Toric Tropical Vector Bundles

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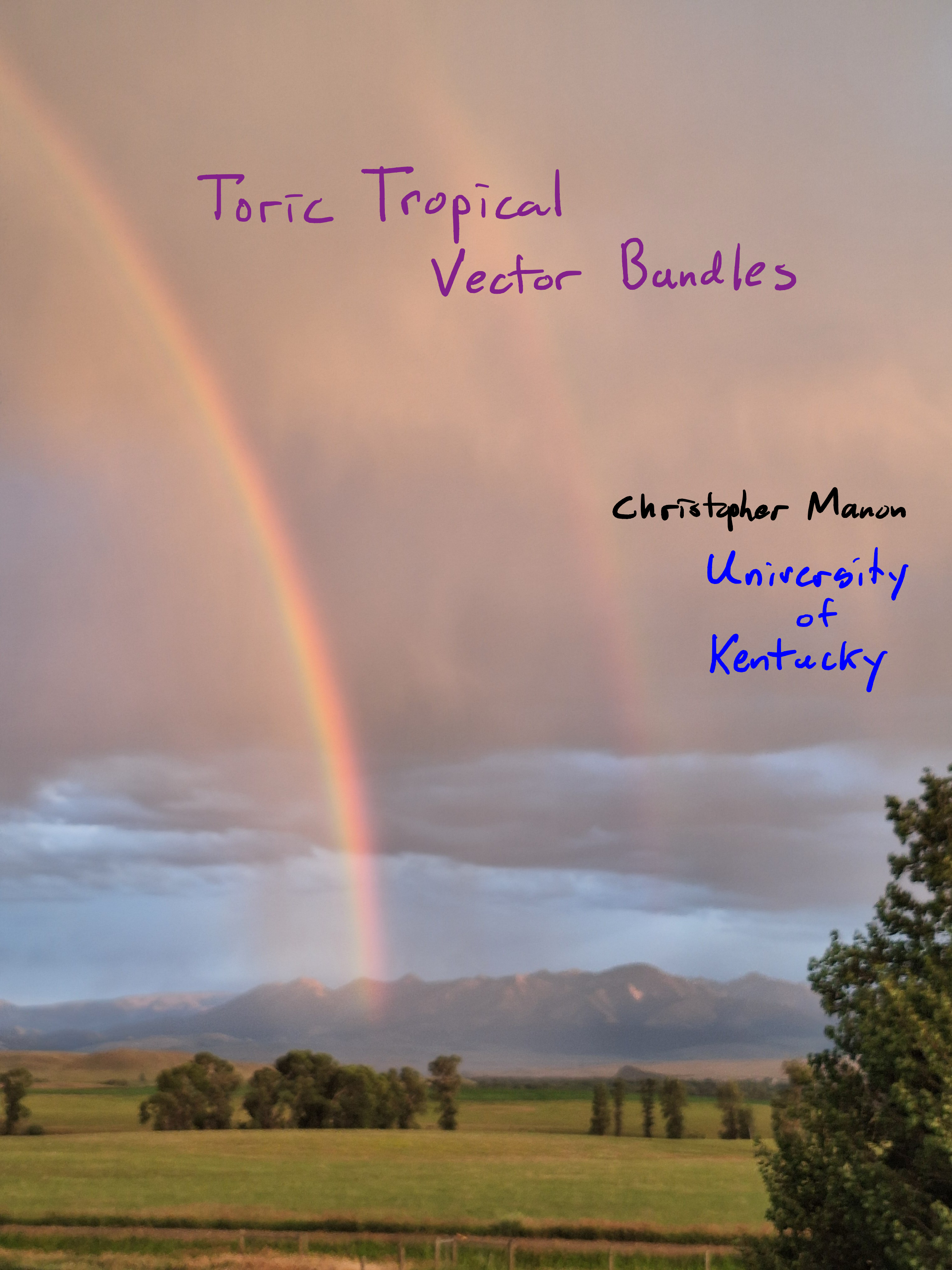
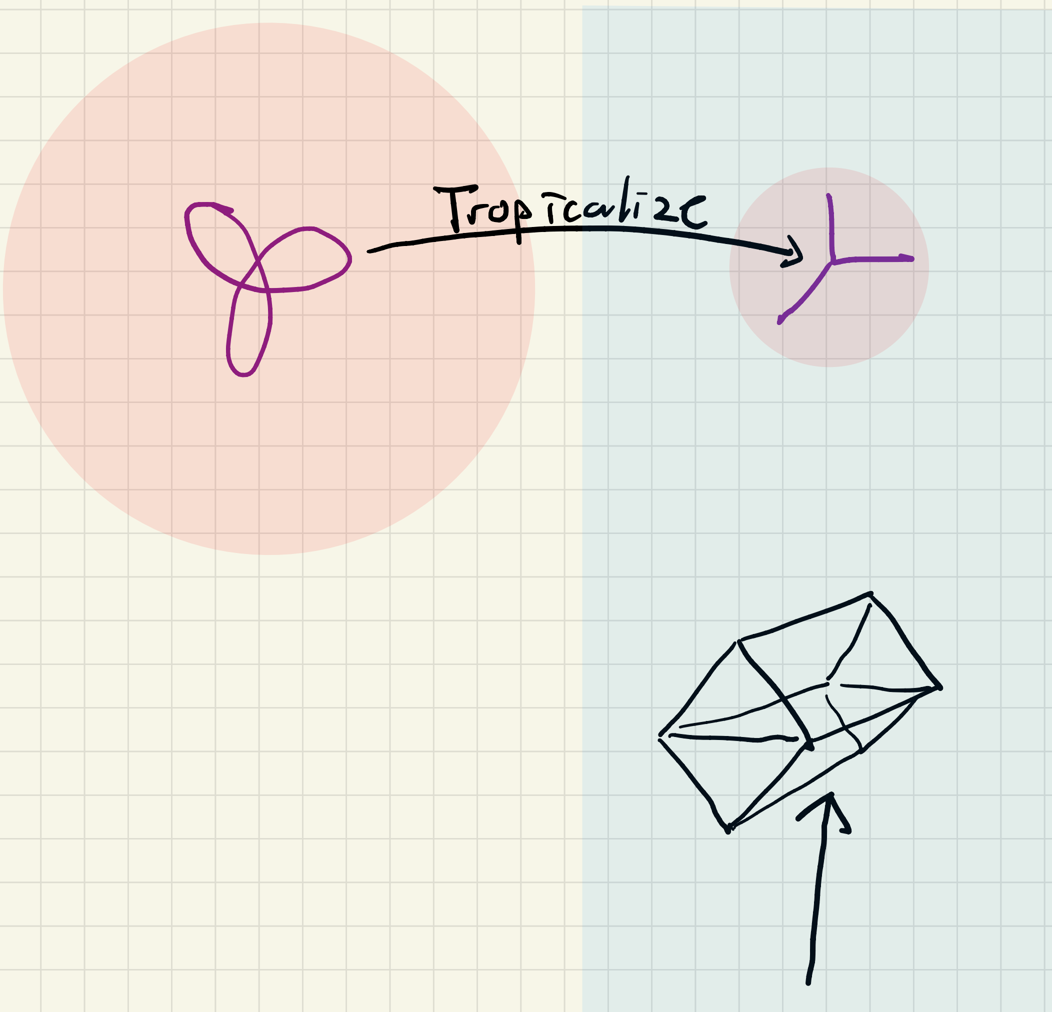


Algebraic

Tropical



"not liftable"

Linear Spaces and Matroids

$L \subseteq k[y_1, \ldots, y_m]$ a linear ideal.
involves all variables

$\mathrm{Trop}(L) \subseteq \mathbb{R}^m$ Tropicalized linear space

- tropical cycle of degree 1

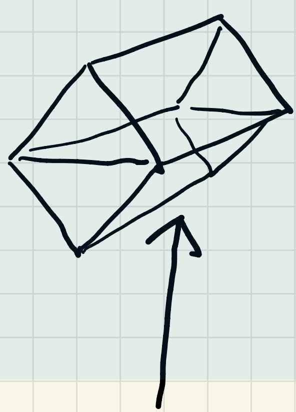
$\mathcal{M}_L$ is the matroid on $[m] = \{1, \ldots, m\}$
where $I \subseteq [m]$ is independent
iff $\{y_i \mid i \in I\}$ is independent
modulo L

(Feichtner - Sturmfels):

$\mathrm{Trop}(L)$ is the support of $\mathrm{Berg}(\mathcal{M}_L)$

Linear space cut out by L $\xrightarrow{\text{Tropicalize}}$ $\mathrm{Berg}(\mathcal{M}_L)$

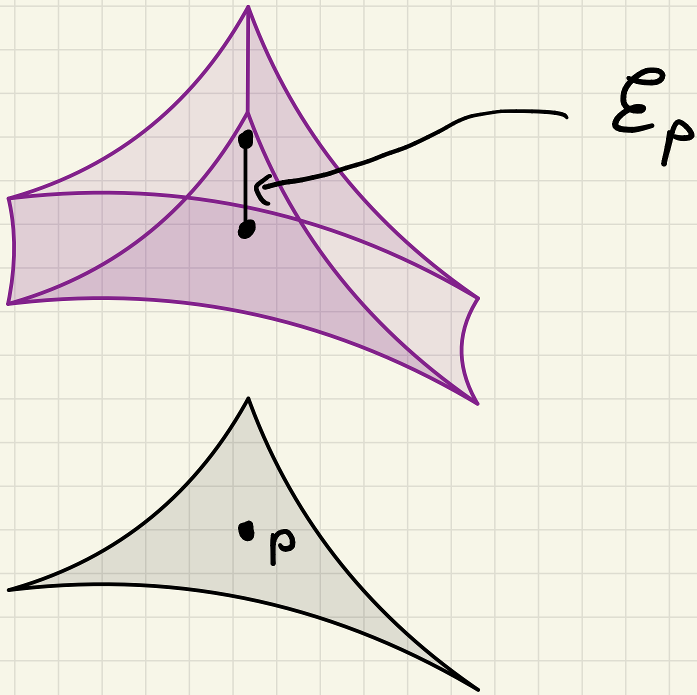
-Eur's talk



"non representable"

Vector Bundles

$$\begin{array}{c} \mathcal{E} \\ \big\downarrow{\scriptstyle \pi} \\ X \end{array}$$



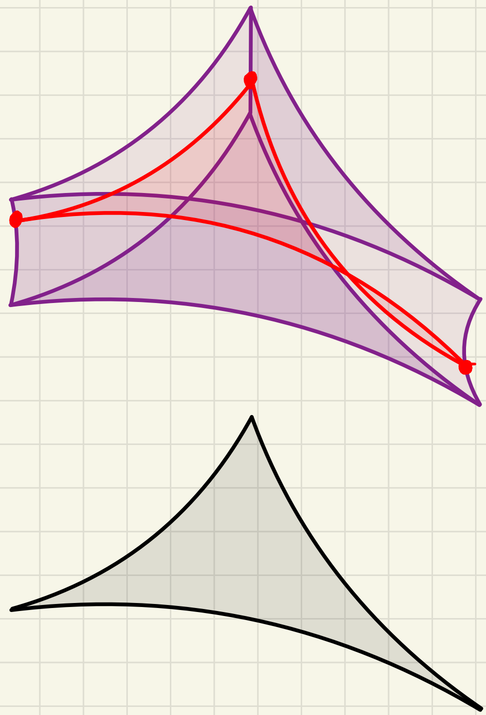
projective
smooth
dimension = d
over $k = \bar{k}$

$$\operatorname{rank}(\mathcal{E}) = \dim_k \mathcal{E}_p$$

Vector Bundles

Global Sections:

$$\begin{tikzcd} \mathcal{E} \arrow[d, "\pi"] \\ X \arrow[u, bend left, "s"] \end{tikzcd}$$



$$\pi s = \mathrm{id}_X$$

Space of Global Sections $\qquad H^0(X, \mathcal{E})$

Vector Bundles

Euler Characteristic:

$$\chi(\mathcal{E}) = \sum (-1)^i h^i(X, \mathcal{E})$$

$$h^i(X, \mathcal{E}) = \dim_k H^i(X, \mathcal{E})$$

Vector Bundles

Let $\mathcal{L} \in \mathrm{Pic}(X)$ be ample

$N \to \chi(\mathcal{E} \otimes \mathcal{L}^N)$ is computed by the Hilbert Polynomial $p_{\mathcal{E},\mathcal{L}}(N)$

for $N >> 0$

$$\chi(\mathcal{E} \otimes \mathcal{L}^N) = h^0(X, \mathcal{E} \otimes \mathcal{L}^N)$$

Vector Bundles

Chern Classes

$A^*(X)$ - chow cohomology ring

$c_i \mathcal{E} \in A^i(X)$

Hirzebruch-Riemann-Roch:

$$\chi(\mathcal{E}) = \int_X ch(\mathcal{E}) Td(X)$$

$Td(X)$ - Todd Class

$$ch(\mathcal{E}) = 1 + c_1\mathcal{E} + \frac{c_2\mathcal{E}^2}{2!} + \cdots$$

Vector Bundles (Positivity)

$\mathcal{E}$ is

① globally generated if

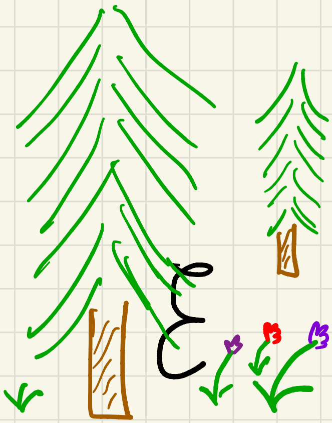
$$ev_p : H^0(X, \mathcal{E}) \longrightarrow \mathcal{E}_p$$

is surjective for all $p \in X$.

② ample (nef) if

$\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ is ample (nef) on $\mathbb{P}\mathcal{E}$

Questions for the vector bundle who you met in the woods.



What is your rank?

What is your space of global sections?

What is your Euler characteristic?

What are your Chern classes?

Are you globally generated? ample? nef?

Toric Vector Bundles

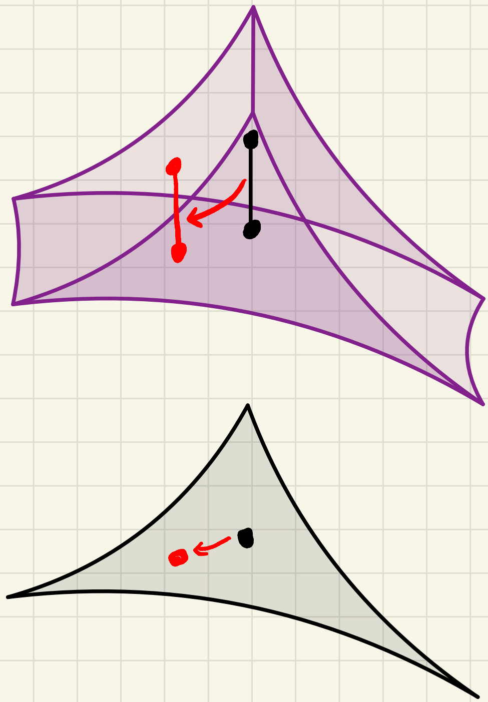
T - algebraic torus

$N = \mathrm{Hom}(\mathbb{G}_m, T)$
$M = \mathrm{Hom}(T, \mathbb{G}_m)$
$\Sigma \subseteq N \otimes \mathbb{R}$ polyhedral fan
$n = |\Sigma(1)|$

X_Σ - toric variety / k

Toric Vector Bundles

$$T \curvearrowright \begin{array}{c} \mathcal{E} \\ \downarrow{\scriptstyle \pi} \\ X_\Sigma \end{array}$$



Line Bundles

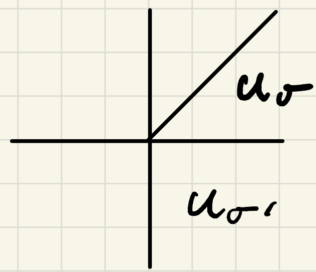
For $\sigma \in \Sigma$ $\quad U_\sigma \subseteq X_\Sigma$

$\mathcal{L}|_{U_\sigma} \cong U_\sigma \times \mathbb{C}$

$T \curvearrowright U_\sigma \times \mathbb{C} \longleftrightarrow u_\sigma \in M$

If $\sigma \cap \sigma' = \tau$

$u_\sigma|_\tau = u_{\sigma'}|_\tau$



$$T \curvearrowright \begin{array}{c} \mathcal{L} \\ \big\downarrow \pi \\ X_\Sigma \end{array} \longleftrightarrow \begin{array}{l} \text{integral} \\ \text{piecewise linear} \\ \text{functions} \\ \psi : \Sigma \to \mathbb{R} \end{array}$$

Klyachko Data

E k-vector space

$\dim E = \operatorname{rank} \mathcal{E}$

① For each ray $\rho \in \Sigma(1)$

$$E \cdots \supseteq F^\rho_r \supseteq F^\rho_{r+1} \supseteq \cdots 0$$

② For each face $\sigma \in \Sigma(d)$

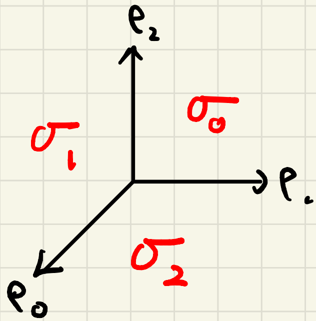
$$E = \bigoplus_{u \in M} L^\sigma_u$$

③ If $\rho \in \sigma(1)$ $\qquad \rho \cap N = \mathbb{Z}_{\geq 0} v_\rho$

$$F^\rho_r = \bigoplus_{\langle v_\rho, u \rangle \geq r} L^\sigma_u$$

Example: $T\mathbb{P}^2$

Fan of $\mathbb{P}^2$



$$e_0 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \quad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{C}^2$$

$$F_r^{e_i} = \begin{cases} 0 & r > 1 \\ \mathbb{C} e_i & r = 1 \\ \mathbb{C}^2 & r < 1 \end{cases}$$

Toric Vector Bundles and Buildings

Klyachko data can be repackaged into a piecewise-linear map*

$$\Phi_{\mathcal{E}} : |\Sigma| \longrightarrow \mathcal{B}_{sph}(E)$$

- Payne
- Kaveh, M

spherical building of $GL(E)$

This pattern has been generalized to:

Toric Principal G-Bundles (Kaveh, M)

TV. Bundles / toric schemes (Kaveh, M, Tsvelikhovskiy)

TV. Bundles / complexity 1 varieties (Dasgupta, Gangopadhyay, Kaveh, M)

(L, D) data for a Toric Vector Bundle

For any linear ideal L with

$$0 \longrightarrow L \longrightarrow k[y_1, \dots, y_m] \longrightarrow Sym(E) \longrightarrow 0$$

There is a 1-1 map

$$Trop(L) \xrightarrow{\eta} B_{sph}(E)$$

(L, D) data for a Toric Vector Bundle

For a toric vector bundle $\mathcal{E}$,

we can always find L so that

$$|\Sigma| \xrightarrow{\Phi_{\mathcal{E}}} \mathcal{B}_{sph}(E)$$

has $\Phi_{\mathcal{E}}(\Sigma) \subseteq \eta(\mathrm{Trop}(L))$

(L, D) data for a Toric Vector Bundle

So we may replace

$$|\Sigma| \xrightarrow{\Phi_\mathcal{E}} B_{sph}(\mathcal{E})$$

with a piecewise linear map

$$|\Sigma| \xrightarrow{\Phi_{\mathcal{E},L}} Trop(L) \subseteq \mathbb{R}^m$$

This map can be recovered from the values on the ray generators, which we organize into an integral $|\Sigma(1)| = n$ by m matrix D.

Example: $T\mathbb{P}^2$

$$e_0 + e_1 + e_2 = 0$$

$$\tilde{T}\mathbb{P}^2 \longleftrightarrow \left(\langle y_0 + y_1 + y_2 \rangle, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

(L, D) data for a Toric Vector Bundle

(L, D) data is equivalent to choosing a certain kind of presentation by a split bundle:

$$\bigoplus_{j=1}^{m} \mathcal{O}(D_j) \longrightarrow \mathcal{E} \longrightarrow 0$$

columns of D are Weil divisors on X_Σ

rows of D are integral points on $\mathrm{Trop}(L)$

Tropical Vector Bundles

see :

Khan, Maclagan

Tropical Vector Bundles

Kaveh, M

Tropical Vector Bundles and Matroids

hopefully coming soon!

$(KM)^2$

Tropical Vector Bundles

Structure of $\mathrm{Berg}(\mathcal{M})$

Let $\mathbb{B} \subseteq [m]$ be a basis of $\mathcal{M}$.
The apartment $A_{\mathbb{B}} \subseteq \mathrm{Berg}(\mathcal{M})$ is the collection of faces $\sigma_{\mathcal{F}}$ such that $\mathbb{B} \cap F_i$ is a basis of F_i for each flat F_i in $\mathcal{F}$.

- $A_{\mathbb{B}}$ is piecewise-linear isomorphic to the permutahedral fan Σ_r, where $r = \mathrm{rank}(\mathcal{M})$.
- Rincôn calls these "local tropical linear spaces."

Tropical Vector Bundles

Definition: Fix a matroid $\mathcal{M}$ on $[m]$.

A tropical vector bundle $\mathbb{E}$ over X_Σ with general fiber $\mathcal{M}$ is a piecewise-linear map:

$$\Phi : |\Sigma| \longrightarrow \mathrm{Berg}(\mathcal{M})$$

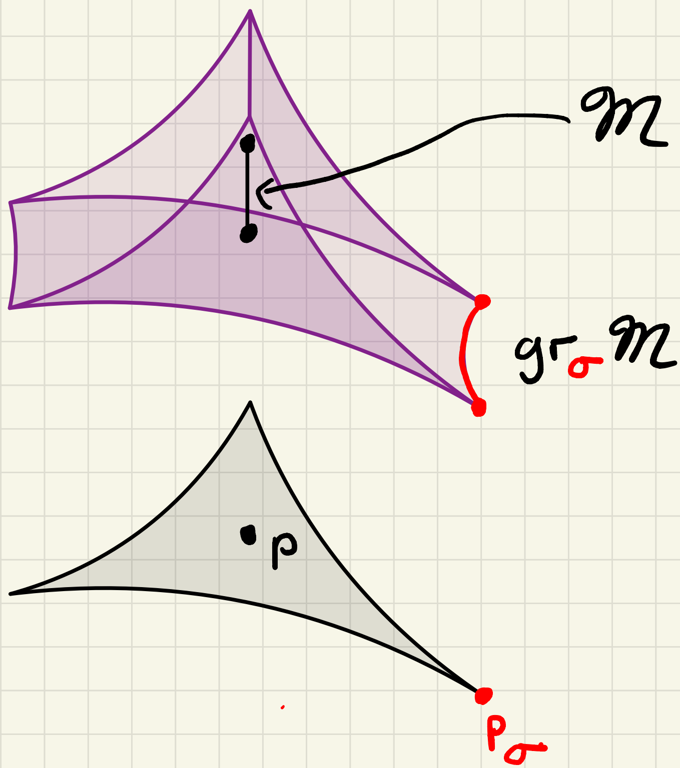
such that for every $\sigma \in \Sigma$ there is an apartment A with $\Phi(\sigma) \subseteq A$.

- Φ is determined by values on ray generators
- We organize this data into a matrix D.

$$\mathbb{E} \longleftrightarrow (\mathcal{M}, D)$$

Tropical Vector Bundles

$$\begin{array}{c}\mathbb{E}\\ \big\downarrow{\scriptstyle \pi}\\ X_\Sigma\end{array}$$



$$\operatorname{rank} \mathbb{E} = \operatorname{rank} \mathcal{M}$$
$$= \operatorname{rank} \mathrm{gr}_\sigma \mathcal{M}$$

Tropical Vector Bundles

Tropicalization

$$\begin{tikzcd}
\mathcal{E} \arrow[d, "T \circlearrowright"'] \arrow[r] & \mathbb{E} = \operatorname{trop} \mathcal{E} \arrow[d] \\
X_\Sigma & X_\Sigma
\end{tikzcd}$$

$$(L, D) \longrightarrow (\mathcal{M}_L, D)$$

Khan, Maclagan do this by tropicalizing Cox equations for $\mathcal{E}$.

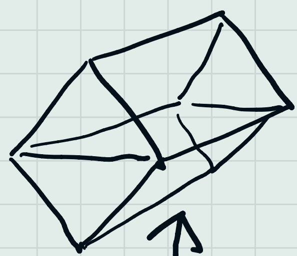
Example: $\mathbb{TP}^2$

$$\widetilde{\mathbb{TP}}^2 \longleftrightarrow \left(\langle y_0 + y_1 + y_2 \rangle, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$\operatorname{trop} \widetilde{\mathbb{TP}}^2 \longleftrightarrow \left(U_{2,3}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

Tropical Vector Bundles

$$T \curvearrowright \begin{array}{c}\mathcal{E}\\ \downarrow \\ X_\Sigma\end{array} \xrightarrow{\text{Tropicalize}} \begin{array}{c}\operatorname{Trop}\mathcal{E} = \mathbb{E}\\ \downarrow\\ X_\Sigma\end{array}$$



"non representable"
Tropical Vector Bundles

Tropical Vector Bundles

Global Sections

Fix $\mathbb{E}$ with $(\mathcal{M}, D)$

- $\mathbb{E}$ defines a "sheaf of matroids on X_Σ"
- The global sections are computed by a parliament of polytopes:

Each column D_j of D defines a polytope $\Delta_j \subseteq M \otimes \mathbb{R}$

for $u \in M$

$$H^0_u(X, \mathbb{E}) = \{ j \mid u \in \Delta_j \} \subseteq [m]$$

- This is a flat of $\mathcal{M}$
- can be used to give a criterion for global generation

Tropical Vector Bundles

Global Sections and tropicalization

If $\mathcal{E} \leftrightarrow (L, D)$ $\quad \mathbb{E} \leftrightarrow (m_L, D)$

ie $\mathbb{E}$ is a tropicalization of $\mathcal{E}$,

then $$h^0(X_\Sigma, \mathcal{E}) \geq h^0(X_\Sigma, \mathbb{E})$$

$$- h^0(X_\Sigma, \mathbb{E}) = \operatorname{rank} H^0(X_\Sigma, \mathbb{E})$$

Equality holds if (L, D) is a Di Rocco, Jabbusch, Smith (DJS) presentation of $\mathcal{E}$.

Tropical Vector Bundles

Equivariant Chern classes

Payne:

$$A^*_T(X_\Sigma) \simeq \begin{array}{l}\text{integral} \\ \text{piecewise polynomial} \\ \text{functions on } \Sigma\end{array}$$

For $1 \leq i \leq r = \operatorname{rank} \mathcal{M}$

There is a piecewise polynomial function

$$\epsilon_i : \operatorname{Berg}(\mathcal{M}) \longrightarrow \mathbb{R}$$

with the property that for any apartment A, $\epsilon_i|_A$ is the i-th elementary symmetric function in r variables.

$$u_1 + \cdots + u_r$$

$$u_1u_2 + \cdots + u_{r-1}u_r$$

$$\vdots$$

$$u_1 \cdots u_r$$

Tropical Vector Bundles

Equivariant Chern classes

Let $\Phi : |\Sigma| \longrightarrow \mathrm{Berg}(\mathcal{M})$ correspond to $\mathbb{E}$

Definition:

$$c_i \mathbb{E} = \varepsilon_i \circ \Phi \in A^i_T(X_\Sigma)$$

If $\mathbb{E}$ is a tropicalization of $\mathcal{E}$

Then

$$c_i \mathcal{E} = c_i \mathbb{E}$$

by Payne's computation of $c_i \mathcal{E}$.

Tropical Vector Bundles

Equivariant Euler characteristic

for $u \in M$

$$\chi_u(\mathbb{E}) = \sum_{\sigma \in \Sigma} (-1)^{\operatorname{codim} \sigma} h_u^0(U_\sigma, \mathbb{E}|_{U_\sigma})$$

$$\chi(\mathbb{E}) = \sum_{u \in M} \chi_u(\mathbb{E})$$

- "Čech cohomology" definition

- If $\mathbb{E}$ is a tropicalization of $\mathcal{E}$

Then

$$h_u^0(U_\sigma, \mathcal{E}|_{U_\sigma}) = h_u^0(U_\sigma, \mathbb{E}|_{U_\sigma})$$

so

$$\chi(\mathcal{E}) = \chi(\mathbb{E})$$

Tropical Vector Bundles

Tensoring with a line bundle

Let $\bar{a} = (a_1, \ldots, a_n)^T$

$\mathcal{O}(\bar{a})$ - the T-linearized line bundle on X_Σ

If $\mathbb{E} \longleftrightarrow (M, D)$

Then $\mathbb{E} \otimes \mathcal{O}(\bar{a}) \longleftrightarrow$

$$(M, [D_1 + \bar{a}, \ldots, D_m + \bar{a}])$$

- Compatible with tropicalization

Tropical Vector Bundles

Let $\mathcal{L}$ be an ample linebundle,

Theorem the function

$$N \longrightarrow \chi(\mathbb{E} \otimes \mathcal{L}^{\otimes N})$$

is polynomial of degree $d = \dim X_\Sigma$.

Theorem For $N >> 0$

$$\chi(\mathbb{E} \otimes \mathcal{L}^N) = h^0(X_\Sigma, \mathbb{E} \otimes \mathcal{L}^{\otimes N})$$

Example: Bundles on $\mathbb{P}^2$ with matroid $U_{2,3}$

$$\mathcal{E} \longleftrightarrow (U_{2,3}, D)$$

$$D = \begin{bmatrix} d_{00} & d_{01} & d_{02} \\ d_{10} & d_{11} & d_{12} \\ d_{20} & d_{21} & d_{22} \end{bmatrix}$$

$$h^0(\mathbb{P}^2, \mathcal{E}) =$$

≥ 0

$$\sum_{j=0}^{2} \binom{d_{0j} + d_{1j} + d_{2j} + 2}{2}$$

$$- \binom{\sum_{i=0}^{2} \min\{d_{i0}, d_{i1}, d_{i2}\} + 2}{2}$$

Tropical Vector Bundles

The techniques used for these results involve the Khovanskii-Pukhlikov Theory of convex chains.

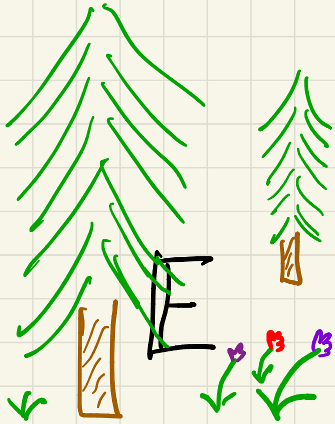
Their theory includes a variant of HRR, which leads to!

Theorem (Chong, Kaveh)

$$\chi(\mathbb{E}) = \int_{X_\Sigma} ch(\mathbb{E})\, Td(X_\Sigma)$$

Questions for the Tropical vector bundle

who you met in the woods.



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