Fix $g$ a Borel $s.s.$ l.a.  \[ \text{Rep}(g) = \text{Full\ U(gl)}-\text{reps is an "easy" semisimple category.} \]

But what about $\infty$-dim reps?  Ireland $V$ has a hw. vector, its weight class is $V$.

$\mathfrak{so}$ is a class of "nice" $\infty$-dim reps that tries to preserve the usual notion of highest weight.

Ex: $\mathfrak{sl}_2 = \langle H, E, F \rangle$

\[ [H, E] = 2E, \quad [H, F] = 2F, \quad [E, F] = H \]

$V_+^{+}V$ is hw. of weight $\lambda$

If $H \lambda = \lambda$

If $E \lambda = 0$

Generate a subrep of $V$.

\[ \begin{array}{ccccccc}
 & 4 & 3 & 2 & 1 & 0 & 1-1
 \end{array}
\]

\[ \begin{array}{ccccccc}
 F & V_4 & V_3 & V_2 & V_1 & V_0
 \end{array}
\]

$4 \leftarrow 3 \leftarrow 2 \leftarrow 1 \leftarrow 0$

Usual arg: If $F_{k, r} \not\in \mathbb{N}$, $F_{k, r}^2 V_+ = 0$ for first time. But this $\Rightarrow$ must have coeff. 0.

$\Rightarrow k = \lambda$, $\lambda \in \mathbb{N}$.

But in $\infty$-dim, allowed for all $F_{k, r} \neq 0$, and $\lambda$ can be arbitrary $\lambda \in \mathbb{C}$.

Get the Verma module $\Delta(\lambda)$.

General defn: $\Delta(\lambda) = \text{U(gl)} \otimes \mathbb{C}_{\lambda}$ when $\mathbb{C}_{\lambda}$ is $\lambda$ rep of $\mathfrak{su}$ $(V_+)$

\[ \text{Ind}_{\mathfrak{su}}^{\mathfrak{gl}} \mathbb{C}_{\lambda} \]

Properties:

1. $\Delta(\lambda)$ is indecomposable. Pf: Any subalgebra splits via "ss" space (assuming $\mathfrak{h}$ can project to)

2. $\Delta(\lambda)$ is simple when $\lambda \in \mathbb{Z}_{\geq 0}$. Pf: Any subalgebra splits via ss space

3. If $\lambda \in \mathbb{Z}_{\geq 0}$ then $\Delta(-\lambda - 2) \Delta(\lambda)$

4. $\Delta(\lambda)$ has weak self-extension which are NOT weight modules!

\[ X = \text{U(gl)} \otimes \mathbb{C}^2 \]

\[ \text{Ind}_{\mathfrak{su}}^{\mathfrak{gl}} \mathbb{C}_{\lambda} \]
We don't like this.

**Def:** \( \mathcal{O}C(X) \)-top is the full subcat whose objects are
- \( f \)-generated
- \( h \)-weight
- \( U(h)^{-\text{finite}} = V \), \( U(h)_V \) is fil.

**Ex:** Includes \( \Delta(n) \) \( L(n) \), let \( \mathcal{X} \), not \( U(h) \otimes \mathcal{O}(x) \).

\( \mathcal{O} \) has duality \( M \to M^* \). Can just take \( M^* = \text{Hom}_E(M, \mathcal{O}) \) become this reverse weights, bold below, compare that with automorphism of \( g \) \( \text{K} \to -\text{K} \).

Effectively reverse roles of arrows!

\( \Delta(n)_V = \Delta(n) \) =

If \( x \in \mathbb{Z}_2 \), \( \Delta(1) \simeq \Delta(0) \).

But if \( x \in \mathbb{Z}_3 \),

\( \Delta(-n-2) \to 0 \) dual ss's.

\( \mathcal{O} \) is not normal. \( V \otimes W \) not fin.

But \( \text{Rep}_G \mathcal{O} \) \( V \) \( \oplus W \) is fin, etc.

Action of \( E \) is \( E \cdot \mathcal{O} \cdot E \).

Let's do an new example?

\( E = \text{sum of two arrows} \)

\( V^t = \text{hw vector} \)

\( F(V) \) \( \cap \) left linear basis of top + bottom but gives copy of \( \Delta(6) \).

\( \text{Spin} \) with \( w' + 4 \) with \( E(\pi) \) gives copy of \( \Delta(4) \).

In fact, \( \Delta(1) \)-module, \( \Delta(6) \to \Delta(4) \to 0 \) \( \Delta(1) \).

\( \text{Induction rdy,} \) \( \text{Induction rdy,} \)

Actually, it splits! \( \Delta(4) \) not indep \( \Delta(6) \).

\[ \Delta(6) \to \Delta(4) \to 0 \]
Ex 1: \( A(\text{dO}L(1)) \)

\[
\begin{align*}
\Delta(0) & \rightarrow \Delta(-1) \rightarrow \Delta(-2) \\
\Delta(0) & \rightarrow \Delta(-1) \rightarrow \Delta(-2) \rightarrow \text{O} & \text{NOT SPLIT}
\end{align*}
\]

What's going on...

---

I better reason for sequence to split:

\[ Z = Z(U(1)) \subset U(1) \]

---

HC Inv: \( Z \cong \text{Sym}[U^*]^n \)

---

M, N Different \( Z \)-equiv. \( \Rightarrow \) no morphisms, no extensions, etc.

---

\( \Delta(6), \Delta(4) \) have no extns... different blocks!

---

Put! \( Z \) does not act by scalars, but gen. equiv. Scale on \( \Delta(1) \)

---

Thus is an attendant thing, \( O' \), where \( Z \) acts by scalars, but \( H \) acts by gen. wt.

---

So \( O = O' \oplus \bigoplus_{k \neq 0} O_k^{[k]} \) blocks.

---

\( O' = \text{block containing } \Delta(\theta) \)

---

Put: Actually, blocks are even smaller. If \( \lambda, \mu \) do not differ by a root, no possible map \( Y_H(\lambda) \rightarrow \Delta(\theta) \)

---

Plan 2: A integral.
Ex: \( \Delta(2) \otimes \Delta(7) = 5 -3 -1 1 3 5 \quad ? \quad 9 \)

L(5) has \( \text{Hert} \) as \( \text{U}(5) - \lambda \)

\[ \begin{array}{c}
\text{C}_\lambda \text{ sub} \\
\text{C}_\lambda \text{ next} \\
\end{array} \]

but different blocks must split.

\[ \begin{array}{c}
\text{not split, similar to} \quad (\Delta(0)) \\
\end{array} \]

Prop: \( \lambda \text{ max in } W_0 \text{ orbit} \implies \Delta(\lambda) \text{ is projective in } \mathcal{O} \)

Proof:
\[ \text{Hom}(\Delta(\lambda), M) = \text{space of } \lambda \text{ vec tors in } M = \begin{cases} \mathbb{C} & \text{if } M \text{ in every block} \\
\text{null} \text{ (when does } \lambda \text{ go?)} & \text{in single block} \\
\end{cases} \]

Now, \( \text{dim } M_\lambda \text{ add in } \mathfrak{g}, \text{ so } \text{Hom}(\Delta(\lambda), \mathcal{O}) \text{ exact.} \)

Rem: \( \mathcal{O}(1) \text{ not exact in } \mathcal{U}(g)^- \text{ reps, we make self-extend } X \text{ earlier.} \)

Ex: \( \Delta(-1) \text{ is both proj + simple,} \)
\[ \begin{array}{c}
\mathcal{O} \cong \text{Vect} \\
\begin{array}{c}
\mathfrak{sl}_2 \\
\end{array} \\
\end{array} \]

\[ \begin{array}{c}
\mathfrak{sl}_2 \text{ integral has simples/verma/projection classified by } \text{W-oh} \\
\begin{array}{c}
\text{by } W/\text{Stab} \text{ (W) } \\
\end{array} \\
\end{array} \]

When \( \lambda \text{ dominant, write } P_\lambda = P(\omega_\lambda) \)

\[ \begin{array}{c}
P(\omega_\lambda) \rightarrow \Delta(\omega_\lambda) \rightarrow L(\omega_\lambda) \\
\end{array} \]

\[ \text{BREAK} \]
How to construct projectives? Using an functor \( \mathfrak{O}_{\bullet} \),

**Def:** A **projective functor** is a functor \( \mathfrak{O}^a \rightarrow \mathfrak{O}^b \) given by

\[
M \rightarrow P^a(M \otimes \mathfrak{O}^b) = P^a \mathfrak{O}^{b+1}
\]
or a submap thereof.

**Prop:** Projective functors in exact and preserve projective projectives.

If actually, they have duals! \( \mathfrak{O} \leftrightarrow \mathfrak{O}^a \)

**Def:** A **translation functor** \( \mathfrak{O} \rightarrow \mathfrak{O}^a \), when \( a \) integral, domain is

\[
P^a \mathfrak{O}^{b+1}
\]

Is at hand, then \( \mathfrak{O}^b \) (loose end)

**Ex:** \( T^5_2 \Delta(2) = P_5(\Delta(2) \otimes V_3^a) = P_5 \left( \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right) = \Delta(5) \)

\( T^2_1 \Delta(5) = P_2(\Delta(5) \otimes V_3^a) = P_2 \left( \begin{pmatrix} 8 \\ 4 \\ 2 \end{pmatrix} \right) = \Delta(2) \)

**Thm:** An element integral regular, then \( T^a \) is an equivalence. What work of \( \mathfrak{O}_0 \).

Can get interesting functors on \( \mathfrak{O} \) by translating to a nonregualr closed end back!

\[
\Theta_0 = T^{-7}_0 \Theta \text{ for } \alpha \text{ on the small } (\text{irrelevant which) }
\]
preserves projectives, acts on \( \Theta_0 \) as described.
Ex. \( A \text{ for } \mathfrak{sl}_2 \text{ arch} \) \[ \Delta(\mathfrak{g}) = \Delta(\mathfrak{g}) \]
\[ \Delta(\mathfrak{g}) = \Delta(\mathfrak{g}) \]
\[ \Gamma(\mathfrak{g}) = L(-2) = \Delta(-2) \]
\[ P(\mathfrak{g}) = \text{Ext}(\Delta(\mathfrak{g}), \Delta(\mathfrak{g})) \]

Ségal Theory

Thm. (Endomorphism sets): \( \text{End}(\mathfrak{P}^{\text{proj}}(\mathfrak{g})) = C = \mathbb{R}/\mathbb{R}^+ \)

Def. The Ségal functor \( \mathcal{V} = \text{Hom}(\mathfrak{P}^{\text{proj}}(\mathfrak{g})) \) exact

a very nasty functor!

\[ \mathcal{V}(\mathfrak{L}_0) = \mathbb{C} \]
\[ \mathcal{V}(\mathfrak{L}_x) = 0 \text{ x-free} \]
\[ \mathcal{V}(\Delta(\omega)) = C \]
\[ \mathcal{V}(\mathfrak{A}(\omega)) = C \]

Thm: \( \Psi(\text{Struktortasche}) \) \( \mathcal{V} \) is fully faithful on projectives

(see every project has some \( L_0 \), inside it, and all morphisms detected by this part)

\( \Rightarrow \) \( \text{Proj } \mathcal{O} \rightarrow \) image in \( C \)-mod \( \cong \) Ségal nodules

What are they?

\[ \mathcal{V}(\mathfrak{P}_L) = C = C/\mathbb{C}^+ \]

Prop.: \( \mathcal{V}(\mathfrak{Q}_M) = \mathbb{C} \otimes \mathcal{V}(M) \Rightarrow \mathcal{V}(\text{BSProj}(\mathfrak{g})) = \mathcal{V}(\text{Proj}(\mathfrak{g})) = \mathbb{R} \otimes \mathbb{C} \]

All proj or sums of \( \text{BSProj} \)
\( \Rightarrow \mathcal{V} \) is sum of \( \mathcal{V}(\text{BSProj}) \)

Thm: \( \mathcal{V}(\mathfrak{P}_\omega) = \mathbb{B}_\omega \cong \mathbb{B}_\mathbb{R} \otimes C = \mathbb{B}_\mathbb{R}/\mathbb{R}^+ \)

The bubbles themselves are some kind of "equivariant lift"

"deformed category \( \mathcal{O} \)"

Also: \( \text{Hom}_\mathfrak{g}(\mathfrak{B}, \mathfrak{B}) \otimes \mathbb{R} \)

\( \text{Hom}_\mathfrak{g}(\mathfrak{B}, \mathfrak{B}^+) \)
How to deal with non-projective in $\mathcal{O}_0$?

Take a prof resolution, ad then apply $\mathcal{V}$.

Get a couple of $\text{SMod} \rightarrow \text{HAs}$ to complex of $\text{SBri}$.

Which category of $\text{SBri}$ came from $\mathcal{O}_0$? Thursday!