Interested in: Showing LIFs are nondegenerate. 

Sudden switch to: studying GIT on \( \overline{B} \). How do they relate?

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Analogous to: learned from: In geometry, Decomposition Theorem: (a statement about how things decompose into direct sums) \( \Lefschetz \) and GIT are dual (and analogous) in nature. 
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Let \( H \) be a field. Define \( L \) as a \( \mathcal{L} \)-valued hom fom: \( \langle \alpha, \beta \rangle = \alpha \beta \). 
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\textbf{Definition:} \( L : H \rightarrow H(\mathbb{R}) \) is a \textbf{Lefschetz operator} if \( \forall i \in \mathbb{Z}, \forall \alpha \in H^i(X), \langle L^i \alpha, \lambda \rangle = \langle \alpha, L^i \lambda \rangle \)
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Example: \( H = \mathbb{R} \), \( \langle \alpha, \beta \rangle = \alpha \beta \) is \( \mathbb{R} \)-valued hom fom.

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\textbf{Example:} \( X \) a smooth variety, \( \langle \alpha, \beta \rangle = \text{Tr}(\alpha \beta) = \sum_{x \in X} \alpha \beta \) \( \in \text{Hom}(H^i(X), H^j(X)) \).
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\text{Analogy is no accident. When} \text{Weyl (Cohomology),} \text{} \mathbb{R} \text{\-valued hom fom:} \text{\} = H^* (\mathbb{R}) \text{\-valued hom fom.}
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\text{As usual, if not, then there is no cohomology.}
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\textbf{Definition:} \( L \) induces a form on each \( H^i, i \geq 0 \), called the \textit{Lefschetz form} \( \langle v, w \rangle_L = \langle v, L^i w \rangle \).
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\textbf{Example:} \( L = 0 \).
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\textbf{Definition:} \( L \) satisfies \textbf{hard Lefschetz} (HL) if \( \forall i \geq 0, L : H^i \rightarrow H^{i+1} \) is injective.
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\Rightarrow \text{isom} \Rightarrow (\_)^{\ast} \text{ is non-degenerate}.
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\text{(vacuous for} \( i = 0 \))
Exercise: $L_f$ on $\mathbb{R}^3$ never has (HL). When do $L_f + M_g$ have HL?

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Claim: $L$ a cont. family of operators $\omega(hL) \rightarrow f$ so $L_0$ has HR, then $L_1$ has HR. Lecture 3.3

**Hint:**
Signature constant in family of moduli forms.

Fix $\mu$, $\omega(h\mu) > 0$ vs $\omega$. You may need to extend $\mu$ a bit to ensure one exists.

Draw an analogy $\mathcal{W}$: $\mathcal{O} \mathcal{B} \mathcal{U} + \text{geometry}$, try expand better tomorrow.

**Geometry** (only when $\mathcal{W}$ is Wayl/syst.)

$H^\ast(\mathcal{B} \mathcal{S}(\mathcal{U})) \rightarrow \mathcal{B} \mathcal{S}(\mathcal{U}) \rightarrow \mathcal{G} \mathcal{B} = F\ell$

$\mathcal{B} \mathcal{S}(\mathcal{U})$ a specific ample bundle

$\mathcal{L}$ relatively ample form

(Not ample.)

$\mathcal{B} \mathcal{S}(\mathcal{U}) \rightarrow \mathcal{P} \mathcal{N}$

$\mathcal{L} \rightarrow \mathcal{O}(1)$

Then: (hard leftshakes thin) $\chi$ smp proj $\mathcal{O}$ alg why, $\mathcal{L}$ ample

Then $H^\ast(X) \otimes \mathcal{O} \mathcal{L}$ was $\omega L_1 \mathcal{H} \mathcal{R}$.

Then: (Improved $h\mathcal{L}$) $X$ not nes. smooth $\text{IH}^\ast(X)$ intersection cohomology

Then $\text{IH}^\ast(X), \mathcal{O}(1)$ was $\omega L_1 \mathcal{H} \mathcal{R}$ when $\mathcal{L}$ ample.

Invented for this purpose, to fix PD etc. when $X$ not smooth.

$\mathcal{B} \mathcal{U}$

Expect $(\mathcal{B} \mathcal{U}, L_\mathcal{B})$ to have $\omega L_1 \mathcal{H} \mathcal{R}$. Also, direct sum

$(\mathcal{B} \mathcal{U} \mathcal{B} \mathcal{B} \mathcal{U}, L_\mathcal{B})$ since $\mathcal{B} \mathcal{U} \mathcal{B} \mathcal{B} \mathcal{U} \cong \mathcal{B} \mathcal{U} \otimes \mathcal{B} \mathcal{U}$, $m \geq 3$

but not $\mathcal{B} \mathcal{B} \mathcal{U} \cong \mathcal{B}(0) \oplus \mathcal{B}(1)$, shifted $h\mathcal{L}$ does not have $h\mathcal{L}$.

$\mathcal{L}$ has $h\mathcal{L}$ for $L_\mathcal{B} + \mathcal{M}_\mathcal{B}$, but that doesn't repeat $\otimes$ decompose.
Theorem: Space $X \xrightarrow{\pi} Y$ is proper + seminormal. The $H^s(X)$ has HL for $L$ only relatively ample, i.e. $L = \pi^*L'$ ample.

BS(w) seminormal, i.e.
no LL of negative degree.

This is core, but same principle should apply
to other seminormal things.

Expect $(\overline{\text{Bun}_S, L^s})$ has HL, HR.

Why is all this useful? We've seen: LITF on $Hom(B_S, \overline{\text{Bun}_S})$ nondegenerate.
Embedding Theorem (next time): LITF $\hookrightarrow$ GIT on $\overline{\text{Bun}_S}$, living inside primitives in degree $-L(y)$.

Restriction of GIT to primitive subspace is $\pm$ definite $\Rightarrow$ nondegenerate!