POLYTOPES AND COXETER MATROIDS: EXERCISES

ALEX FINK

(1) Draw all the polytopes of rank 2 matroids on four elements, up to symmetries of the ground set.

(2) Does the matroid polytope of the Fano matroid have an octahedral face?

(3) Prove that the standard permutohedron in $\mathbb{R}^n$ is the Minkowski sum of
   (a) all of the hypersimplices in $\mathbb{R}^n$.
   (b) a collection of segments.

(4) *Loday's realisation of the associahedron* is the generalised permutohedron which is the Minkowski sum of all simplices of the form $\operatorname{conv}\{e_k, e_{k+1}, \dots, e_\ell\}$ in $\mathbb{R}^n$. What is its rank function? What are its vertices?

(5) (a) A *polytope subdivision* Δ of a polytope P is a geometric polyhedral complex whose total space is P. Given such a polytope subdivision, prove that

$$\sum_F (-1)^{\dim P - \dim F}\, 1_F = 1_P,$$

   where the sum is over cells F of Δ not contained in ∂P, and 1_F is the indicator function of F.
   (b) Let C_n be the Catalan matroid on $\{1, \dots, 2n\}$, and let $\tilde{C}_n$ be $C_n \setminus \{1, 2n\}$. Let σ be the permutation of $\{2, \dots, 2n-1\}$ given by $\sigma(i) = i + 2 \bmod 2n - 2$. Prove that $P(\tilde{C}_n)$, $P(\sigma\tilde{C}_n)$, …, $P(\sigma^{n-2}\tilde{C}_n)$ are the maximal cells in a polytope subdivision of a hypersimplex.

(6) Construct the stratification of $Gr(2, \mathbb{K}^4)$ into matroid realisation spaces, and the poset formed by the strata under containment of closures. Does the result depend on $\mathbb{K}$?

(7) Let W be a dihedral group. Characterise the Coxeter matroids for W whose vertices lie in a single W-orbit.

(8) A matroid M on ground set E is a *projection* of a matroid N on E if there is some matroid M' on a ground set $E \amalg F$ such that $M = M'/F$ and $N = M' \setminus F$.

   Fix a finite set E and naturals $0 < r_1 < \cdots < r_k < |E|$. A *flag matroid* on E of ranks $(r_1, \dots, r_k)$ is a collection of matroids on E of respective ranks $r_1, \dots, r_k$, each of which is a projection of the next. Describe how to associate a type A Coxeter matroid, i.e. a polymatroid, to a flag matroid. Prove that distinct flag matroids have distinct polymatroids.

(9) A *ribbon graph* is a two-dimensional manifold with boundary obtained as the neighbourhood of a graph embedded in a smooth surface, not necessarily orientable. More precisely, a ribbon graph is obtained by taking a collection of discs, some of them *vertices* and the others *edges*, and repeatedly identifying a closed segment on the boundary of an edge with a closed segment on the boundary of a vertex, such that all these segments are disjoint, and each edge is involved in exactly two identifications.

A *basis* of a ribbon graph G is a subset S of its edges such that the union of S and all the vertices of G has exactly one boundary component. Prove that the set of bases of a ribbon graph is a Coxeter matroid for the type C maximal parabolic subgroup excluding the long root.

(Note that, if the ribbon graph comes from a graph embedded in the plane, then its bases are the usual spanning trees of the graph.)

(10) Take a concrete $x \in Gr(r, \mathbb{C}^n)$: perhaps $(r, n) = (2, 4)$ and

$$x = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}.$$

Define the morphisms

$$\begin{array}{ccc} & Fl(1, r, n-1; \mathbb{C}^n) & \\ {\scriptstyle\pi}\swarrow & & \searrow{\scriptstyle\rho} \\ Gr(r, \mathbb{C}^n) & & \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \end{array}$$

given by retaining only the r-dimensional, respectively the 1 and $(n-1)$-dimensional, spaces in the flags. Let $\mathcal{O}(1)$ be the line bundle on $Gr(r, \mathbb{C}^n)$ determining the Plücker embedding.

Compute $\rho_* \pi^*([\mathcal{O}(1)] \cdot [\mathcal{O}_{\overline{(C^\times)^n x}}])$ in algebraic K-theory, and express it in the basis of powers of structure sheaves of hyperplanes in the two $\mathbb{P}^{n-1}$ factors. You should get the Tutte polynomial of the matroid of x.

Equivariant localisation is the best tool for this, I reckon.

(11) (Open, but perhaps not difficult?) Let $\Sigma_{A_{n-1}}$ be the permutahedral fan. Let MW^c be the group of codimension c Minkowski weights on $\Sigma_{A_{n-1}}$, i.e. associations of weights to the codimension c faces of $\Sigma_{A_{n-1}}$ satisfying the balancing condition.

Let $ch : MW^c \to MW^1$ be the linear map given by Minkowski sum with the $(c-1)$-dimensional skeleton F of the normal fan of the simplex $\operatorname{conv}\{-e_1, \ldots, -e_n\}$, with the weight on a cone σ of $ch(m)$ being the sum of the weights on cones τ of $m \in MW^c$ such that the Minkowski sum $\tau + F$ contains σ. (In general, to make this operation yield a balanced fan, one needs a correction factor coming from an index of a sublattice, but in the present situation this is trivial.)

Characterise the kernel of ch.