How matroids behave like smth proj. toric varieties

(w/ A. Fink & M. Larson)
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Let X be a smth proj. toric variety over $k = \overline{k}$.

(H) If ~~$k = \mathbb{C}$~~, ~~singular cohomology $H^*(X)$~~ Chow ring $A^\bullet(X)$ satisfies the Kähler package.
(H') (comb. description) (PD, HL, HR)

(S) For D ~~ample~~ nef / base-pt-free divisor, $H^i(X, \mathcal{O}_X(\cancel{\ell} D)) = 0 \ \forall i > 0$ for ~~$\ell \gg 0$~~.
(D)

(K) If ~~$k = \mathbb{C}$~~, $H^i(X, \mathcal{O}_X(-D)) = 0$ unless $i = \dim X$ for D ~~ample (big & nef)~~ nef / base-pt-free
(BB) $i = \dim \operatorname{image} \varphi_D(X)$

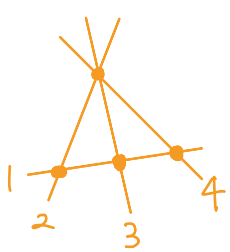
Let $[n] = \{1, \cdots, n\}$.

$L \subseteq k^{[n]}$ r-dim'l linear subspace (not contained in a coord. hyperpl.)

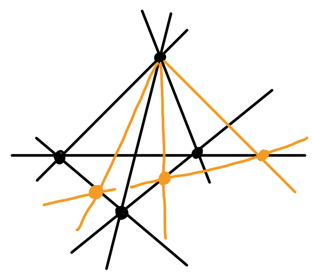
$\mathbb{P}L \subseteq \mathbb{P}^{n-1} \rightsquigarrow$ arrangement of hyperplanes $\mathcal{A} = \{\mathbb{P}L \cap \{X_i = 0\}\}_{i \in [n]}$

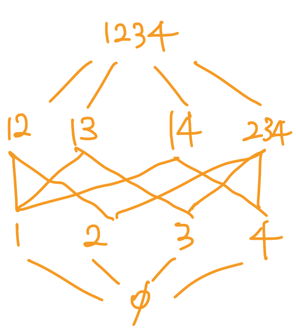
E.g. $L = \text{rowspan} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \simeq k^3 \subset k^{[4]}$

$\mathcal{A}$ on $\mathbb{P}L$

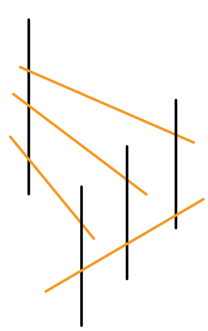


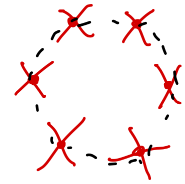
in $\mathbb{P}^3$:





∂W_L:





Matroid $M(L)$ of L remembers only the intersection lattice of $\mathcal{A}$.

$\{$(simple) matroids$\} = \{$geometric lattices$\} \supsetneq \{$intersection lattices of hyperplane arrangements$\}$

Defn Permutohedral (toric) variety $X_n :=$ sequential blow-up of all (strict transforms of) coordinate subspaces of $\mathbb{P}^{n-1}$.

Wonderful variety $W_L :=$ strict transform of $\mathbb{P}L$ under this blow-up.

$$\begin{tikzcd} \mathrm{Bl}_{\substack{\text{hyperpl's}\\ \text{of } \mathcal{A}}} \cdots \mathrm{Bl}_{\substack{\text{lines}\\ \text{of } \mathcal{A}}} \mathrm{Bl}_{\substack{\text{pts}\\ \text{of } \mathcal{A}}} \mathbb{P}L = W_L \arrow[r, hook] \arrow[d] & X_n = \mathrm{Bl}_{\substack{\binom{n}{n-1}\\ \text{hyperpl's}}} \cdots \mathrm{Bl}_{\binom{n}{2}\text{ lines}} \mathrm{Bl}_{n \text{ pts}} \mathbb{P}^{n-1} \arrow[d] \\ \mathbb{P}L \arrow[r, hook] & \mathbb{P}^{n-1} \end{tikzcd}$$

$\partial W_L =$ sum of exceptional divisors

[de Concini–Procesi '95][Feichtner–Yuzvinsky '04]: $A^\bullet(W_L)$ depends only on $M(L)$.

$\rightsquigarrow$ Chow ring $A^\bullet(M)$ of a matroid M.

[Adiprasito–Huh–Katz '18] $A^\bullet(M)$ satisfies the Kähler package.

[Berget–E.–Spink–Tseng '23] $[\mathcal{O}_{W_L}] \in K_0(X_n)$ depends only on $M(L)$.

[Larson–Li–Payne–Proudfoot '24] $K(M)$ and $\chi(M,-): K(M) \to \mathbb{Z}$.

(D) & (BB) for W_L ?

No (D). (E.g. take 9 pts arising as intersection of 2 plane cubics...)

But maybe for $D|_{W_L}$ with D nef on X_n ?

Rem Speyer's f-vector conj. $\overset{\text{reduces to}}{\rightsquigarrow}$ $(-1)^{r-1}\chi(W_L, \mathcal{O}(-K_{W_L} - \partial W_L)) \geq 0$

[Speyer '09][Fink–Shaw–Speyer] when $K_{W_L} + \partial W_L$ big & nef.

Partial progress for (D) & (BB): [E. '24][E.–Larson][Berget–Fink]

resolved Speyer's conj.

Thm [E.–Fink–Larson] For a nef divisor D on X_n, we have:

$$H^i(W_L, \mathcal{O}(D)) = 0 \quad \forall i > 0,$$

$$H^i(W_L, \mathcal{O}(-D)) = 0 \quad \text{unless} \quad i = \dim \mathcal{S}_D(W_L), \quad \text{and}$$

$$H^0(X_n, \mathcal{O}(D)) \twoheadrightarrow H^0(W_L, \mathcal{O}(D)).$$

Cor The section ring $\bigoplus_{\ell \geq 0} H^0(W_L, \mathcal{O}(\ell D))$ is Cohen–Macaulay and gen. in deg 1.

Thm [E.–Fink–Larson] For a matroid M and a nef divisor D on X_n, the h^*-vector defined by $\displaystyle\sum_{\ell \geq 0} \chi(M, \ell D)\, q^\ell = \frac{h^*(q)}{(1-q)^{d+1}}$ is a Macaulay vector.

$(d = \deg.\text{ of } \chi(M; tD))$

Proof for D ample:

(1) $T = (k^*)^n$ acts on X_n. Pick general $\lambda \in \mathbb{Z}^n$ and consider

$\mathrm{in}_\lambda(W_L) := \lim_{t \to 0} (t^{\lambda_1}, \ldots, t^{\lambda_n}) \cdot W_L$ (a union of toric boundary strata).

(2) [Katz '09] + intersection thry of tropical linear spaces [Speyer '07] [Li '19] [Backman–E.–Simpson '24] [E.–Larson '24]

$\Downarrow$

$[\mathrm{in}_\lambda(W_L)]$ is multiplicity-free

(3) $\left.\begin{array}{l} \text{locally “Kindred” (special subschemes of } (\mathbb{P}^1)^m) \\ \{e_i - e_j\}_{i,j \in [n]} \text{ is a unimodular collection} \end{array}\right\} \Rightarrow \mathrm{in}_\lambda(W_L)$ reduced

(4) matroid complexes are CM $\Rightarrow$ $\mathrm{in}_\lambda(W_L)$ CM

~~(4)~~ (5) (toric) Frobenius splitting

Rem For D ample, (3) follows also from [Brion '03] and [Conca–De Negri–Gorla '24] Cartwright–Sturmfels ideals

Addendum (Correction)

Contrary to what I stated during the talk, the question "When is the h^*-vector symmetric?" may be very interesting to pursue. Apologies to the inquirer...