§1 (Red Path Morphism) Recall from Alex’s lecture: From a path in the expression

\[ g \text{ of } \omega W \text{, one could construct a morphism in } SLW^2 \text{.} \]

Every morphism in } SLW^2 \text{ is an isomorphism. So } \omega \text{ is simply } \omega \text{.}

Relation \[ \equiv \text{ loops in expression graph} \]

\[ \Rightarrow \text{ any two path morphisms } \sim \text{ are equal.} \]

Now consider } BS\beta \text{ instead. Could do the same thing using } \beta \text{.}

Issues:

1. } U \text{ is nowhere close to being an isomorphism. We shouldn’t use it. Stick to reduced expression.} \]

\[ \text{Example: } [0] = 0 \text{ degree above } = 0. \]

Def: Given a path in the red graph for } \omega W \text{, the corresponding morphism in } BS\beta \text{ is called the path morphism. Or red wave.}

2. } \sim \text{ is also not an isomorphism! Projection to a common summand.} \]

\[ \text{Example: } \frac{1}{2} \text{ is not identically zero, but not the identity.} \]

\[ \text{So two path morphisms will NOT be equal.} \]

\[ \text{Nonetheless…} \]

Fact: Two path morphisms } W \Rightarrow 2\text{ will be equal modulo higher terms. } \]

Localisation: Any } BS(\omega) \Rightarrow Q\omega \text{ after localisation.}

Def: Lower terms } \Rightarrow \text{ zero on } Q\omega, \text{ after localisation.}

Factorisation: Lower terms (are divisor content of things which) factor through shorter expressions.

\[ \text{Example: } \frac{1}{2} = 1 + \frac{1}{2} \Rightarrow \text{ Fact } \Rightarrow \text{ Loc since shorter expression have no } Q\omega \text{ elements.} \]

1-tensor: Lower terms are maps which kill } C_1 = 1002000 \text{ BS}(\omega) \Rightarrow \text{ Fact } \Rightarrow 1 \text{-tensor.} \]

Ex: Above. Exercise! All 1-tensors in } JW \text{ kill Cost.
Why is fact true? JW stuff $\Rightarrow$ equal mult for $N$ loops. But what about Zams?

This requires a new relation!

So for each $\frac{1}{3}S$, we will have a relation saying $w = w$ opposite are equal up to some precise law terms

Examples: $A_1$ $A_2$ $A_3$

 Miracle (unexplained): In types $A + I_2(m), A_3, B_3$ there is a choice of $w$ such that the two paths agree on the nose! (Not all choices of $w$ work!!!)

Possible explanation in type A: higher Borchert order. But why?

In type $B_3$, lower terms are necessary (and also uncomputed...)

§21 Gen Rel Relation Redux


Input (E-W): Let $D$ be monoidal act $w$ presentation

Ob: $S = S$

Mor: $\alpha \eta_Y \eta_X$ $\eta_X \eta_Y$

Old Relatin: 1-color 2-color Zam.

Let $\mathbf{F}: D \rightarrow \mathbf{BS Bm}$ be defined as before. Then $\mathbf{F}$ is an equiv.

So pictorial computation for the win!

But many crazy pictures can be drawn, and it's not always easy to simplify. What a basis for morphism space.

§31 Lubinski's Light Leaves

Recall $H(w) = \sum_{g,c} \nu_{\text{def}(e)} H_{g,e}$

$\Rightarrow$ $(H(w), H(y)) = \sum_{\begin{array}{c} g,c \end{array}} \nu_{\text{def}(e) + \text{def}(f)} H_{g,e}$

get rank Hom $\circ (\mathbf{BS}(w), \mathbf{BS}(y))$
So basis should be parameterized by triples \((e, f, x)\) where \(e C W \& C Y \quad w^e = y^f = x \in W\).

Actually, the basis splits into two halves, \((e, x)\) and \((f, x)\).

**Construction:** \((w, e, x)\)
- Fix arbitrary \(r e x \& x \text{ for } x\).
- We build inductively.

**Example**
\[
\begin{align*}
w &= \text{sts} \\
e &= 001 \\
&\quad \text{UUU} \\
t &\quad \text{11} \\
&\quad 10 \\
&\quad 01 1 \quad \text{ok so far}
\end{align*}
\]

\[
w_2 = \text{sts} \\
&\quad \text{11 \text{00}} \\
&\quad \text{000DD} \quad \text{can get nasty}
\]

Which \(r e x \) path do you choose?? It matters!!

No canonical choice.

**Inductive Formula:**

- Procedure is well defined up to various choices at each move.
- Just choose arbitrarily, but in practice, we call anything constructible in this way by the name "light leaf."

**Thm (Lb)**

**Def:** For \((w, e, x)\) let \(ll_{e,f}^{w,x}\) denote a double leaf.

Then \(\{ll_{e,f}^{w,x}\}\) forms a basis for \(\text{Hom}^2(\text{rS}(e), \text{rS}(f))\).

Let \(I\) be an ideal in \(W\) for Borel order. Let \(I = \text{SW} \text{ on } \text{CU}\).

This is an ideal in \(D\).

When \(W\) is understood, \(D < W\) is lower terms.

\(\beta, \beta^i\) induce \(W\), so \(\text{LL is canon modulo } D\).