GRAND INDUCTION (+ other lemmas) Fix $x, z$ with $z < x$.

1. $S(z) \Rightarrow F_x$ has diagonal miracle. $\text{DM}(x)$.

2. $\text{DM}(x) \Rightarrow \text{RoHR}(x)$
   
3. For all $\, y < x$, $\, \text{HL}(y, z_t) = \text{HL}(x, z_t), \quad z \geq 0$

4. We prove $\text{HR}(y, z_t) \leq \text{HL}(y, z_t), \quad z \geq 0$ gives $\text{HR}(y, z_t) \leq \text{HL}(x), \quad z \geq 0$.

5. Setting $z = 0$ gives $\text{HR}(x)$. The embedding theorem then shows the
   nondegeneracy of all LIF on $B_x B_y$.

6. Finally, $S(x) + \text{HR}(z) \Rightarrow \text{HL}(x)$. The loop is complete.

Let's do (2) Prop:

$S(z) \Rightarrow \text{RoHR}(z)$

Proof: $x = y, z \geq 0$, $F_x \otimes F_y F_z$, a homotopy equivalence.

$\text{HL}(z_t) = \text{HL}(x, z_t), \quad z \geq 0$

$\text{RoHR}(z) \Rightarrow \text{RoHR}(x)$
Write \( F_r^j(\cdot) = \Theta B_Z(\cdot) = B^{r} \Theta B^{l} \) when \( B^{r} \Theta B^{l} \) \( Z > Z \) \( B^{r} \Theta B^{l} \) \( Z < Z \).

\[ F_r^j(\cdot) B_S = B^{r} B_S \Theta B^{l} B_S \] is NOT small, can't lower HR... stage suggested we restrict to the "on-shifted" panel part.

\[ B^{r} B_S \Theta B^{l} B_S \]

\[ B^{r} (\Theta B^{l} (\cdot)) \]

small

\( R \) local stuff

has HR by HR(s)

\[ \text{Claim: The map } F_r^j(\cdot) \rightarrow (F_r^j)^{-1}(\cdot) \rightarrow B^{r} B_S \Theta F_r^{-1}(\cdot) \]

\[ \text{L-stable is an } \] split inclusion and an isometry for the Leslieit form.

\[ \text{Pf: This is obvious for the first map about the projection kills stuff...} \]

\[ \begin{align*}
\text{1. By } S(x), \text{ there are no maps } & F_r^j(\cdot) \rightarrow B^{l}(\cdot) \text{ using } S(\cdot), \text{ (negative degree 1)} \\
\text{so that term didn't contribute.} \\
\text{2. Any map } & F_r^j(\cdot) \rightarrow B^{l}(\cdot) \text{ is pos def map in max'd ideal, so killing} \\
& \text{it won't affect the map being a split inclusion.} \\
\text{3. Exercise: } (\cdot) \rightarrow B^{l}(\cdot) = 0 \text{ so killing this term doesn't affect the} \\
& \text{Leslieit form.} \\
\end{align*} \]

"Unspotted" \( \Theta \) \( L \) is left with so commits of all these banal local maps, decamps etc.

As an L-stable sum of HR, \( F_r^j(\cdot) \) has HR.

Now we apply a similar trick to \( F_r^j F_r^j \) in order to prove.

On something of the form \( B B_S \), let \( L_S = A_0 + iA_B \) \( (3^+) \).

Assume ergopy so far. \( F_r^j F_r^j = B_x B_S \Theta \rightarrow B_x(\cdot) \Theta F_r^j B_S \rightarrow \ldots \)

Think of \( \Theta \) as a degree +1 map \( B_x(\cdot) \Theta \rightarrow B_x(\cdot) \Theta F_r^j(\cdot) B_S \).

Write \( F_r^j(\cdot) = B^{r} \Theta B^{l} \) just as before. (or \( L_S \) but \( S_A \) action vanishes, will die in \( B_x \)).
Recall: Lemma: \( \Phi \) is a factorization of \( L \) up to positive renormalization.

I.e., \( L \circ \cbs(x) \) is equal to \( \sum x_i \frac{1}{1 + x_i} \) for \( x_i > 0 \).

Last time we suggested factoring \( L \) as a composition of maps

\[
\sigma : \cbs(\mathbb{S}) \to \bigoplus_i \cbs_i
\]

but alternatively, we consider \( \Phi = \bigoplus_i \left( 1 + x_i \right) \) and renormalize the intersection form on \( \mathbb{S} \).

Restricting to the submanifold \( B \times B \times \cbs(\mathbb{S}) \), the restricted map \( \Phi \) still satisfies this property.

The positive scales \( x_i \) are irrelevant—they don't affect \( HR \), etc. We will ignore.

Moreover, claim: \( \Phi \) is injective from negative degrees.

Proof: \( \ker \Phi = H^p(F(x)) = B_{\mathbb{S}}(-\mathbb{S}) \) does not exist in strictly positive degrees.

Theorem: \( R \circ HR(x) \implies hL(x, s)_5 \) for \( s > x \), \( s > 0 \).

Proof: Case 1: \( s > 0 \). \( B \times B \to B \times B \) by \( \Phi \) (has HR by \( HR(x) \))

Key remark: Can NOT split \( B \times B \) into \( B \times B \) as before. Splitting does NOT commute with \( L \), because middle mult don't commute (see example of \( B \times B \)).

Now the \( W \) sub \( = B \times B \) has \( hL \).

Case 2: \( s = 0 \). \( L \to L \times L \times L \) can split now.

But can't apply the same argument. \( \deg \Phi = +1 \) and \( 0 \circ \Phi \) are not maps to \( B \times B \).

In fact, they must be! \( B \times B \) is the same, but \( F(x) \) is lower terms must contract, and they contract against \( g(-1) \)!!
However, we ignore $B^0(1)$ for two reasons.

- $C_2 \mid B^0(1) = 0$
- $B^0(1)$ must contract against something in hom degree 2 in order for $\text{Fix}$ to have DM. (Less cunning.)

So two possibilities.

(a) The map

\[
\overline{B} \times \overline{B} \xrightarrow{\Phi} (\cdot) \Rightarrow \overline{B}(-1) \text{ is zero.}
\]

Then

\[
\overline{B} \times \overline{B} \Rightarrow \overline{B} \times \overline{B}
\]

satisfy $\langle \nu, L\nu \rangle = \langle \Phi(\nu, \nu) \rangle$

and $\text{tr} \ of \ RHS \Rightarrow \text{hl of LHS}$ by W-L rule.

(b) The map

\[
\overline{B} \times \overline{B} \xrightarrow{\Phi} \overline{B}(-1) \text{ is nonzero. Now we argue analogously to W-L's proof. Fix } \nu \in \overline{B} \times \overline{B}_k,
\]

$\Phi(\nu) \neq 0$, $\Rightarrow$ hl on $\overline{B}$ implies $L^k \Phi(\nu) \neq 0 \Rightarrow L^k \nu \neq 0$, as desired. \[\blacksquare\]