Outline/Recap

Let $S(w) := \{ B_w \} = H_w$. We've seen $S(w) \Rightarrow \exists ! \text{ nonzero root form on } B_w = B_{\text{reg}}$ up to scalar, it is unique.

\[
\langle \cdot, \cdot \rangle_{B_w} \text{ is nonzero, so } H \text{ is indivisible.}
\]

For state $\omega \\
S(w)$
\[\begin{cases}
hr(\omega) := B_w \text{ equipped with } \langle \cdot, \cdot \rangle_{B_w} \text{ has } hr
\text{ has } \text{hr}
\wedge
\text{true for all } \omega \in W. \quad S(w) \Rightarrow \langle hr(\omega), hr(\omega) \rangle
\end{cases}
\]

\[
\text{Rank 1 } \langle \cdot, \cdot \rangle_{B_w} \text{ is already normalized s.t. } (c_{\text{top}}, c_{\text{bot}})^{-\langle \cdot, \cdot \rangle_{B_w}} > 0 \text{ via exercises.}
\]

Really, we want to prove $S(w)$, and our induction step will be to assume $S(<w,s>)$ and prove $S(w,s)$.

\[
\begin{align*}
hr(w,s) := B_{w,s} & \subset B_{w,s} \text{ has } hr \\
hr(w,s) &
\end{align*}
\]

Now $H_{ws} = H_{w,s} + \mathbb{R} \mu(w,s)$, by

\[
\mu(w,s) = \text{dim } \text{Hom}^0(B_y, B_{w,s}) = \text{dim } \text{Hom}^0(B_{w,s}, B_y)
\]

and no negative degree maps.

\[
The \text{ LI pairing } \quad \text{Hom}^0(B_y, B_{w,s}) \times \text{Hom}^0(B_{w,s}, B_y) \rightarrow \text{End}^0(B_y) = \mathbb{R}
\]

has rank $= \# \text{ of submods } B_y \text{ inside } B_{w,s}$. So LI pairing $\Rightarrow S(w,s)$.

We're been identifying the two sides (by flipping diagrams upside down), let's do abstractly.

Both $B_y, B_{w,s}$ have nonzero forms $\langle \cdot, \cdot \rangle_{B_y}$, $\langle \cdot, \cdot \rangle_{B_{w,s}}$ and indeed form from exercises.

For $\psi, \text{Hom}(B_y, B_{w,s})$ define $\psi' : B_{w,s} \rightarrow B_y$ via $\langle \psi(b), b \rangle_{B_{w,s}} = \langle b, \psi'(b) \rangle_{B_y}$.

This transfers the LI pairing to a LI form on $\text{Hom}^0(B_y, B_{w,s})$

\[\langle \psi, \psi' \rangle_{B_{w,s}} = \text{coeff of 1 in } \psi \ast \psi'.
\]

Embedding Thm: $\text{Hom}(B_y, B_{w,s}) \rightarrow \text{End}^0(B_{w,s})$ has image inside $\text{P}(B_{w,s})$ is injective, is isometry up to pos scalar.

$\text{Proof:}$ $A \in \text{P}(B_{w,s})$ for degree reasons. Thus $A \rightarrow \left( c_{\text{bot}} = 0 \right)$, image is in $\text{P}(B_y)$.
Injectivity comes from the fact that $\Gamma(C_{\text{bot}})$ was a basic.

unraveling this is an exercise.

Now, $\langle C_{\text{bot}}, C_{\text{top}} \rangle _{\text{by}} = 1 \quad < C_{\text{bot}}, P_C(B_{\text{bot}}) > = N > 0$. Thus

$$(\psi, \chi)_{\text{by}} = \text{off of } 1 \in \chi_{\text{by}} = < \psi, P_C(B_{\text{bot}}), C_{\text{top}} > = \frac{1}{N} < \psi, P_C(B_{\text{bot}}), C_{\text{bot}} >$$

$$= \frac{1}{N} < \psi(C_{\text{bot}}), \psi(P_C(B_{\text{bot}}) C_{\text{bot}}) > = \frac{1}{N} (\psi(C_{\text{bot}}), \psi(C_{\text{bot}})) \chi_{\text{by}}$$

Con: $\text{HR}(w, s) \Rightarrow \text{LIF}$ is non-deg (by atoms) $\Rightarrow s(w, s)$.

Also, $B_{w, s} \in B_{w, s}$. preserved by $L_y$, restriction of $\text{HR}$ to $L_y$-inv subspaces has $\text{HR}$ (so long as restriction of $\langle \rangle$ is non-degenerate) $\Rightarrow hL(w, s), \text{HR}(w, s)$.

To finish the induction, use $S(w, s), hL(w, s), \text{HR}(w, s)$ to prove $\text{HR}(w, s)$.

Showing $\text{HR}$ directly is hard. We use a limiting argument.

Let $L_5 \subset C_{B_{w, s}}$ dense.

For $\text{HL}(w, s)$, same but for $L_5$. $L_0 = \text{our previous operator}$.

$\text{Lk}$: $L_5$ will NOT commute with all basis maps, nor with $L_0$, so will not restrict to $B_{w, s}$ on any of our summand.

Rank: Can even define them when $w < w$!!

In fact, an easy exercise (very $B_{w, s}$ example or prototype) shows $\text{HL}(w) \equiv \text{HL}(w, s)$ when $w < w$ and $\mathbb{Z}^{+0}$.

Limit Thm: $\text{HR}(w) \Rightarrow \text{HR}(w, s)$ for $s > 0$ (either $w < w$ or $w < w$).

PF: $L_5 = \beta + \mathbb{Z} M^k$ where $M$ is middle mult. $L_0$ has bounded expansion.

Bit: $M^2 = 0$ on $B_{w, s}$

$$M = M + M^k$$

if $f$ begian then linear

then $k$, if are at least linear.

$\Rightarrow L_5 = \beta + \mathbb{Z} k + M$
In exercises, get a basis for $\overline{B_{w}B_{w}^{-k}}$ in terms of $\overline{B_{w}^{-k+1}}$ and $\overline{B_{w}^{-k+1}}$.

\[ \text{let } \alpha_{i}^{x} \in B_{w}^{-k+1}\text{ project to an ONB of } \overline{B_{w}^{-k+1}} \]

so $\overline{B_{w}^{-k+1}}$ is an ONB for $B_{w}^{-k+1}$.

\[ \text{get basis } \left\{ \alpha_{i}^{x}, \beta_{i} \right\} \text{ for } B_{w}^{-k+1} \]

\[ \text{Exerci: Compute } \left\langle v_{j}, \beta^{k-1}Mw \right\rangle \text{ for this basis.} \]

It has signature equal to signature of $\left( , \right)^{-k+1}$ on $\overline{B_{w}^{-k+1}}$ (only $\beta_{i}$ matters, $\alpha_{i}$ not $\beta_{i}$).

Exerci: Correct signature on $HR$ on $\overline{B_{w}B_{w}^{-k}}$.

So, if we can show $hL(\omega_{s}) \geq 0$ for all $\geq 0$ (indeed 0) then continuity gives us $HR(\omega_{s})$, and we win. How to get $hL(\omega_{s}) \geq 0$?

We've been roughly following dC+NL's geometric proof:

A is relatively ample, $X + SM$ is truly ample. (rel. is on boundary of ample cone)

How to show $HL$ in geometry? Have the weak lifchetzes theorem.

\[ X \text{ is ample section of } \Gamma(L) \text{, zeros form a hyperplane } Y \subseteq X, \text{ dim } Y = \text{dim } X - 1 \]

\[ H^{*}(Y) \cong H^{*}(X) \text{ and } H^{*}(X) \cong H^{*}(Y) \]

after recentering, $i^{*}$ and $i^{\flat}$ have degree $t$. Cones of respective lifchetze operators.

Weak Lifchetze Theorem:

\[ i^{*} \text{ is injective from negative degree} \]

\[ i^{\flat} \text{ to degree zero} \]

\[ i^{*} \text{ is surj to pos degree} \text{ from pos degree.} \]

And they are adjoint.

\[ \Rightarrow \left\langle v_{j}, w \right\rangle = \left\langle i^{*}v_{j}, w \right\rangle \]
Prop: Suppose \((V_{ij}, W_{ij})\) are Liebich's spaces.

\[ \sigma: (V_{ij} \rightarrow W_{ij}) \text{ satisfies } \langle v_{ij}, w_{ij} \rangle = \langle \sigma(v_{ij}), \sigma(w_{ij}) \rangle \text{ for all } v_{ij}, w_{ij} \cap L_{ij} = 0 \]

Then \(H_{R} \text{ for } W_{ij} \Rightarrow H_{L} \text{ for } V_{ij} \).

Claim: \(\sigma\) is injective from neg degree.

Proof: Suppose \(v_{ij} \in P_{k}^{c} V_{ij}\), and \(k > 0\) (\(k = 0\) trivial). Then \(v_{ij} \in 0\).

1. \(v_{ij} \in \text{ primitive} \Rightarrow L_{k}^{*} v_{ij} = 0 \Rightarrow \sigma L_{k} v_{ij} = 0 \Rightarrow L_{k} v_{ij} = 0\).
2. \(v_{ij} \in \text{ primitive} \Rightarrow \sigma(L_{k} v_{ij}) = \langle v_{ij}, L_{k}^{-1} \rangle_{L_{1}} \in \langle \gamma_{ij}, L_{k}^{-1} \rangle_{V_{ij}} \Rightarrow L_{k} v_{ij} = 0\).

So weak Liebich \(+ HR\) induction \(\Rightarrow\) hard Liebich, \(+\) LHS \(\Rightarrow HR\).

Unfortunately, we have no algebraic notion of a generic hyperplane section (aka no combination for \(\delta_{ij}(y)\)). We need to find some other map \(\sigma^{*}\).

Factor \(L_{ij}\) as \(\sigma^{0} \cdots \).

Final step in setup: \(L_{ij} \text{ on } BS(\omega_{3})\) is

\[ \sum_{i=1}^{n} \gamma_{i} B_{i} \]  

break the \(i\) th strand.

Claim (Exercise): \(\gamma_{i} > 0\) for all \(i\) (when \(\sum_{i} \gamma_{i} > 0\)).

So consider

\[ BS(\omega_{3}) \text{ for square roots} \]

Then \(\sigma^{*} \sigma = L_{ij} \in \{ v_{ij} \}_{\ell} \Rightarrow < v_{ij}, L_{ij} w_{ij} > = \sum_{i} \gamma_{i} < v_{ij}, L_{ij} w_{ij} > \).

So want to show: \(\sigma\) is injective from neg degree. \(\text{RHS has } HR\).

But! \(\sigma\) is the first differential in the regular complex.

Unfortunately, \(\sigma\) is very false! Not summable!

But! We only care about \(B_{ij} B_{i} \subset BS(\omega_{3})\). So why not restrict to a minimal complex?

Diagonal miracle gets rid of all the trash \(\delta_{ij}\). To be continued.