BSBm was built by taking the Frob Ext $R^e \circ R$ and making a Frobenius $R \circ R(1)$ over $R$ bilinear.

Why not look at the Frob Ext itself?

**Def:** (Type A) $SSBm \subset Bim$, a 2-act.

$\phi : \otimes \mapsto \otimes R, R^e$

1-mor: Generated by the simple $R^e$

2-mor: bilinear maps.

$1 \otimes 0$

$\psi : \phi \circ \otimes \mapsto \otimes R, R^e$

Relation: Idempotent

$[\text{Eval} + \text{Coneal}]$

Decomposition

$R^e = R \oplus R(2)$

Thin (E-W): These give a presentation of $SSBm$. (a la pesuar theories)

In fact, this gives a presentation of the analogous 2-act for ANY Frob ext, provided the relation are written in general form: $\phi = \otimes R^e$

For $\otimes R^e$ over the dual basis, etc.

Def: $SSBm = \text{Ker}(SSBm)$; we apply $\text{Ker}$ to each category $\text{Hom}(\phi, \phi)$ $\text{Hom}(\phi, s)$ $\text{Hom}(s, \phi)$ $\text{Hom}(s, s)$.

**Link:** 4 categories here... but most are being $\otimes R^e$.

What is this category? Some algebraic category with 2 objects

End $(\phi) = [\text{End}(\phi)] = \mathbb{H}$

End $(s) = [\text{End}(s)] = \mathbb{H}$.
Best way to view this is as ideals inside $H_w$:

$$\text{Hom}(\phi,\psi) = H \xleftarrow{\psi H_u} H_{w_1} \xrightarrow{\phi H_u} H_w$$

$$\text{Hom}(s, s) = H_s H_w \cap H_{w_1}$$

$$[\phi] = H_s$$

Composition:

$$a * b = ab$$

$$a * b = \frac{ab}{v^{1+1/2}}$$

Check: This makes sense. Called the Harker algebroid.

52 | Second Part

Def: The Harker Algebroid $H$ has $Q = I_E S$.

Let $\ell(I) = l(w)$ and $H_{\ell} = H_{w_{\ell}}$ and $[E] = \text{Poncet poly of } W_1$

$$H_{\ell} = [E] H_{\ell}$$

$$\text{Hom}(I,J) = H_{\ell} H_w \cap H_{w_1}$$

Composition:

$$a * b = \frac{ab}{[I,J]}$$

Def: $SSBi$ has $Q: I_E S \to R^I$ because it gives a Frobenius.

1st Ver: Generated by $\text{Ind}_{I_{s^*}} \to R^{I_{s^*}}$ $\text{Res}_{I_{s^*}}$ $I - I_{s^*} = I_{J^*}$

(remark: don't need $\text{Ind}_{I_{J^*}}$ for $I_{J^*}$ since $J = I_{s^*}$, $\text{Ind}_{I_{J^*}} = I_{I_{s^*}}$ $\text{Ind}_{I_{s^*}}$)

(remark: $\text{Ind}_{I_{J^*}}$ represents the factor, depending on the specific labeling.)

Def: $SSBi = \ker(S\triangleright SBi)$

Thus $[SSBi] = \phi H$

Diagonals $R^{I_{s^*}} \to R^I$ a Frobenius, so have

Also, $\text{Ind}_{I_{s^*}} \circ \text{Ind}_{I_{s^*}} = \text{Ind}_{I_{s^*}} \text{Ind}_{I_{s^*}}$ $\text{Ind}_{I_{s^*}}$ $\text{Ind}_{I_{s^*}}$

Let $\text{Ind}_{I_{s^*}} \circ \text{Ind}_{I_{s^*}}$ $\text{Ind}_{I_{s^*}}$ for $R^I$

Then simply the generic Frobenius relation as above.
Restate: In the circled arc two separate hunters, as seen by region labels. But nice notations makes it look like a crossing of 1-manifolds!!

Claim: If cyclic arc the previously given caps+caps.

Theorem (E-W) These are all the generators. I.e., any binormal map by Isotopy!

Heated Ind, Res can be drawn as colored 1-manifolds of bigles in a planar box.

**Proof:** Not written up. The susceptibility part was always easy though — the localization arguments.

**Relations:** 1. Generic relations for "compatible" space of Frob ext.

\[ \begin{array}{c}
\text{BA} \\
\text{CD} \\
\text{BC} \\
\end{array} =
\begin{array}{c}
\text{BA} \\
\text{CD} \\
\text{BC} \\
\end{array} \]

I.e. for deleted maps \( R \).

\[ \begin{array}{c}
\text{BA} \\
\text{CD} \\
\text{BC} \\
\end{array} =
\begin{array}{c}
\text{BA} \\
\text{CD} \\
\text{BC} \\
\end{array} \]

for \( f \) in \( \text{Mod}_{\text{A}} \).

+ a couple more.

2. Generic relations for a Yale of Frob ext.

3. Very specific S15m Relations, only known in deleted type, and almost proven in types A, X. REALLY NASTY. But you'll see why...

Talking Points:

1. Space \( W \) finite, so \( \text{I} \in \text{S} \) is finite. \( \text{H} \) has a maximal object. \( \text{H} \) is 1-dimensional.

   Similarly, \( \text{Hom}(\phi, S) \) is just the category of free graded \( R \)-modules. Yawn. But does it sound you of something?
Yes, of course... if \( SB_{\phi} = \text{Hom}(\phi, \iota) \iff \text{Proj} O \).

then \( \text{Hom}(\phi, S) \iff \text{Proj} O \) (most single block)

and more generally, \( \text{Hom}(\phi, I) \iff \text{Proj} O \) \( I \) is a block for some reason.

More specifically, these categories have a right limit (they are right, really, it's the same)

and \( \text{Hom}(\phi, I) \leftrightsquigarrow \text{Proj} O \).

Meanwhile, what is \( \text{Hom}(I, \iota) \)? It is "dual" in some sense. It is the dual category of \( \text{End}(\iota) \).

\( \text{Hom}(I, \iota) \) = projective singular \( O \).

\( \text{Hom}(I, \iota) \) acts locally finitely.

Need to take \( \text{Ind}(\iota, \iota) \) to each other.

At least I didn't cut it off at all.

2) So diagrammatics seems harder. But it is also easier, in some ways.

(a) Expressing hom in \( SB_{\phi} \) on planar graphs makes you want to use planar graph theory -

like an Euler characteristic argument - but it's hard and it isn't easy.

Expressing hom in \( SB_{\phi} \) as planar \( \chi \)-fields lets you use topology - Morse theory.

Euler characteristics - and they work much better!!

(b) Singular diagrams are more "local."

Ex 1: \( \chi = 1 \) \( \text{End}(\phi) \)

\( \chi = 1 \) needed a relation

\( \chi = 1 \)

\( \chi = 1 \)

This is a vast improvement over the 2-color + 3-color relations.

Ex 1: \( m_{\iota} = 3 \)

\[ \chi = \chi_{\iota} \]

implies both

\[ \chi_{\iota} = \chi_{\iota} \]

leads to only formula & 6-valent vertex in bubbles... once you know what \( \chi \) is.