Much talk of malachite by Bobo. Now we return to viewing them as vector spaces/R-modules.

Interested in elements. First control 2 bases for BS(\omega). How like $G_0$.

Recall: $G_5$ has basis $C_5 = \{ \emptyset \} = C_3$, $C_5 = 1$.

With $f_{G_5} = g_{G_5} + f_{G_3}$

Also, note that $C_5 = (\emptyset)(C_3)$

So chosen $\omega$, BS(\omega) has basis $\{ C_{\omega} \}$ for each $\omega$.

Claim: $C_5 = C_{\omega} \oplus C_{\omega} \oplus \cdots \oplus C_{\omega}$. By $G_0$.

Obvious Corollary: Every elt of BS(\omega) is $\psi(C_{\text{base}})$ for some $\psi \in \text{End}(\text{BS}(\omega))$.

But we have a basis for $\text{End}(\text{BS}(\omega))$, double letter. What is $\Lambda(C_{\text{base}})$?

Claim: $\Lambda(C_{\text{base}}) = 0$ if $e$ has any clause.

Pf: See diagram.

If all $\omega$ we have expression for each $\omega$, called the canonical exp.

So

Claim: $\Lambda(C_{\text{base}}) = 0$ if $\omega \not\in \text{Con}_x$.

Pf: See diagram.

Not canonical because LLL are not canonical.

Adapted well for certain things.

Key conclusion $\Lambda(C_{\text{base}})$ are all linearly independent.
\textbf{BS}(w) = R_{0} \cdot \cdots \cdot R_{d}(R(\omega)) \quad \text{and} \quad \text{for } d \in \mathbb{N}, \quad R_{d} = \underbrace{R \cdots R}_{d \text{ times}} + \underbrace{0 \cdots 0}_{d \text{ times}} + \underbrace{\cdots + 0}_{d \text{ times}} \quad \text{so } \text{BS}(w) \text{ is a shifted ring (w is in degree -d)}$

\text{Ex: } \quad \text{BS}(01) \quad C_{0}^{0}C_{0}^{0} = \frac{1}{\psi_{2}^{0}}(\psi_{2}^{0}(C_{0}^{0}(t_{a}))) = \frac{1}{\psi_{2}^{0}}(\psi_{2}^{0}(C_{0}^{0}(t_{a}))) = \frac{1}{\psi_{2}^{0}}(\psi_{2}^{0}(t_{a})) = \frac{1}{\psi_{2}^{0}}(c_{0}^{0}) + \frac{1}{\psi_{2}^{0}}(C_{0}^{0}(t_{a}))$

$= C_{0}^{0}(t_{a}) + C_{1}^{0}a_{t_{a}}$

\text{Why? Have nice commutative subgroup } \{0, 1, 2, 3\} \subset \text{End } (\text{BS}(w))

\text{for which } \psi(C_{0}^{0}) \cdot \psi(C_{1}^{0}) = \psi(C_{1}^{0}) \cdot \psi(C_{0}^{0})$

\text{Things of the form } \left( \begin{array}{c} 0 \\ 1 \\ 1 \\ 1 \end{array} \right) \text{ are inside the subgroup } \Rightarrow C_{2} \text{ basis is nicely adapted to multiplication.}$

\text{Warning: } \quad \psi(C_{0}^{0}) \cdot \psi(C_{1}^{0}) \neq \psi(C_{1}^{0}) \cdot \psi(C_{0}^{0}) \text{ in general, is harder with}$

\text{Finding what } C_{0}B \text{ is hard!}$

\textbf{Def: Global intersection form } \quad \text{C}_{\text{top}} = C_{1111} = (\psi - I)(C_{0}^{0}) = \Gamma_{4} \text{ BS}(w)$

$<, > : \text{BS}(w) \times \text{BS}(w) \rightarrow \mathbb{R}$

\text{Easy "upper-triangular" argument } \Rightarrow \text{BS is non-deg.}$

\begin{array}{c|c|c|c|c}
 \text{BS}(w) & C_{0}^{0} & C_{0}^{1} & C_{0}^{2} & \text{C}_{\text{top}} \\
 1 & 1 & 0 & 0 & 1 \\
 1 & 0 & 1 & 0 & 0 \\
 1 & 0 & 0 & 0 & 1 \\
 1 & 0 & 0 & 1 & \xi \\
 1 & 0 & 0 & \alpha_{t_{a}} & \alpha_{t_{a}} \\
 1 & 1 & 1 & 1 & 1 \\
 \text{C}_{\text{top}} & 1 & \alpha_{t_{a}} & \alpha_{t_{a}} & \alpha_{t_{a}}
\end{array}$

\text{What kind of thing is } <, >?
The pairing $\langle $, $\rangle$ is

1. graded
2. invariant

So $\langle af, b \rangle = \langle a, gb \rangle = \langle a, gb \rangle$ for $R$.


non-deg. to degree 0

Abstraction: How right $R$-mod $(B, R) = DB$. It's an $R$-bidual $D(B) = B(-1)$

$Hom_{R-mod}(B, DB) = \text{Space of invariant graded form on } B$.

Let $\lambda \mapsto \text{non-deg. to degree 0}$

Thus BS(\omega) has non-deg. form $\implies$ it is self-dual.

Easy argument: $B \in BS(\omega) \implies \{ DB \in BS(\omega) \text{ s.t. } B \cong DB \} \implies \text{Hom} = \{ \text{non-deg. forms} \}$

Meanwhile, $B \in BS(\omega)$ inherits GIF as restriction. Exercise! Check Grp $E(B)$ is so

$\langle , \rangle_{|B}$ is non-zero.

Cor: If $SC_{\text{Conf}}$ is true, then $End^0(B) = R$ so any non-zero $\langle , \rangle$ is non-degenerate.

Expect $\langle , \rangle_{|B}$ to be non-degenerate. This is the GIF on $B$.

But! Non-deg. to degree 0 $\implies$ descends to non-deg. form on

$B \times R \rightarrow R = R$. GIF on $B$.