The Classical Approach to Soergel Bimodules

**Jacob Matherne (LSU)**

Setup: \((W,S)\) Coxeter system

- \(\mathfrak{h}\) (geometric) realization of \((W,S)\)
- \(R = S(\mathfrak{h}^*)\) = polynomial functions on \(\mathfrak{h}\)
- \(W \otimes R\) & \(R^s = \text{s-invt polys}\)

Consider all of \(R\)-Bim for the moment.

**Standard bimodules:** graded \(R\)-bimodules

Let \(x \in W\).

**Def:** Define \(R_x \in R\)-Bim by

1. \(R_x = R\) as a left \(R\)-module
2. As a right \(R\)-module, \(m \cdot r = x(r)m\).

**Visualization:**
Recall \(B_5\) can be viewed as "border patrol"

<table>
<thead>
<tr>
<th>Semi-porous wall where s-invt polys can slide through</th>
</tr>
</thead>
<tbody>
<tr>
<td>For (R_x), we have &quot;puberty&quot;</td>
</tr>
</tbody>
</table>

\[ f = x(f) \]

| Completely-porous wall where anything can slide through but act by \(x\) |
Facts:
1. \( R_x \otimes_R R_y \cong R_{xy} \)
2. \( \text{Hom}(R_x, R_y) = \begin{cases} R & \text{if } x=y \\ 0 & \text{else} \end{cases} \) since \( h \) is a faithful rep.

Define \( \text{StdBim} \) = full \((\oplus, (\cdot), (\otimes))\) subcat of \( R\text{-Bim} \) gen by \( \mathcal{E} R^3_{x \times y} \).

1 & 2 \Rightarrow \text{StdBim} is a realization of the additive 2-groupoid of \( W \) over \( R \).

- Tensor is multiplication, but
- Now we have more than just id maps, have mult by elts of \( R \) (polys in boxes)

\[ f = \begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array}
\end{array}
\]
\[ A \in \mathcal{S}. \]

Filtrations:
Consider the SES
\[ \begin{array}{c}
\begin{array}{c}
1 \\
\rightarrow d_0 := \frac{1}{2}(a_0 \otimes 1 + 1 \otimes a_0)
\end{array}
\end{array}
\]

(\(\Delta\)) \[ R_{s(-1)} \rightarrow B_s \rightarrow R_i d(1) \]
\[ f \circ g \rightarrow f g \]

(\(\nabla\)) \[ R_i d(-1) \rightarrow B_s \rightarrow R_{s(1)} \]
\[ f \circ g \rightarrow f s(g) \]

In alternate notation: \[ f \circ g \rightarrow f g \leq \text{here } \otimes \text{ is the wall from before.} \]
So, $R_s$ is filtered by both $R_{id}$ and $R_s$, but with no particular order (and grading shifts depend on this order).

Soon, we will fix this by specifying an order.

**Def.** A bimodule $M_{R}$-$R$-Bim is said to have property $(\ast)$ if it has a finite filtration with subquotients $\left\{ \bigoplus_{x} R_x(\cdot) \right\}$.

**Facts:**

1. Having $(\ast)$ is closed under $\otimes$.

   - Suppose $0 \leq B' \leq \ldots \leq B^m = B$ with subquotients $\left\{ \bigoplus R_x(\cdot) \right\}$
   - \& $0 \leq C' \leq \ldots \leq C^n = C$

   **Lemma:** If $M$ can be filtered so that subquotients have $(\ast)$, then $M$ has $(\ast)$.

   - For $i, j$, $B_{i+1}/B_i \otimes C_{j+1}/C_j$ is a std bimod,
   - So $B \otimes C_{j+1}/C_j$ has $(\ast)$.
   - By Lemma, $B \otimes C$ has $(\ast)$.

2. Having $(\ast)$ is closed under direct summands.

   - We are working in a Krull-Schmidt cat.
   - A filt of $B \otimes C$ by indec subquotients $R_x(\cdot)$ (is indec) will give a filt of $B \otimes C$ separately.
Observations:

i) Bott-Samelsons have (★) by SESs & ①

ii) Summands of Bott-Samelsons (and therefore Soergel bimods) have (★) by②

iii) $R_x$ are not Soergel bimods.

They can appear as submods & quotients of Bott-Samelsons (as in SESs from before), but not as summands. (Except $R = R_{id} = B_{id}$).

Time to pin down this order/grading shift issue!

$\Delta$- & $\forall$-filtrations:

\begin{equation}
\text{Def: A } \Delta \text{-filtration on } B \in SBim \text{ is a finite filtration by } R-Bim \text{ of } \mathcal{O} \subset B^m \subset \ldots \subset B_i \subset B_0 = B \text{ such that } B_i^j / B_i^{j+1} = \bigoplus R_x(\nu) \text{ with } l(\nu) = i \text{ & } \nu \in \mathbb{Z}.
\end{equation}

Analogously, $\forall$-filtration on $B \in SBim$ is a finite filtration by $R-Bim$ of $\mathcal{O} \subset B^0 \subset \ldots \subset B^m = B$ such that $B^i / B^{i-1} = \bigoplus R_x(\nu)$ with $l(\nu) = i$ & $\nu \in \mathbb{Z}$.

\textbf{NOTE:} The SES ($\Delta$) gives a $\Delta$-filtration on $B$ & ($\forall$) gives a $\forall$-filtration on $B$. 

Thm (Soergel 2006)
Any $BSBim$ has a unique $\Delta/\nabla$-filt.

Pf (Sketchy Sketch):
Having $(\Delta)$ is preserved under taking summands
(same reason as for $(\ast)$).

Having $(\Delta)$ is closed under $\otimes$ with $B_s$.
\[ \Rightarrow \text{Gives } (\Delta) \text{ for all Bott-Samelsons}. \]

The subquotients do not depend on the choice of refinement,
but order & grading shifts are very different!

Remark: Support filtrations give an explicit construction of
$\Delta/\nabla$-filtls!

One can define two notions of character.
We will use $\text{ch}_\Delta$.

\[ \text{ch}_\Delta : S_{Bim} \to H \]
\[ \text{ch}_\Delta(B) = \sum_{x \in \mathbb{W}}^e(x) + \text{shift on } R_x \cdot H_x \]

E.g.: From $SES$, $\text{ch}_\Delta(B_s) = VH_{id} + H_s = H_s$.

Remark: (1) This gives an inverse to the isomorphism

\[ E : H \to [SBim] \]
\[ H_s \mapsto [B_s] \]

(2) In the course of the proof of the theorem
above, Soergel showed

\[ \text{ch}_\Delta(M \otimes B_s) = \text{ch}_\Delta(M) \cdot H_s \]

Recall: Soergel conj: $\forall x \in W$, $\text{ch}_\Delta(B_x) = H_x$. 

Localization:

Soergel bimodules become much simpler after localization, (become "like" standards).

Let \( Q \) be the homogeneous fraction field of \( R \).
Even though \( Q \) is graded, \( Q \cong Q(2) \) in this case.

\textbf{Def:} The "localization" of \( B \in \text{R-Bim} \) is \( B \otimes_R Q \).

\textbf{Notation:} \( B^Q_s := B_s \otimes_R Q \), \( BS^Q(w) := BS(w) \otimes_R Q \),
\( Q_s := R_s \otimes_R Q \).

\textbf{Note that:} \( BS^Q(w) \cong Q_{Q_s} \cdots \otimes Q_{Q_s(w)} \) if \( w = s_1 \cdots s_n \).

Observation:
Suppose \( M \) is free as an \( R \)-module.

Then, it includes into its own localization (injection since \( M \) is free).

But \( \text{Hom}(B_x, B_y) \) is free as a right \( R \)-module,
so get injection:
\( \text{Hom}(B_x, B_y) \hookrightarrow \text{Hom}(B_x, B_y) \otimes_R Q \).

Localization is a faithful functor!

Back to \( \text{SESs} \):

After localization, these \( \text{SESs} \) split each other:

\[ R_s(-1) \rightarrow B_s \rightarrow R_s(1) \]
\[ 1 \rightarrow d_s = \frac{1}{2}(x_s \otimes 1 - 1 \otimes x_s) \rightarrow \alpha_s \]

and

\[ R_{id}(-1) \rightarrow B_s \rightarrow R_{id}(1) \]
\[ 1 \rightarrow c_s = \frac{1}{2}(x_s \otimes 1 + 1 \otimes x_s) \rightarrow \alpha_s \]

But \( \alpha_s \) is invertible now!
So,
\( B_s^Q \cong Q(1) \oplus Q_s(1) \).
By expansion:
\[ BS^w(w) = \bigoplus_{\text{ec} w} Q_{we}(l(w)) \]

This is key! Bott-Samelsons are easy after localization.

Why is localization useful?

1. **Proposition:** For a rex \( w, \exists! \) indec summand \( B_w \) in \( BS(w) \) which contains \( Q_w \) after localization, and \( B_w \) is not a summand of any shorter sequence.

   **Proof:** Look at the decomposition of \( BS^w(w) \) above. Since \( Q_w \) is indec and appears exactly once, there is certainly a unique summand which contains it. Again by looking above, it is clear that \( Q_w \) does not appear in the localization of any shorter sequence.

   **Note:** This does not prove that all the other summands in \( BS(w) \) do appear in some shorter sequence.

2. Localization is a faithful functor, so can check if two morphisms of Soergel bimods are equal by checking after localization.

   - Geordie writes computer programs exploiting this fact.
There are nice diagrammatics for playing with localization!