $\pi_n(X)$ under its natural mapping into $H^n(X)$.

In the cohomology group $H_n(S_m(X), G)$ of $S_m(X)$ the corresponding subgroup $\Lambda_n(S_m(X), G)$ consists of all the cohomology classes that annihilate the subgroup $\Sigma^n(S_m(X))$, under the Kronecker index as multiplication. In particular $\Lambda_n(X, G)$ is the annihilator of $\Sigma^n(X)$.

If X is arcwise connected, the transformation $\eta_1 : S_1(X) \to S(X)$ is a chain equivalence and under the isomorphisms of the homology and cohomology groups induced by η_1 the subgroups $\Sigma^n(S_1(X), G)$ correspond to $\Sigma^n(X, G)$ and similarly $\Lambda_n(S_1(X), G)$ correspond to $\Lambda_n(X, G)$.

7. The chain transformation κ

The basic tool used in this paper for the study of the influence of the fundamental group $\pi_1(X)$ upon the structure of the space X is a chain transformation

$$\kappa : S_1(X) \to K(\pi_1(X))$$

The definition of this transformation proceeds as follows.

Let $s = \langle p_0 \cdots p_n \rangle$ be an n-dimensional ordered simplex and let $T: s \to$

X be a singular simplex in $S_1(X)$. Since every vertex p_i of s is mapped into the base point $x_0 \in X$ each edge $\overline{p_ip_j}$ of s maps into a closed path in X and therefore determines uniquely an element d_{ij} of $\pi_1(X)$. If $i = j$, we define $d_{ij} = 1$. It is clear that $d_{ij} = d_{ji}^{-1}$. For any three vertices p_i, p_j, p_k the interior of the triangle $p_ip_jp_k$ is mapped into X and therefore

$$d_{ij}d_{jk}d_{ki} = 1.$$

This implies that

$$d_{ij}d_{jk} = d_{ik}, \qquad i, j, k = 0, 1, \cdots, n,$$

so that the matrix

$$\Delta = || d_{ij} ||$$

satisfies the conditions (2.1) and thus is an n-cell of the complex $K(\pi_1(X))$. We define

$$\kappa(T) = \Delta.$$

This gives a mapping of the cells of the complex $S_1(X)$ into those of $K(\pi_1(X))$ and leads to homomorphisms of the integral chains in $S_1(X)$ into those of $K(\pi_1(X))$. These homomorphisms will also be denoted by the letter κ. In order tc show that κ is a legitimate chain transformation we must show that κ and the boundary operator ∂ commute. Let

$$\partial T = \Sigma(-1)^i T^{(i)}$$

where $T^{(i)}$ is the face of T opposite the i^{th} vertex. Let $\Delta = \kappa(T)$ and let $\Delta^{(i)}$ be the matrix obtained from Δ by erasing the i^{th} row and the i^{th} column. As remarked in §2, we have

$$\partial\Delta = \Sigma(-1)^i\Delta^{(i)}.$$

It is immediately clear from the definition of κ that $\kappa(T^{(i)}) = \Delta^{(i)}$. Consequently,

$$\partial\kappa T = \partial\Delta = \Sigma(-1)^i\Delta^{(i)} = \Sigma(-1)^i\kappa T^{(i)} = \kappa\Sigma(-1)^iT^{(i)} = \kappa\partial T$$

which proves that $\partial\kappa = \kappa\partial$.

The chain transformation κ induces homomorphisms (see [1, §4])

$$\kappa : H^n(S_1(X), G) \to H^n(\pi_1(X), G)$$

$$\kappa^* : H_n(\pi_1(X), G) \to H_n(S_1(X), G).$$

These homomorphisms will be used to compare the groups of the space X with those of the complex $K(\pi_1(X))$.

8. Properties of κ

Before we proceed with the proof of our main theorem we shall list a few properties of the chain transformation κ.

LEMMA 8.1. *If z^n is a cycle (with any coefficients) in the subcomplex $S_2(X)$ of $S_1(X)$ then the cycle κz^n bounds in $K(\pi_1(X))$.*

PROOF. Let Δ_0^n denote the n-cell of $K(\pi_1(X))$ represented by the matrix with $d_{ij} = 1$. Clearly $\partial\Delta_0^n = 0$ if n is odd and $\partial\Delta_0^n = \Delta_0^{n-1}$ if n is even.

Now let $T:s \to X$ be a singular n-simplex in X belonging to $S_2(X)$. This means that all the edges of S are mapped into the base point x_0 of X. Consequently, by the definition of κ, it follows that $\kappa(T) = \Delta_0^n$.

If now z^n is any cycle in $S_2(X)$ then $\kappa(z^n) = k\Delta_0^n$ where k is some element of the coefficient group. If n is odd then $\partial\Delta_0^{n+1} = \Delta_0^n$ and $\kappa(z^n) = \partial(k\Delta_0^{n+1})$. If n is even then $\partial\kappa(z^n) = k\partial\Delta_0^n = k\Delta_0^{n-1}$. Since z^n is a cycle, we have $\partial\kappa(z^n) = 0$, hence $k = 0$ and $\kappa(z^n) = 0$. In either case $\kappa(z^n)$ bounds in $K(\pi_1(X))$, q.e.d.

LEMMA 8.2. *Under the homomorphism*

$$\kappa:H^n(S_1(X), G) \to H^n(\pi_1(X), G)$$

the subgroup $\Sigma^n(S_1(X), G)$ *maps into zero.*

PROOF. Let z^n be a cycle in $S_1(X)$ belonging to a homology class in $\Sigma^n(S_1(X), G)$. By the definition of a spherical cycle in $S_1(X)$, the cycle z^n is homologous in $S_1(X)$ to a cycle z_2^m in $S_2(X)$. Hence $\kappa(z^n)$ is homologous to $\kappa(z_2^n)$. But $\kappa(z_2^n)$ bounds in $K(\pi_1(X))$ in virtue of the preceding lemma. Hence $\kappa(z^n)$ bounds, q.e.d.

LEMMA 8.3. *Under the homomorphism*

$$\kappa^*:H_n(\pi_1(X), G) \to H_n(S_1(X), G)$$

the image of $H_n(\pi_1(X), G)$ *is contained in the subgroup* $\Lambda_n(S_1(X), G)$.

PROOF. Let $f^n \in H_n(\pi_1(X), G)$ and let $z^n \in \Sigma^n(S_1(X))$. We have the Kronecker intersection

$$KI(\kappa^*f^n, z^n) = KI(f^n, \kappa z^n),$$

but $\kappa z^n = 0$ by Lemma 8.2. Hence κ^*f^n annihilates the group $\Sigma^n(S_1(X))$ and consequently $\kappa^*f^n \in \Lambda_n(S_1(X), G)$.

9. Proof of Theorem I

We shall now study the transformation κ in the case when X is aspherical; i.e., when $\pi_i(X) = 0$ for $i > 1$.

THEOREM Ia. *If X is aspherical then the chain transformation*

$$\kappa:S_1(X) \to K(\pi_1(X))$$

is a chain equivalence.

This result contains Theorem I as a corollary. For, if X is arcwise connected, the chain transformation

$$\eta_1: S_1(X) \to S(X)$$

also is a chain equivalence; it follows that the complexes $S(X)$, $S_1(X)$, and $K(\pi_1(X))$ are all chain equivalent. Consequently the complexes $S(X)$ and $K(\pi_1(X))$ have isomorphic homology and cohomology groups, as asserted by Theorem I.

PROOF OF THEOREM Ia. We shall define a chain transformation

$$\bar{\kappa}: K(\pi_1(X)) \to S_1(X)$$

subject to the following conditions:

(9.1) *for each cell* Δ *of* $K(\pi_1(X))$, $\bar{\kappa}(\Delta)$ *is a singular simplex of* $S_1(X)$,

(9.2) $\kappa\bar{\kappa}(\Delta) = \Delta$,

(9.3) *if* $\bar{\kappa}(\Delta) = T$, *then* $\bar{\kappa}(\Delta^{(i)}) = T^{(i)}$.

Since $K(\pi_1(X))$ and $S_1(X)$ each have only one 0-cell Δ^0 and T^0, we define $\bar{\kappa}(\Delta^0) = T^0$. Let Δ^1 be a 1-cell of $K(\pi_1(X))$, so that

$$\Delta^1 = \begin{Vmatrix} 1 & d \\ d^{-1} & 1 \end{Vmatrix},$$

where $d \in \pi_1(X)$. Take a 1-simplex s^1 with ordered vertices and let $T: s^1 \to X$ be a continuous function mapping s^1 into a closed path about x_0 belonging to the element d of the fundamental group. Define $\bar{\kappa}(\Delta) = T$. Next, let

$$\Delta = \begin{Vmatrix} 1 & d & de \\ d^{-1} & 1 & e \\ (de)^{-1} & e^{-1} & 1 \end{Vmatrix}$$

be a 2-cell of $K(\pi_1(X))$ and let $s = < v_0v_1v_2 >$ be a 2-simplex with ordered vertices. The faces of Δ are

$$\Delta^{(0)} = \begin{Vmatrix} 1 & e \\ e^{-1} & 1 \end{Vmatrix} \qquad \Delta^{(1)} = \begin{Vmatrix} 1 & de \\ (de)^{-1} & 1 \end{Vmatrix} \qquad \Delta^{(2)} = \begin{Vmatrix} 1 & d \\ d^{-1} & 1 \end{Vmatrix}.$$

We consider the 1-simplexes

$$s^{(0)} = < v_1v_2 > \qquad s^{(1)} = < v_0v_2 > \qquad s^{(2)} = < v_0v_1 >$$

and since $\bar{\kappa}$ has already been defined for the 1-cells of $K(\pi_1(X))$ we have three mappings

$$\bar{\kappa}(\Delta^{(i)}): s^{(i)} \to X, \qquad i = 0, 1, 2,$$

which give closed paths about x_0 belonging to the elments e, de, and d of $\pi_1(X)$ respectively. Jointly these three mappings give a mapping T of the boundary $b(s)$ of into X, and this mapping is nullhomotopic. Consequently T can be extended to a mapping $T: s \to X$. We define $\bar{\kappa}(\Delta) = T$.

From now on, we proceed by induction. Suppose that $\bar{\kappa}(\Delta)$ has been defined for all cells Δ of dimension $< k (k > 2)$. Let Δ be a k-cell of $K(\pi_1(X))$. Let s be a k-simplex with ordered vertices. If $s^{(i)}$ is the i^{th} face of s, we have mappings

$$\bar{\kappa}(\Delta^{(i)}): s^{(i)} \to X$$

and in view of (9.3) these mappings agree on the common parts of any two faces of s. Consequently they combine and give a mapping $T: b(s) \to X$ of the boundary $b(s)$ of s. Since $b(s)$ is homeomorphic to a $(k - 1)$-sphere and $\pi_{k-1}(X) = 0$ because $k - 1 > 1$, the mapping T can be extended to a mapping $T:s \to X$. We define $\bar{\kappa}(\Delta) = T$. From the construction it is clear that (9.1)–(9.3) are satisfied, so that the definition of $\bar{\kappa}$ is complete.

It follows from (9.3) that $\bar{\kappa}$ and ∂ commute so that $\bar{\kappa}$ is a chain transformation. From (9.2) it follows that

$$\kappa\bar{\kappa} = 1.$$

In order to show that κ is a chain equivalence it is now sufficient to show that

$$\bar{\kappa}\kappa \simeq 1 \tag{9.4}$$

which means that the chain transformation $\bar{\kappa}\kappa$ is chain homotopic with the identity mapping of $S_1(X)$ into itself.

We shall now define for each singular simplex $T:s \to X$ of $S_1(X)$ a "singular prism"

$$R_T:s \times I \to X,$$

where $s \times I$ is the cartesian product of the simplex s with the closed interval $[0, 1]$, subject to the following conditions:

$$R_T(p, 0) = T(p), \tag{9.5}$$

$$R_T(p, 1) = (\bar{\kappa}\kappa T)(p), \tag{9.6}$$

$$R_T(p, t) = R_{T^{(i)}}(p, t) \qquad \text{for } p \in s^{(i)}, \tag{9.7}$$

$$\textit{If } T_1 \equiv T_2, \quad \textit{then} \quad R_{T_1} \equiv R_{T_2}. \tag{9.8}$$

Here $R_{T_1} \equiv R_{T_2}$ means that these two singular prisms are equivalent, as defined in [1, §11].

If T has dimension 0 we define R_T by setting $R_T:s \times I \to x_0$. If T has dimension 1 then $s \times I$ is a rectangle with $b(s \times I)$ as boundary. We define a mapping

$$R:b(s \times I) \to X$$

by setting

$$R(p, 0) = T(p),$$
$$R(p, 1) = (\bar{\kappa}\kappa T)(p),$$
$$R(p, t) = x_0 \quad \text{for} \quad p \in b(s).$$

This mapping is continuous. According to (9.2) we have

$$\kappa\bar{\kappa}\kappa T = \kappa T$$

and therefore by the definition of κ the paths $T:s \to X$ and $\bar{\kappa}\kappa T:s \to X$ represent the same element of the fundamental group $\pi_1(X)$. Consequently the mapping R is nullhomotopic and can be extended to a mapping

$$R_T:s \times I \to X.$$

From now on we proceed by induction. Suppose that R_T has been defined for singular simplexes T of dimension $< k(k > 1)$. Let $T:s \to X$ be a singular k-simplex of $S_1(X)$. We define a mapping

$$R:b(s \times I) \to X$$

by setting

$$R(p, 0) = T(p),$$
$$R(p, 1) = (\bar{\kappa}\kappa T)(p),$$
$$R(p, t) = R_T(i)(p, t) \qquad \text{for } p \in s^{(i)}.$$

This mapping is continuous. Since $b(s \times I)$ is homeomorphic with the k-sphere and since $\pi_k(X) = 0$ because $k > 1$, the mapping R can be extended to a mapping $R_T:s \times I \to X$. It is easy to see that conditions (9.5)–(9.8) are still satisfied, and therefore the definition of R_T is completed.

We shall now consider the standard subdivision of the prism $s \times I$ and the basic chain $d(s \times I)$ of this subdivision, as defined in [1, §16]. For each $T \in S_1(X)$ the continuous mapping R_T applied to the chain $d(s \times I)$ in the polyhedron $s \times I$ yields a chain in $S(X)$ which we shall denote by $\mathfrak{D}T$ (this is the chain $(s \times I, d(s \times I), R_T)$ in the "triple" notation of [1, §15]). Since the standard subdivision of $s \times I$ introduces no new vertices in addition to those of $s \times 0$ and $s \times 1$ and since R_T maps all those vertices into x_0 it follows that $\mathfrak{D}T$ is a chain in $S_1(X)$. Furthermore it follows from (9.5)–(9.8) and from the properties of $d(s \times I)$ [1, §16] that

$$\partial\mathfrak{D}T = \bar{\kappa}\kappa T - T - \mathfrak{D}\partial T.$$

This proves (9.4) and completes the proof of Theorem Ia.

10. Proof of Theorem II

We now proceed to examine the case when

$$(10.1) \qquad \pi_n(X) = 0 \quad \textit{for} \quad 1 < n < r.$$

If we examine the construction of $\bar{\kappa}$ in the previous paragraph we notice that in order to define $\bar{\kappa}\Lambda$ for a k-dimensional cell of $K(\pi_1(X))$ we used the fact that $\pi_{k-1}(X) = 0$. Consequently the definition of $\bar{\kappa}$ can be duplicated for all cells Δ of dimension $\leqq r$. We summarize the properties of $\bar{\kappa}$ which will be used:

$$(10.2) \qquad \bar{\kappa}\partial = \partial\bar{\kappa},$$

$$(10.3) \qquad \kappa\bar{\kappa} = 1.$$

(10.4) *If Δ is an $(r + 1)$-cell of $K(\pi_1(x))$ then $\bar{\kappa}(\delta\Delta)$ is a spherical cycle in $S_1(X)$.*

Next we shall examine the definition of the operator $\mathfrak{D}$ in the last part of the preceding section. In defining the continuous mappings R_T we have used the fact that $\pi_k(X) = 0$ where k is the dimension of the singular simplex T. Consequently the definition of R_T and therefore also that of $\mathfrak{D}T$ carries over without change, provided that the singular simplex T has dimension $<r$. In particular, we have

$$(10.5) \qquad \partial\mathfrak{D}T = \bar{\kappa}\kappa T - T - \mathfrak{D}\partial T \qquad \dim T < r.$$

If $\dim T = r$, then $\mathfrak{D}T$ is not defined, but the definition of $\mathfrak{D}\partial T$ readily implies that

(10.6) *If T is a singular r-simplex of $S_1(X)$, then the chain*

$$\bar{\kappa}\kappa T - T - \mathfrak{D}\partial T$$

is a spherical r-cycle in $S_1(X)$.

These facts will be used in the proof of the following theorem.

THEOREM IIa. *If*

$$\pi_n(X) = 0 \qquad \text{for } 1 < n < r$$

then the chain transformation

$$\kappa : S_1(X) \to K(\pi_1(X))$$

induces the following isomorphisms

$$(10.7) \qquad \kappa : H^n(S_1(X), G) \leftrightarrow H^n(\pi_1(X), G) \quad \text{for } n < r,$$

$$(10.8) \qquad \kappa^* : H_n(\pi_1(X), G) \leftrightarrow H_n(S_1(X), G) \quad \text{for } n < r,$$

$$(10.9) \qquad \kappa : H^r(S_1(X), G)/\Sigma^r(S_1(X), G) \leftrightarrow H^r(\pi_1(X), G),$$

$$(10.10) \qquad \kappa^* : H_r(\pi_1(X), G) \leftrightarrow \Lambda_r(S_1(X), G).$$

If X is arcwise connected, these isomorphisms combined with the isomorphisms induced by the chain equivalence $\eta_1 : S_1(X) \to S(X)$ produce the isomorphisms required for Theorem II.

PROOF OF THEOREM IIa. We shall denote by

$$[S_1(X)]^r \quad \text{and} \quad [K(\pi_1(X))]^r$$

the subcomplexes of $S_1(X)$ and $K(\pi_1(X))$ which consist of all cells of dimension $\leqq r$. For these two complexes we have the chain transformations κ and $\bar{\kappa}$, and because of (10.3) and (10.5) they form an equivalence pair. Hence κ induces isomorphisms of the groups of the complex $[S_1(X)]^r$ onto those of $[K(\pi_1(X))]^r$. However for the dimensions $n < r$ the complexes $[S_1(X)]^r$ and $S_1(X)$ have identical homology and cohomology groups, and similarly for the complexes $[K(\pi_1(X))]^r$ and $K(\pi_1(X))$; this implies formulae (10.7) and (10.8).

We shall now turn to the discussion of the homology and cohomology groups of dimension r. Let z^r be an r-dimensional cycle in $K(\pi_1(X))$, with coefficients in a group G. From (10.2) it follows that $\partial\bar{\kappa}z^r = \bar{\kappa}\partial z^r = 0$. Hence it follows that $\bar{\kappa}z^r$ is a cycle in $S_1(X)$. From equation (10.3) the conclusion follows that $\kappa\bar{\kappa}z^r = z^r$. This implies that the homomorphism

(10.11) $$\kappa: H^r(S_1)(X), G) \to H^r(\pi_1(X), G)$$

is a homomorphism onto. In order to complete the proof of (10.9) we must show that the kernel of the homomorphism (10.11) is the subgroup $\Sigma^r(S_1(X), G)$ of $H^r(S_1(X), G)$. We have already seen from Lemma 8.2 that the group $\Sigma^r(S_1(X), G)$ is contained in the kernel. Now let z^r be a cycle in $S_1(X)$ with coefficients in G such that $\kappa z^r \sim 0$. Let

$$c^{r+1} = \sum_i g_i \Delta_i^{r+1}$$

be an $(r + 1)$-chain in $K(\pi_1(X))$ such that

$$\partial c^{r+1} = \kappa z^r,$$

and consider the r-chain in $S_1(X)$:

$$\bar{\kappa}\kappa z^r = \bar{\kappa}(\partial c^{r+1}) = \bar{\kappa}\Sigma g_i \partial \Delta_i^{r+1} = \Sigma g_i \bar{\kappa}(\partial \Delta_i^{r+1}).$$

According to (10.4), $\bar{\kappa}\kappa z^r$ is a spherical cycle in $S_1(X)$. From (10.6) we deduce that $\bar{\kappa}\kappa z^r - z^r$ is a spherical cycle. Consequently z^r is a spherical cycle in $S_1(X)$. This concludes the proof of (10.9).

Before we proceed with the proof of (10.10) we should recall the definition of κ^*. Given any cochain f in the complex $K(\pi_1(X))$ we can define a cochain $\kappa^* f$ in $S_1 X)$ by setting

$$(\kappa^* f)(T) = f(\kappa T).$$

This maps the cohomology groups of $K(\pi_1(X))$ homomorphically into those of $S_1(X)$. In particular, we have

(10.12) $$\kappa^*: H_r(K(\pi_1(X)), G) \to H_r(S_1(X), G).$$

We shall show first that the kernel of this homomorphism is zero. To this end, let f be an r-cocyle in $K(\pi_1(X), G)$ such that

$$\kappa^* f = \delta h,$$

where h is an $(r - 1)$-cochain in $S_1(X)$. Define

$$h'(\Delta^{r-1}) = h(\bar{\kappa}\Delta^{r-1})$$

for every $(r - 1)$-cell Δ^{r-1} of $K(\pi_1(X))$. Clearly h' is an $(r - 1)$-cochain in $K(\pi_1(X))$. We have

$$(\delta h')(\Delta^r) = h'(\partial \Delta^r) = h(\bar{\kappa}\partial \Delta^r) = h(\partial \bar{\kappa}\Delta^r) =$$
$$(\delta h)(\bar{\kappa}\Delta^r) = (\kappa^* f)(\bar{\kappa}\Delta^r) = f(\kappa\bar{\kappa}\Delta^r) = f(\Delta^r)$$

since by (10.3) $\bar{\kappa}\kappa\Delta^r = \Delta^r$. Consquently $\delta h' = f$, which proves that the kernel of (10.12) is zero.

We have shown already (Lemma 8.3) that the homomorphism (10.12) maps $H_r(K(\pi_1(X)), G)$ into the subgroup $\Lambda_r(S_1(X), G)$. In order to complete the proof of (10.10) it is sufficient therefore to show that κ^* maps $H_r(K(\pi_1(X)), G)$ onto $\Lambda_r(S_1(X), G)$.

Let f be a cocycle belonging to a cohomology class in $\Lambda_r(S_1(X), G)$. Define a cochain f' of $K(\pi_1(X))$ by setting

$$f'(\Delta^r) = f(\bar{\kappa}\Delta^r)$$

for Δ^r in $K(\pi_1(X))$. We have

$$(\delta f')(\Delta^{r+1}) = f'(\partial\Delta^{r+1}) = f(\bar{\kappa}\partial\Delta^{r+1}) = 0$$

since by (10.4) $\bar{\kappa}\partial\Delta^{r+1}$ is a spherical cycle in $S_1(X)$ and since f annihilates all the spherical cycles by the definition of the subgroup Λ_r. It follows that f' is a cocycle in $K(\pi_1(X))$. We can further define an $(r - 1)$-cochain h in $S_1(X)$ by setting

$$h(T^{r-1}) = f(\mathfrak{D}T^{r-1}).$$

According to (10.6) $\bar{\kappa}\kappa T^r - T^r - \mathfrak{D}\partial T^r$ is a spherical cycle in $S_1(X)$, and since f belongs to a cohomology class in Λ_r, f annihilates all the spherical cycles, consequently

$$f(\bar{\kappa}\kappa T^r - T^r - \mathfrak{D}\partial T^r) = 0.$$

We now compute the coboundary δh of h as

$$\begin{aligned}(\delta h)(T^r) &= h(\partial T^r) = f(\mathfrak{D}\partial T^r)\\ &= f(\bar{\kappa}\kappa T^r) - f(T^r) = f'(\kappa T^r) - f(T^r)\\ &= (\kappa^* f')(T^r) - f(T^r)\end{aligned}$$

and therefore

$$\delta h = \kappa^* f' - f$$

This shows that $\kappa^* f'$ and f are cohomologous, and therefore the cohomology class of f is the image under κ^* of the cohomology class of f'. This concludes the proof of Theorem IIa.

11. Discussion of naturality

In the case when $X = P$ is a locally finite connected polyhedron P, given in a definite simplicial decomposition, it is useful to have an explicit connection between the complex $K(\pi_1(P))$ and the homology groups of P defined from the simplicial decomposition.

In [1, Ch. II] the complexes $k(P)$ and $K(P)$ were defined; $k(P)$ was the abstract complex obtained from P by assigning an orientation to any simplex of P, while $K(P)$ was the much larger abstract complex in which every simplex of P is taken with a definite order of vertices and two different orderings of vertices lead to two distinct cells in $K(P)$. The two complexes are compared by means of chain transformations

$$\alpha : K(P) \to k(P) \qquad \bar{\alpha} : k(P) \to K(P),$$

which form an equivalence pair. The chain transformation α was defined in a natural and, in a sense, in a unique fashion, while the definition of $\bar{\alpha}$ involved an element of choice and was unique only to within a chain homotopy.

Next we shall consider the singular complex $S(P)$ and the natural chain transformation

$$\beta : K(P) \to S(P),$$

which was proved in [1, Chap IV] to be a chain equivalence. Further we have the subcomplex $S_1(P)$ of $S(P)$ and the chain transformations

$$\eta_1 : S_1(P) \to S(P) \qquad \bar{\eta}_1 : S(P) \to S_1(P)$$

which form an equivalence pair. Again, as in the case of $(\alpha, \bar{\alpha})$ the transformation η_1 is natural, while the definition of $\bar{\eta}_1$ involves some choices and is unique only to within a chain homotopy. Finally, we have the natural chain transformation

$$\kappa : S_1(P) \to K(\pi_1(P)).$$

All told we have the following natural chain transformations

$$k(P) \xleftarrow[\alpha]{} K(P) \xrightarrow[\beta]{} S(P) \xleftarrow[\eta_1]{} S_1(P) \xrightarrow[\kappa]{} K(\pi_1(P)).$$

This diagram shows that the natural chain transformations on hand do not permit the comparison of the complexes $k(P)$ and $K(\pi_1(X))$ directly. However, if we use $\bar{\alpha}$ and $\bar{\eta}_1$, we have

$$k(P) \xrightarrow[\bar{\alpha}]{} K(P) \xrightarrow[\beta]{} S(P) \xrightarrow[\bar{\eta}_1]{} S_1(P) \xrightarrow[\kappa]{} K(\pi_1(P)),$$

so that we get a chain transformation

$$\kappa\bar{\eta}_1\beta\bar{\alpha} : k(P) \to K(\pi_1(P)). \tag{11.1}$$

We shall show now how the transformation (11.1) can be obtained directly without the use of the complexes $K(P)$, $S(P)$, and $S_1(P)$.

The direct definition of (11.1) will involve two choices:

1° We choose a definite order of the vertices of the polyhedron P.

2° For each vertex v of P we choose a definite path $\overline{x_0 v}$ leading from the base point $x_0 \in P$ to v.

Given a simplex s in P with the vertices $v_0 < v_1 < \cdots < v_n$, we denote by d_{ij} the element of $\pi_1(P)$ determined by the closed path

$$(\overline{x_0 v_i})(\overline{v_i v_j})(\overline{x_0 v_j})^{-1}$$

where $\overline{v_i v_j}$ denotes the rectilinear path from v_i to v_j. It is apparent that the matrix $\Delta = \| d_{ij} \|$ is an n-cell in the complex $K(\pi_1(P))$. By setting $\theta(s) = \Delta$, we can get the required chain transformation:

$$\theta : k(P) \to K(\pi_1(P)).$$

An analysis of the definitions of $\bar{\alpha}$ and $\bar{\eta}_1$ shows readily that $\bar{\alpha}$ and $\bar{\eta}_1$ can be chosen so as to have

$$\theta = \kappa\bar{\eta}_1\beta\bar{\alpha}.$$

This implies that θ is unique to within a chain homotopy, a fact which could be proved directly.

The concept of naturality is used here in a vague sense, which, however, could be made quite rigorous using the theory of functors [4].

CHAPTER III PRODUCTS

12. Products in $K(\Pi)$

Let F_0 be the 0-dimensional cochain in $K(\Pi)$ with integral coefficients defined as

$$F_0(x) = 1 \qquad \text{for all} \qquad x \in \Pi.$$

Since, as we have remarked in §4, every 0-cochain is a cocycle, it follows that F_0 is a cocycle. Consequently, in the terminology of [1, §24], the complex $V(\Pi)$ is augmentable.

We shall follow now the pattern of [1, Ch. V] and define products in the complex $K(\Pi)$. Let two homogeneous cochains

$$F_p \in C_p(\Pi, G_1), \qquad F_q \in C_q(\Pi, G_2)$$

be given with the coefficient groups G_1 and G_2 paired to a third group G. We define

$$(12.1) \quad (F_p \cup F_q)(x_0, \cdots, x_{p+q}) = F_p(x_0, \cdots, x_p)F_q(x_p, \cdots, x_{p+q}).$$

We verify that this definition gives a homogeneous cochain

$$F_p \cup F_q \in C_{p+q}(\Pi, G),$$

and that the cup product so defined satisfies axioms ($\cup$1) – ($\cup$5) of [1, §25]. This establishes $K(\Pi)$ as a complex with products.

Let f_p, f_q, and $f_p \cup f_q$ be the non homogeneous cochains corresponding to the homogeneous cochains F_p, F_q, and $F_p \cup F_q$. According to (3.4) we have

$$\begin{aligned} f_p \cup f_q(x_1, \cdots, x_{p+q}) &= F_p \cup F_q(1, x_1, x_1x_2, \cdots, x_1 \cdots x_{p+q}) \\ &= F_p(1, x_1, \cdots, x_1 \cdots x_p)F_q(x_1 \cdots x_p, \cdots, x_1 \cdots x_{p+q}) \\ &= F_p(1, x_1, \cdots, x_1 \cdots x_p)F_q(1, x_{p+1}, x_{p+1}x_{p+2}, \cdots, x_{p+1} \cdots x_{p+q}) \\ &= f_p(x_1, \cdots, x_p)f_q(x_{p+1}, \cdots, x_{p+q}). \end{aligned}$$

Hence we have the formula

$$(12.2) \qquad f_p \cup f_q(x_1, \cdots, x_{p+q}) = f_p(x_1, \cdots, x_p)f_q(x_{p+1}, \cdots, x_{p+q})$$

for the cup product in the non-homogeneous description.

We follow with the matric description. Let a $(p + q)$-cell of $K(\Pi)$ be given as a $(p + q + 1) \times (p + q + 1)$ matrix

$$\Delta = || d_{ij} || .$$

We shall denote by ${}_p\Delta$ the matrix obtained from Δ by erasing the last q rows and columns of Δ, and by Δ_q the matrix obtained by erasing the first p rows and columns of Δ. According to (3.5) we have

$$\Delta \to (d_{01}, d_{12}, d_{p+q-1,p+q})$$

$${}_p\Delta \to (d_{01}, d_{12}, \cdots, d_{p-1,p})$$

$$\Delta_q \to (d_{p,p+1}, d_{p+1,p+2}, \cdots, d_{p+q-1,p+q}).$$

Consequently (12.2) implies the following definition of the cup product in the matric description

$$f_p \cup f_q(\Delta) = f_p({}_p\Delta)f_q(\Delta_q). \tag{12.3}$$

The definition of a cup product carries with it [1, §26] a corresponding definition for the cap product

$$f_q \cap c^{p+q} \in C^p(\Pi, G)$$

of a cochain $f_q \in C_q(\Pi, G_1)$ and a chain $c^{p+q} \in C^{p+q}(\Pi, G_2)$, provided that G_1 and G_2 are suitably paired to G. The cap product corresponding to our definition of the cup product in $K(\Pi)$ can be described as follows. Let $c^{p+q} = g\Delta$, where $g \in G_2$, then

$$f_q \cap c^{p+q} = [f_q(\Delta_q)g]_p\Delta.$$

13. Products in $S(X)$ and $S_1(X)$

We shall briefly review the definition of the cup product in the complexes $S(X)$ and $S_1(X)$, as introduced in [1]. Let

$$T: s \to X \qquad \text{where } s = < p_0, \cdots, p_{p+q} >$$

be a singular simplex in X of dimension $p + q$. We shall consider the faces

$${}_ps = < p_0, \cdots, p_p > \qquad s_q = < p_p, \cdots, p_{p+q} >$$

of s and the corresponding singular simplexes defined by the partial mappings

$${}_pT = T \mid {}_ps \qquad T_q = T \mid s_q .$$

For two cochains

$$f_p \in C_p(X, G_1), \qquad f_q \in C_q(X, G_2)$$

where G_1 and G_2 are paired to G, the cup product

$$f_{p'} \cup f_q \in C_{p+q}(X, G)$$

is defined by the formula

(13.1) $$(f_p \cup f_q)(T) = f_p({}_pT)f_q(T_q).$$

The corresponding cap product can be defined as follows. Let

$$f_q \in C_q(X, G_1), \qquad c^{p+q} \in C^{p+q}(X, G_2),$$

where G_1 and G_2 are paired to G. Further let c^{p+q} be of the form gT, where $g \in G_2$, then

$$f_q \cap c^{p+q} = [f_q(T_q)g]_pT.$$

If the simplex T is the subcomplex $S_1(X)$, then both ${}_pT$ and T_q are in $S_1(X)$. Formula (13.1) can therefore be used to define the cup product in the complex $S_1(X)$.

The chain transformation

$$\eta_1 : S_1(X) \to S(X)$$

was defined as the identity mapping $\eta_1 T = T$ of the complex $S_1(X)$ into $S(X)$. Given a cochain f of $S(X)$, the cochain $\eta_1^* f$ of $S_1(X)$ is defined by

$$(\eta_1^* f)(T) = f(\eta_1 T) = f(T).$$

This means that the cochain $\eta_1^* f$ is obtained from f by cutting down the domain of definition of f to include just simplices of $S_1X)$. This implies that for the cup products we have

$$\eta_1^*(f_p \cup f_q) = \eta_1^* f_p \cup \eta_1^* f_q,$$

so that in the terminology of [1, §27], the chain transformation η_1 preserves the products. Since η_1 is a chain equivalence (when X is arcwise connected), it follows that the parings which the cup and the cap products induce on the homology and cohomology classes of $S_1(X)$ will be isomorphic with the corresponding pairings in $S(X)$.

14. Reduced products

It was shown in [1, §35] that if G_1 and G_2 are paired to G, and if $p > 0$ and $q > 0$, then for any two cohomology classes

$$f_p \in H_p(X, G_1), \qquad f_q \in H_q(X, G_2)$$

we have

$$f_p \cup f_q \in \Lambda_{p+q}(X, G)$$

where the subgroup Λ_{p+q} of H_{p+q} is defined as in §8 above. Consequently, by a trivial restriction of the range, we may consider the product as a pairing of the groups H_p and H_q to the group Λ_{p+q}. This pairing will be called the *reduced cup product*.

For the cap product it was shown in [1, §35) that if $p > 0$, $q > 0$, and

$$f_q \in H_q(X, G_1), \qquad z^{p+q} \in \Sigma^{p+q}(X, G_2),$$

then

$$f_q \cap z^{p+q} = 0,$$

where the subgroup Σ^{p+q} of H^{p+q} is defined as in §8 above. Consequently the pairing of the groups $H_q(X, G_1)$ and $H^{p+q}(X, G_2)$ with values in $H^p(X, G)$ induces a pairing of the groups $H_q(X, G_1)$ and $H^{p+q}(X, G_2)/\Sigma^{p+q}(X, G_2)$ with values in $H^p(X, G)$. This pairing will be called the *reduced cap product.*

The reduced products are defined in the complex $S_1(X)$ in an identical fashion. They lead to pairings isomorphic with the corresponding pairings in $S(X)$.

15. Comparison of products in X and K $(\pi_1(X))$

We shall show first that the chain transformation

$$\kappa: S_1(X) \rightarrow K(\pi_1(X))$$

preserves the products. In fact, if $\kappa(T) = \Delta$, then it is clear from the definitions involved that $\kappa({}_pT) = {}_p\Delta$ and that $\kappa(\check{T}_q) = \Delta_q$. Consequently,

$$\begin{aligned} \kappa^*(f_p \cup f_q)(T) &= (f_p \cup f_q)(\kappa T) = f_p \cup f_q(\Delta) = f_p({}_p\Delta)f_q(\Delta_q) \\ &= f_p(\kappa_p T)\, f_q(\kappa T_q) = \kappa^* f_p({}_pT)\kappa^* f_q(T_q) \\ &= (\kappa^* f_p \cup \kappa^* f_q)(T), \end{aligned}$$

hence we get

$$\kappa^*(f_p \cup f_q) = \kappa^* f_p \cup \kappa^* f_q \tag{15.1}$$

for any two cochains $f_p \in C_p(\pi_1(X), G_1)$ and $f_q \in C_q(\pi_1(X), G_2)$, provided that G_1 and G_2 are paired to some group G.

This implies that if, for the dimensions p, q, and $p + q$, the transformation κ establishes the isomorphisms

$$H_p(S_1(X), G_1) \cong H_p(\pi_1(X), G_1)$$

$$H_q(S_1(X), G_2) \cong H_q(\pi_1(X), G_2)$$

$$H_{p+q}(S_1(X), G) \cong H_{p+q}(\pi_1(X), G),$$

then for these groups the cup products also will be mapped isomorphically. A similar statement applies to the cap product. Since the products in the complexes $S(X)$ and $S_1(X)$ are isomorphic, Theorem Ia implies the following:

ADDENDUM TO THEOREM I. *If X is aspherical, then the cup and the cap products of the homology and cohomology classes in X are determined by the fundamental group $\pi_1(X)$. More precisely, the products in X are isomorphic with the corresponding products in the complex $K(\pi_1(X))$.*

In order to get a similar addendum to Theorem II we notice that if κ induces the isomorphisms

$$H_p(S_1(X), G_1) \cong H_p(\pi_1(X), G_1) \qquad p > 0$$

$$H_q(S_1(X), G_2) \cong H_q(\pi_1(X), G_2) \qquad q > 0$$

$$\Lambda_{p+q}(S_1(X), G) \cong H_{p+q}(\pi_1(X), G),$$

then the reduced cup product of the groups on the left is isomorphic with the cup product of the groups on the right. Similarly, if κ induces the isomorphisms

$$H_q(S_1(X), G_1) \cong H_q(\pi_1(X), G_1) \qquad q > 0$$

$$H^{p+q}(S_1(X), G_2)/\Sigma^{p+q}(S_1(X), G_2) \cong H^{p+q}(\pi_1(X), G_2)$$

$$H^p(S_1(X), G) \cong H^p(\pi_1(X), G) \qquad p > 0,$$

then the reduced cap product of the groups on the left is isomorphic with the cap product of the groups on the right. Since the products in the complexes $S_1(X)$ and $S(X)$ are isomorphic, Theorem IIa implies the following:

ADDENDUM TO THEOREM II. *If*

$$\pi_n(X) = 0 \quad for \quad 1 < n < r,$$

then for $p + q < r$ the cup products and the cap products in X are isomorphic with the corresponding products in the complex $K(\pi_1(X))$. If $p > 0$, $q > 0$, and $p + q = r$, then the reduced cup and cap products in X are isomorphic with the corresponding products in the complex $K(\pi_1(X))$.

CHAPTER IV. GENERALIZATION TO HIGHER DIMENSIONS

16. Preliminaries

In the preceding chapters we have studied the influence of the fundamental group $\pi_1(X)$ of a space upon the homology structure of X. A similar discussion can be carried out, replacing the group $\pi_1(X)$ by a higher homotopy group $\pi_m(X)$. The group constructions involved are rather complicated, and consequently the final results have less interest. We shall treat the entire topic very sketchily; we shall give all the necessary definitions and shall state the theorems, but we shall omit the proofs, since they are strictly analogous to the ones in Chapter II.

Let X be a topological space with base point x_0. We shall consider the total singular complex $S(X)$ and the subcomplex $S_m(X)$ $(m > 1)$ consisting of all those singular simplexes

$$T = s \to X \qquad s = < p_0, \cdots, p_n >,$$

which map all faces of s of dimension $<m$ into the point x_0. We shall consider the m^{th} homotopy group $\pi_m(X)$ of X with x_0 as the base point. If s' is any m-dimensional subsimplex of s (with ordered vertices) the mapping $T: s' \to X$ maps the boundary of s' into x_0 and therefore defines an element α of $\pi_m(X)$. If

$s' = < p_{a_0}, \cdots, p_{a_m} >$, then the definition $\phi(a_0, \cdots, a_m) = \alpha$ gives a function ϕ of $m + 1$ variables $a_i = 0, \cdots, n$ with values in $\pi_m(X)$. This defines ϕ only when $a_0, a_1, \cdots, a_m$ are distinct; we set

(16.1) $\qquad \phi(a_0, \cdots, a_m) = 0 \qquad$ *if* $a_0, \cdots, a_m$ *are not distinct.*

This function ϕ has the following important property:

(16.2) $$\sum_{i=0}^{m+1} (-1)^i \phi(a_0, \cdots, \hat{a}_i, \cdots, a_{m+1}) = 0$$

for any system of $m + 2$ *numbers* $a_i = 0, \cdots, n$.

If $a_0, \cdots, a_{m+1}$ are distinct, the vertices $p_{a_0}, \cdots, p_{a_{m+1}}$ determine an $(m + 1)$-dimensional face s' of the simplex s, and (16.2) asserts that the mapping T of the boundary of s' gives the zero element of the homotopy group. If $a_i = a_j$ for $i < j$, then in view of (16.1) formula (16.2) reduces to

$$(-1)^i\phi(a_0, \cdots, \hat{a}_i, \cdots, a_{m+1}) + (-1)^j\phi(a_0, \cdots, \hat{a}_j, \cdots, a_{m+1}) = 0,$$

which means that the function ϕ is alternating in its variables.

17. The complexes $K(\Pi, m)$

Let Π be an abelian group written additively. Guided by the preceding discussion we shall define an (n, m)-cell of the group Π to be a function of $m + 1$ variables

$$\phi(a_0, \cdots, a_m) \in \Pi \qquad \text{where } a_i = 0, \cdots, n,$$

subject to the conditions (16.1) and (16.2).

Given an (n, m)-cell ϕ we shall define the $(n - 1, m)$-cell $\phi^{(i)}$ for $i = 0, \cdots, n$, as follows

$$\phi^{(i)}(b_0, \cdots, b_m) = \phi(a_0, \cdots, a_m)$$

where

$$a_j = b_j \qquad \text{if } b_j < i$$

$$a_j = 1 + b_j \qquad \text{if } b_j \geqq i.$$

The number m will be fixed throughout the discussion. The (n, m)-cells will be considered as the n-dimensional cells of an abstract complex $K(\Pi, m)$ with the boundary $\partial\phi$ of an (n, m) cell defined as

$$\partial\phi = \sum_{i=0}^{n} (-1)^i \phi^{(i)},$$

the summation being understood in the free group of the $(n - 1)$-chains generated by the $(n - 1, m)$-cells. The fact that $\partial\partial\phi = 0$ is an easy consequence of the following fact

$$[\phi^{(i)}]^{(j)} = [\phi^{(j+1)}]^{(i)} \qquad \text{for } i \leqq j,$$

which follows directly from the definition of $\phi^{(i)}$.

If $m = 1$, each (n, m)-cell is a function of two variables and therefore can be written as a $(n + 1) \times (n + 1)$ matrix. It is then obvious that (16.2) reduces to the previous condition (2.1), and that the complex $K(\Pi, 1)$ coincides with the complex $K(\Pi)$, as defined in Chapter I. We remark that while the complex $K(\Pi)$ was defined for all groups Π, the definition of $K(\Pi, m)$ assumes that Π is abelian, because of the additive form of (16.2).

18. The chain transformations κ_m

Let T be a singular n-simplex in $S_m(X)$. The function ϕ constructed in §16 is an (n, m)-cell of $\pi_m(X)$ and therefore is an n-dimensional cell of the complex $K(\pi_m(X), m)$. We shall write

$$\kappa_m T = \phi.$$

If $T^{(i)}$ is the face of T opposite the i^{th} vertex, then it is apparent that

$$\kappa_m T^{(i)} = \phi^{(i)}.$$

Consequently κ_m commutes with the boundary operator, and we have a chain transformation:

$$\kappa_m : S_m(X) \to K(\pi_m(X), m).$$

The chain transformation κ_m is a generalization of the chain transformation

$$\kappa : S_1(X) \to K(\pi_1(X));$$

in fact, if we regard the complexes $K(\pi_1(X), 1)$ and $K(\pi_1(X))$ as identical, we we have $\kappa = \kappa_1$. In particular, the proofs of Theorems Ia and IIa carry over without essential changes, and lead to the following theorems:

THEOREM I_a^m. *If* $\pi_n(X) = 0$ *for all* $n > m$, *then the chain transformation*

$$\kappa_m : S_m(X) \to K(\pi_m(X), m)$$

is a chain equivalence.

THEOREM II_a^m. *If*

$$\pi_n(X) = 0 \quad \text{for } m < n < r,$$

then the chain transformation

$$\kappa_m : S_m(X) \to K(\pi_m(X), m) = K$$

induces the following isomorphisms:

$$\kappa_m : H^n(S_m(X), G) \leftrightarrow H^n(K, G) \quad \text{for } n < r$$
$$\kappa_m^* : H_n(K, G) \leftrightarrow H_n(S_1(X), G) \quad \text{for } n < r$$
$$\kappa_m : H^r(S_m(X), G)/\Sigma^r(S_m(X), G) \leftrightarrow H^r(K, G)$$
$$\kappa_m^* : H_r(K, G) \leftrightarrow \Lambda_r(S_m(X), G).$$

The subgroups Σ^r and Λ_r are defined in §6.

Let us now assume that X is arcwise connected, and that $\pi_n(X) = 0$ for all $n < m$. It was shown in [1, §31] that the identity chain transformation

$$\eta_m : S_m(X) \to S(X)$$

is a chain equivalence. Consequently η_m induces isomorphisms of the homology and cohomology groups of $S_m(X)$ and $S(X)$. Under these isomorphisms the groups $\Sigma^r(S_m(X), G)$ and $\Sigma^r(X, G)$ correspond to one another. Similarly, the groups $\Lambda_r(S_m(X), G)$ and $\Lambda_r(X, G)$ correspond. Theorems I_a^m and II_a^m therefore lead to the following theorems:

THEOREM I^m. *If X is arcwise connected and*

$$\pi_n(X) = 0 \quad \text{for } n < m \text{ and } m < n,$$

then the homology and cohomology groups of X are determined by the homotopy group $\pi_m(X)$. More precisely,

$$H^n(X, G) \cong H^n(K, G),$$

$$H_n(X, G) \cong H_n(K, G)$$

where $K = K(\pi_m(X), m)$.

THEOREM II^m. *If X is arcwise connected and*

$$\pi_n(X) = 0 \quad \text{for } n < m \text{ and } m < n < r$$

then

$$H^n(X, G) \cong H^n(K, G) \quad \text{for } n < r$$

$$H_n(X, G) \cong H_n(K, G) \quad \text{for } n < r$$

$$H^r(X, G)/\Sigma^r(X, G) \cong H^r(K, G)$$

$$\Lambda_r(X, G) \cong H_r(K, G)$$

where $K = K(\pi_m(X), m)$.

The results of Chapter III are subject to a similar generalization.

BIBLIOGRAPHY

[1] EILENBERG, S., *Singular homology theory*, Ann. of Math. 45 (1944), pp. 407–447.

[2] EILENBERG, S., AND MACLANE, S., *Group extensions and homology*, Ann. of Math. 43 (1942), pp. 757–831.

[3] ———, *Relations between homology and homotopy groups*, Proc. Nat. Acad. Sc. U. S. A. 29 (1943), pp. 155–158.

[4] ———, *General theory of natural equivalences*, Trans. Amer. Math. Soc. 58 (1945), pp. 231–294.

[5] HOPF, H., *Fundamentalgruppe und zweite Bettische Gruppe*, Comment. Math. Helv. 14 (1942), pp. 257–309.

[6] ———, *Nachtrag zu der Arbeit "Fundamentalgruppe und zweite Bettische Gruppe,"* Comment. Math. Helv. 15 (1942), pp. 27–32.

[$7_{1,2,3,4}$] HUREWICZ, W., *Beitrage zur Topologie der Deformationen*, Proc. Akad. Amsterdam, 38 (1935), pp. 112–119 and 521–528; 39 (1936), pp. 117–125 and pp. 215–224.

[8] LEFSCHETZ, S., *Algebraic Topology*, Amer. Math. Soc. Colloquium Series, vol. 27, New York, 1942.

THE UNIVERSITY OF MICHIGAN
HARVARD UNIVERSITY