Recall: \( BS\text{Bim} = \) full monoidal subcat of (graded) \( R\)-bimodules w/ objects \( BS(\omega) \)

\( \text{Hom}^0 = \) all graded maps.

This is a monoidal cat (via biadjunction) so should draw morphisms as planar diagrams (ribbon category).

Warning: Before we visualized elements \( f \mid g \mid f g \) now we draw morphisms, connect in some but don't get confused.

Type A

\( S = \mathbb{Z}[I] \quad \text{Obj:} \quad B_8 \quad R \)

\( B_8 \) is a Frobenius algebra in \( BS\text{Bim} \) so have 4 strucure maps.

\[ R \quad \xrightarrow{\delta} \quad R \quad \xrightarrow{\epsilon} \quad R \quad \xrightarrow{\beta} \quad R \]

\( \delta \quad \epsilon \quad \beta \)

degree \( +1 \)

\[ \delta = \delta_0 + \delta_1 \quad \epsilon = \epsilon_0 + \epsilon_1 \quad \beta = \beta_0 + \beta_1 \]

EXAMPLES:

a)

\[ \begin{array}{ccc}
    1 & 1 \\
    0 & 1 \\
    0 & 0 \\
    0 & 0 \\
\end{array} \]

b)

Exercise

Another example:

\[ \begin{array}{ccc}
    R & \xrightarrow{f} & R \\
    \uparrow & & \uparrow \\
    R & \xrightarrow{g} & R \\
\end{array} \]

Ex: a)

\[ \begin{array}{c}
    1 \\
    0 \\
\end{array} \]

\[ \Rightarrow \]

Linear combination of diagrams.
Thm: (Eilenberg-Kazhdan) Let $\lambda, \gamma : \mathbf{Y} \rightarrow \mathbf{Z}$ generate all $R$-bimodule morphisms in $\mathbf{BSBim}$.

1. Unit: $\lambda = \lambda \Rightarrow \text{Unit} \quad \gamma = \gamma$
2. Assoc: $\lambda \circ \lambda \Rightarrow \lambda$ (what is this?)
3. Decomposition: $\phi = \frac{\phi_1}{2} + \frac{\phi_2}{2}$ (for more generality)
4. Eval: $i = \text{Eval} \quad \mathfrak{E} = \mathfrak{E}$
5. Polynomiality relation

Moreover, there are all the relations. I.e. let $D$ denote the monoidal co-objects of

and $\text{Hom}(\quad, \quad) = \text{linear combo of}$.

$F : D \rightarrow \mathbf{BSBim}$ as above. Then $F$ is an equivalence of categories.

Most important feature of $\mathbf{BSBim}$ for going go is $\mathbf{BSBim} \cong R \oplus \mathbf{B}_s(-1)$. How to show this "diagonally" i.e. morphism-theoretically?

$id_{\mathbf{BSBim}} = e_1 + e_2$, e.g. factor the $R$:

Now practice!
Goal: Consider case (W, S) Can already construct many morphisms and the relations still hold. Call these Universal  morphisms.

Exercise: Any Universal morphism with empty bdy (\in \text{End}(R)) reduces to a polynomial. \Rightarrow \text{nothing in negative degree.}

Can (A) be true... are these all morphisms? No. Exercise: Set. The minimal degree of any Universal morphism $R_{B \in \mathcal{B}, B \in \mathcal{B}}$ is 2.

\text{But if } M^2 = 5 \text{, there should be a degree 1 map } R_{B(\text{st}t \forall)} \rightarrow R_{B(\text{st}t \forall)} \Rightarrow \text{SHF} \Rightarrow \text{degree 0 maps are 1D.}

Can pin down the precise morphism by asking $|\phi[0,0]| \rightarrow |\phi[0,0]|$. (works since $R_{B(\text{st}t \forall)} \rightarrow B_{stt \forall}$.)

Qb: What is a formula for this map in terms of polys? Ans: Cash, it's awful.

Regardless, we draw this map:

```
2st. vertical vertex
```

I could just write down some relations + ask you to believe me, but actually there's some beautiful deep math at play here + I want you to understand. (Even if you only care about type A.)

Key relation:\[ \text{proj} \rightarrow B_{\text{stt}} \otimes BS(\text{stt}) \in \text{Universal, what is it?} \]

- It's a Jones-Wenzl projector.

\text{Thm (E-W):} (A) \& Y \& (K) \& (C) \& (D) \Rightarrow \text{generate all morphisms.}

Really, it's alluded.

\text{B: What are additional relations?} \text{ In order for:}

\text{C: } F:D \rightarrow B_{\mathcal{B}m}.\]
Next goal: \[ S = \{ s, t \} \]. Can already construct many morphisms

(AD space of) degree 0 map

But none are possible using just the stuff above (exercise)

Exercise 6

If NO. \( sT \mapsto E \),

If not, actually YES...

How to approach...

\[ S^2 \text{-rep} \]

But? Switch? You'll find out.

Denote \( V^0 \).\[ \rightarrow \text{Object of } \text{FILD}_2 \text{rep } \mathbb{L}_2 \]

Now, \( V^2 \cong S^2 V \otimes V^* \)

Check:

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

Crashingly matches (row 2 column 2 is zero)

Thm: \( sT_a \mapsto \text{FILD}_2 \)

Lem: \( T_a \) has objects \( n \in \mathbb{N} \), \( \text{Hom}(m, n) = \begin{pmatrix}
q & 1 \\
q & 1 \\
1 & 1 \\
1 & 1 \\
\end{pmatrix}
\]

Main implication: \( V^0 \otimes V^0 \) so \( E \) idempotent

\[ V^0 \otimes V^0 \]

Jones-Wenzl Projector

Properties:

1. \( \text{JW}_n \) can be defined when quantum braids \([k]\) are invertible for \( k \in \mathbb{N} \)

2. Nice recursion formulas.

3. \( \text{JW}_n \) is a morphism s.t. \[ \text{JW}_n = 0 \text{ for all caps and coref of } \]
Can use recursion formula to give digraph or graph theory proof that $V_0V_n = V_{n+1} \oplus V_{n-1}$ for $n \geq 1$  $\Rightarrow V_0V_0 = V_1$

$[E3] C_m = [n^2 + n]$  
$[E4] [n] = [2]$  

$H_s H_{st} = H_{st} + H_{s_0}$  
$H_{st} H_s = H_{st}$

**Triumphant Return to dihedral case**

$S = \{e, t\}$  
$\frac{TL}{F_t} = BSBim$  

Well defined?  

Thus it is degree 0.

Well defined when  

$$a_{st} = - (q - q^{-1})$$  
$b_{st} = - 2 \cos \frac{\pi}{M_{st}} = -(S_m + S_m^{-1})$

so works when $q$ is the correct root of unity!!

$[M_{st}] = 0$  
$[M_{st-1}] = 1$

Pause, Relish. Fix checkered $qp$ for $m$. Set $q = S_m$. Then $\frac{TL}{F_t} = BSBim$.

when $m = \infty$, $q = 1$.  

**Remark:** Unnatural? Actually comes from a 2-functor which is very natural!

This is (quantum) geometric sort of (at a root of unity!)

Exercise.

Patient until later in the week.
Ex: Type $A_2$
\[ m = 3 \]
\[ \alpha = \frac{2}{3} + \frac{5}{6} = 1 \]
\[ \alpha_{st} = -1 \]
\[ \beta = 0 \]

\[ \text{JW}_1 = \begin{array}{c}
\text{JW}_2 = \begin{array}{c}
\text{JW}_3 \text{ is not defined, } \text{still not a red exp.}\n\end{array}
\end{array} \]

\[ \text{Spacce on define } B_{st(m)} = \text{Im } \text{JW}_n \otimes B(st) \]
\[ V \otimes V = V_{m1} \oplus V_{m2} \quad \Rightarrow \quad B_{st(m1)} = B_{st(m2)} \otimes B_{st(n)} \]

\[ \Rightarrow [B_{st}] = H_H \quad \Rightarrow \quad \text{End}(B_{st}) = \text{id + higher degree} \]

\[ \Rightarrow \quad B_{st} \text{ is indecomposable } \Rightarrow B_{st} \text{ is the } \text{index } \text{Stegel's basis as desired.} \]

One thing is missing. $F_s(\text{JW}_{n-1})$ gives $B(st) \otimes BS(st)$ but $B(st) \neq B^{st}$.

As noted, this isomorphism cannot come from any (with finite, of course, but other)

Read new system:

\[ B_{st} B_{st} \]
\[ \text{???
}\]
\[ B_{st} \]
\[ B_{st} \]

\[ \text{on polynomials} \]

For explicit forms, see my thesis.

\[ m = 5 \]

\[ \text{2m-valent vertex} \]

\[ \text{degree 0.} \]
Thm (Euler): $\{A, P, I, X, Y, M, \ldots\}$ generate all morphisms.

B Relation: Isotopy

Dot:

\[ \text{Diagram: } \quad \text{Degree 2 version, another way to dot retract} \]

\[ \text{Diagram: } \quad \text{With additional dots} \]

C There are all relations, i.e., $F : \mathbb{C} \to \text{BSBrin}$. 