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Higher Homotopy Groups of Complements of Complex Hyperplane Arrangements

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We generalize results of Hattori on the topology of complements of hyperplane arrangements, from the class of generic arrangements, to the much broader class of hypersolvable arrangements. We show that the higher homotopy groups of the complement vanish in a certain combinatorially determined range, and we give an explicit $\mathbb{Z}\pi_1$-module presentation of π_p, the first non-vanishing higher homotopy group. We also give a combinatorial formula for the π_1-coinvariants of π_p. For affine line arrangements whose cones are hypersolvable, we provide a minimal resolution of π_2 and study some of the properties of this module. For graphic arrangements associated to graphs with no 3-cycles, the algorithm for computing π_2 is purely combinatorial. The Fitting varieties associated to π_2 may distinguish the homotopy 2-types of arrangement complements with the same π_1, and the same Betti numbers in low degrees.

Key Words: homotopy groups; hyperplane arrangement; hypersolvable; minimality; $K(\pi, 1)$; characteristic variety; graphic arrangement.

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1. INTRODUCTION

1.1. *Background.* One of the fundamental problems in the topological study of polynomial functions, $f\colon (\mathbb{C}^\ell, \mathbf{0}) \to (\mathbb{C}, 0)$, is the computation of the homotopy groups of the complement to the hypersurface $V(f) = f^{-1}(0)$. A well-known algorithm for finding a finite presentation for $\pi_1(\mathbb{C}^\ell \backslash V(f))$ was given by Zariski and VanKampen in the early 1930's. Much less is known about the higher homotopy groups of the complement, except when $V(f)$ is irreducible, in which case the Zariski–VanKampen method can be extended to give information about $\pi_k(\mathbb{C}^\ell \backslash V(f)) \otimes \mathbb{Q}$, for $k > 1$, see [23].

In this paper, we concentrate on the simplest kind of polynomial f for which the hypersurface $V(f)$ is *not* irreducible. Namely, suppose f factors completely into distinct, degree one factors. Then f is the defining polynomial of a hyperplane arrangement, $\mathscr{A}$, with union $V(f) = \bigcup_{H \in \mathscr{A}} H$, and complement $X(\mathscr{A}) = \mathbb{C}^\ell \backslash \bigcup_{H \in \mathscr{A}} H$. The cohomology ring of $X = X(\mathscr{A})$ was computed by Brieskorn [3]. Orlik and Solomon [24] expressed $H^*(X)$ in terms of the combinatorics of $\mathscr{A}$, encoded in the intersection lattice, $L(\mathscr{A})$. In particular, the Poincaré polynomial, $P_{\mathscr{A}}(T) = \sum_{k=1}^{\ell} b_k(X)\, T^k$, admits a simple combinatorial expression, see Orlik and Terao [25]. On the other hand, the fundamental group of the complement, $\pi_1(X)$, is not determined by $L(\mathscr{A})$ alone, as the example of Rybnikov [30] shows.

For certain arrangements, all the higher homotopy groups of the complement vanish. Examples of such $K(\pi, 1)$ arrangements include the *simplicial* arrangements (Deligne [7]), and the *supersolvable* arrangements (Terao [35]). Examples of non-$K(\pi, 1)$ arrangements, and methods for detecting the first non-vanishing higher homotopy group of their complements, were given by Falk [11] and Randell [28] (see also the recent survey [14]).

The first (and, up to now, only) explicit computation of non-trivial higher homotopy groups of arrangement complements was made by Hattori [18]. An arrangement $\mathscr{A}$ in $\mathbb{C}^\ell$, $\ell > 1$, is called *generic* if, for all $\mathscr{B} \subset \mathscr{A}$, the intersection $\bigcap_{H \in \mathscr{B}} H$ has codimension $|\mathscr{B}|$ when $|\mathscr{B}| \leqslant \ell$, and is empty when $|\mathscr{B}| > \ell$. The standard example is the *Boolean* arrangement of coordinate hyperplanes in $\mathbb{C}^n$, with complement $(\mathbb{C}^*)^n$. Hattori used the minimal cell decomposition of $(\mathbb{C}^*)^n \simeq T^n$ to find an explicit, minimal cell decomposition for the complement of an arbitrary generic arrangement. More precisely, if $\mathscr{A}$ is an arrangement of n hyperplanes in general position in $\mathbb{C}^\ell$ ($n > \ell$), then $X(\mathscr{A}) \simeq (T^n)^{(\ell)}$. From this decomposition, Hattori deduced:

(A) $\pi_1(X) = \mathbb{Z}^n$.

(B) $\pi_k(X) = 0$ for $1 < k < \ell$.

(C) $\pi_\ell(X)$ admits a free $\mathbb{Z}\pi_1$-resolution of length $n - \ell$.

The simplest example is that of 3 generic affine lines in $\mathbb{C}^2$. In that case, the complement X has the homotopy type of the 2-skeleton of the 3-torus, $T^3 = K(\mathbb{Z}^3, 1)$. Looking at the universal cover, $\widetilde{X}$, we thus see that $\pi_2(X) = H_2(\widetilde{X})$ is a free $\mathbb{Z}\mathbb{Z}^3$-module, generated by the boundary of a cubical 3-cell from the standard decomposition of $\widetilde{T}^3 = \mathbb{R}^3$.

1.2. *Results.* In this paper, we set out to generalize Hattori's results to the wider class of *hypersolvable arrangements*. This class, introduced in [20], includes both supersolvable arrangements and (cones of) generic arrangements.

A hypersolvable arrangement $\mathscr{A}$ admits a "supersolvable deformation," $\hat{\mathscr{A}}$, which preserves the collinearity relations. For example, if we start with $n \geqslant 3$ generic lines in $\mathbb{C}^2$ and take $\mathscr{A}$ to be the respective central arrangement of planes in $\mathbb{C}^3$, then $\hat{\mathscr{A}}$ is the Boolean arrangement in $\mathbb{C}^n$. In general, $X(\mathscr{A})$ has the same fundamental group as $X(\hat{\mathscr{A}})$; see [20, 21]. Moreover, $\pi_1(X(\hat{\mathscr{A}}))$ is a (special kind of) iterated semidirect product of finitely generated free groups; see [5, 12, 35] and Theorem 4.8. These facts together provide the generalization of Hattori's result (A) to hypersolvable arrangements.

The key tool for generalizing (B) and (C) to complements of hypersolvable arrangements is the existence of *minimal* cell structures, on both $X(\mathscr{A})$ and $X(\hat{\mathscr{A}}) = K(\pi_1(X(\mathscr{A})), 1)$.

To find an explicit presentation for the first higher non-vanishing homotopy group, we thus turn to a general study of minimal cell decompositions. The idea is to get higher homotopy information on a connected, finite-type, CW-space X, by comparing it to its classifying space $K(\pi, 1)$, where $\pi = \pi_1(X)$. We are thus led to introduce a homotopy-type invariant of X, called the *order of π_1-connectivity*, which measures the rational-homology deviation of X from asphericity:

$$p(X) := \sup\{q \mid b_r(X) = b_r(K(\pi, 1)), \forall r \leqslant q\}.$$

(If X is 1-connected, and $H_*(X)$ is torsion-free, then $p(X)$ is the usual order of connectivity of X.)

A (connected, finite-type) CW-space X is said to be *minimal* if it has a CW-structure with b_k k-cells, for all k, where b_k is the kth Betti number of X. Under certain minimality and homological assumptions, it turns out that the universal cover of X and the (contractible) universal cover of $K(\pi, 1)$ behave like sharing the same $p(X)$-skeleton. This leads to the following result, proved in Section 2.

THEOREM 1.3. *Let X be a minimal space, such that $Y = K(\pi, 1)$ is also minimal. Assume that both X and Y have cohomology algebras generated in degree* 1. *Let $(C_*(\widetilde{Y}), \partial_*)$ be the π-equivariant cellular chain complex of the* *universal cover of Y, associated to some minimal cell decomposition of Y. Let $j\colon X \to Y$ be a classifying map, and $\Pi_*\colon H_*(Y) \to H_*(Y, X)$ be the projection onto the cokernel of j_*. Set $p = p(X)$. Then:*

(1) $\pi_k(X) = 0$, *for* $1 < k < p$.

(2) *If* $p < \infty$, *then* $\pi_p(X)$ *is non-trivial and has a finite, minimal, $\mathbb{Z}\pi$-presentation, given by*

$$H_{p+2}(Y) \otimes \mathbb{Z}\pi \xrightarrow{(\Pi_{p+1} \otimes \mathrm{id}) \circ \partial_{p+2}} H_{p+1}(Y, X) \otimes \mathbb{Z}\pi \to \pi_p(X) \to 0. \tag{1.1}$$

The theorem provides a complete generalization of Hattori's result (B) in this setting, and a partial generalization of (C). If $\dim X = p(X)$ and Y is a finite complex, the presentation (1.1) extends to a finite-length, free $\mathbb{Z}\pi$-resolution of $\pi_p(X)$; see Remark 2.12. In particular, if X is the complement of a generic arrangement, then $p(X) = \ell$, and (1.1) may be continued to Hattori's resolution (C).

In Section 3, we follow a standard approach and extract from the above presentation of $\pi_p(X)$ more manageable invariants of homotopy type: the subvarieties of the complex torus $(\mathbb{C}^*)^n$, $n = b_1(X)$, defined by the Fitting ideals of $\pi_p(X) \otimes_{\mathbb{Z}\pi} \mathbb{Z}\mathbb{Z}^n$. In turn, we identify these varieties with the jumping loci for homology with coefficients in rank 1 local systems of the pair $(K(\pi, 1), X)$.

We now return to the case where $X = X(\mathscr{A})$ is the complement of an arrangement $\mathscr{A}$. From recent work of Dimca [8] and Randell [29], we know that X is minimal. We also know (from [3]) that $H^*(X)$ is generated in degree 1. Since X may fail to possess any finite-type $K(\pi, 1)$, our approach does not work in this generality. If $\mathscr{A}$ is hypersolvable, though, we may take $K(\pi, 1) = X(\hat{\mathscr{A}})$, where $\hat{\mathscr{A}}$ is the supersolvable deformation of $\mathscr{A}$, and thus Theorem 1.3 becomes available. Moreover, the chain complex $(C_*(\tilde{Y}), \partial_*)$ may be computed explicitly, using the Fox calculus algorithm from [6].

We devote Section 4 to the description of the explicit computation of the first non-vanishing higher homotopy group of hypersolvable (non-supersolvable) arrangements, $\mathscr{A}$. We give, in Section 4.10, a combinatorial formula for $p(X(\mathscr{A}))$. We also show that the map $\Pi_*\colon H_*(K(\pi, 1)) \to H_*(K(\pi, 1), X)$ is determined by $L(\mathscr{A})$; see (4.6). Our algorithm may be summarized as follows.

THEOREM 1.4. *Let $\mathscr{A}$ be a hypersolvable arrangement, with complement $X = X(\mathscr{A})$, fundamental group $\pi = \pi_1(X)$, and order of π_1-connectivity $p = p(X)$. Then:*

(1) *X is aspherical* $\Leftrightarrow$ *$\mathscr{A}$ is supersolvable* $\Leftrightarrow$ $p = \infty$.

(2) *If $p < \infty$, then the first non-vanishing higher homotopy group of X is $\pi_p(X)$, with finite, minimal, $\mathbb{Z}\pi$-presentation* (1.1).

(3) *If $p < \infty$, then the group of π-coinvariants of $\pi_p(X)$ is free abelian, with (strictly positive) combinatorially determined rank.*

The precise formula for the coinvariants is provided by Corollary 4.10(4). A similar formula was obtained by Randell [28], for generic sections of aspherical arrangements—a class of arrangements which overlaps with the hypersolvable class, but neither includes it, nor is included in it.

1.5. *Applications.* Particularly simple is the case of affine line arrangements in $\mathbb{C}^2$. These arrangements represent both the simplest case of non-irreducible plane algebraic curves, and the simplest case of hyperplane arrangements. As such, they have been the object of intense investigation; see, e.g., [5, 11, 14, 15, 30]. For one, the fundamental group of an arbitrary hyperplane arrangement complement can be identified with the fundamental group of an affine line arrangement (by the Hamm–Lê theorem), thereby making line arrangements key to the understanding of all arrangements. For another, complements of affine line arrangements need not be aspherical (unlike, say, complements of weighted-homogeneous plane curves, which always are), thereby making for a richer object of topological study.

In Section 5, we consider affine line arrangements whose cones are hypersolvable. In Theorem 5.4, we go further, providing a minimal, finite-length resolution for π_2, which completely generalizes Hattori's resolution (C) in this context. We also obtain some finer information about the $\mathbb{Z}\pi_1$-module structure of π_2: it is neither projective (except in a very special, combinatorially decidable case, when it is free, with rank combinatorially determined), nor nilpotent (except if it is trivial).

Another class of arrangements which can be fairly well understood from our point of view is that of (hypersolvable) graphic arrangements. In Section 6, we implement in this setting our algorithm for higher homotopy computations. The class of arrangements associated to graphs without 3-cycles provides a natural, rich supply of hypersolvable arrangements, which are neither supersolvable nor generic, and for which the algorithm for computing π_2 is purely combinatorial. As an illustration, we exhibit two graphic arrangements, whose complements have the same π_1, but non-isomorphic π_2's (when viewed as $\mathbb{Z}\pi_1$-modules). The difference between the π_2's is picked by the number of components of their respective Fitting varieties.

2. MINIMAL CELL DECOMPOSITIONS AND HOMOTOPY GROUPS

2.1. *Minimal Cell Decompositions.* Given a space X, consider the following conditions on its homotopy type:

(i) X is homotopy equivalent to a connected, finite-type CW-complex;

(ii) The integral homology groups $H_*(X)$ are torsion-free;

(iii) The cup-product map $\bigcup: \bigwedge^* H^1(X) \to H^*(X)$ is surjective.

These three conditions abstract some well-known topological properties of complements of complex hyperplane arrangements. Next, we delineate a class of spaces that satisfy condition (i) and a much stronger form of (ii).

DEFINITION 2.2. A space X is called *minimal* if X has the homotopy type of a connected, finite-type, CW-complex K such that

$$\#\{q\text{-cells in } K\} = \operatorname{rank} H_q(X; \mathbb{Z}), \qquad \text{for all} \quad q \geqslant 0. \tag{2.1}$$

This definition implies at once that all the (abelian) groups $H_q(X)$ are finitely generated and torsion-free. Consequently, we may unambiguously speak about the Betti numbers of X, $b_q(X)$, without specifying the coefficients.

Let X be a minimal space, and let $C_*(X)$ be the cellular chain complex of X, corresponding to a minimal CW-decomposition. Let $\pi = \pi_1(X)$ be the fundamental group, $\mathbb{Z}\pi$ its group ring, and $\varepsilon: \mathbb{Z}\pi \to \mathbb{Z}$ the augmentation map. Let $\tilde{X}$ be the universal cover of X, and let $(C_*(\tilde{X}), d_*)$ be the π-equivariant chain complex of $\tilde{X}$, with $C_q(\tilde{X}) = C_q(X) \otimes \mathbb{Z}\pi$ and $d_q: C_q(\tilde{X}) \to C_{q-1}(\tilde{X})$. By minimality, all the boundary maps d_q are *ε-minimal*, in the sense that $d_q \otimes_{\mathbb{Z}\pi} \mathbb{Z} = 0$.

EXAMPLE 2.3. The standard example of a space admitting a minimal cell decomposition is the n-torus, T^n. Identifying $\pi_1(T^n) = \mathbb{Z}^n$, with basis $\{x_i\}_i$, and $C_q(T^n) = \bigwedge^q \mathbb{Z}^n$, with basis $\{\sigma_I = \sigma_{i_1} \cdots \sigma_{i_q}\}_I$, the boundary map $d_q: \bigwedge^q \mathbb{Z}^n \otimes \mathbb{Z}\mathbb{Z}^n \to \bigwedge^{q-1} \mathbb{Z}^n \otimes \mathbb{Z}\mathbb{Z}^n$ is given by $d_q(\sigma_I) = \sum_{r=1}^q (-1)^{r-1} \sigma_{I \setminus \{i_r\}} \otimes (x_{i_r} - 1)$.

EXAMPLE 2.4. More generally, let $\pi = F_{d_n} \rtimes_{\rho_{n-1}} F_{d_{n-1}} \rtimes \cdots \rtimes_{\rho_1} F_{d_1}$ be an iterated semidirect product of free groups, with ρ_i acting as the identity in homology, and $X = K(\pi, 1)$ a corresponding Eilenberg–MacLane space. A finite, minimal cell decomposition of X is given in [6]: The number of cells is read off the Poincaré polynomial, $P_X(T) = \prod_{i=1}^n (1 + d_i T)$, and the

(ε-minimal) boundary maps, $d_q: C_q(\widetilde{X}) \to C_{q-1}(\widetilde{X})$, are given explicitly in terms of Fox Jacobians of the monodromy operators ρ_i; see [6, Theorem 2.10, Proposition 3.3, and Corollary 3.4].

Remark 2.5. Not all manifolds admit minimal cell decompositions. For example, if X is the complement of a non-trivial knot in S^3, then X has no minimal cell decomposition, not even up to $q = 1$. See also the monograph by Sharko [32] for various other definitions of minimality in related contexts.

Now assume X is a minimal space for which there exists a minimal Eilenberg–MacLane space $Y = K(\pi, 1)$. Let $j: X \to Y$ be a classifying map. Without loss of generality, we may assume j respects the given (minimal) CW-decompositions on X and Y. Then the chain map $j_\#: C_*(X) \to C_*(Y)$ lifts to an equivariant chain map $\tilde{j}_\#: (C_*(\widetilde{X}), d_*) \to (C_*(\widetilde{Y}), \partial_*)$, which, by minimality, can be identified homologically, as

$$\begin{array}{ccccc} C_q(\widetilde{X}) & \xrightarrow{=} & C_q(X) \otimes \mathbb{Z}\pi & \xrightarrow{\cong} & H_q(X) \otimes \mathbb{Z}\pi \\ \downarrow{\scriptstyle \tilde{j}_q} & & \downarrow{\scriptstyle j_q \otimes \mathrm{id}} & & \downarrow{\scriptstyle j_{*q} \otimes \mathrm{id}} \\ C_q(\widetilde{Y}) & \xrightarrow{=} & C_q(Y) \otimes \mathbb{Z}\pi & \xrightarrow{\cong} & H_q(Y) \otimes \mathbb{Z}\pi \end{array}$$

for each q.

2.6. *Homotopy Groups.* We now analyze the homotopy groups of certain minimal spaces. In order to state our results, we need to introduce one more notion.

DEFINITION 2.7. Let X be a space satisfying condition (i). Define the *order of π_1-connectivity* of X to be

$$p(X) := \sup\{q \mid b_r(X; \mathbb{Q}) = b_r(K(\pi_1(X), 1); \mathbb{Q}),\ \forall r \leqslant q\}.$$

Remark 2.8. The terminology is borrowed from the simply connected case: if $\pi_1(X) = 0$ and X also satisfies (ii), then $p(X)$ is the usual order of connectivity of X. Note that $p(X)$ is a positive integer, depending only on the homotopy type of X. Furthermore, if both X and $K(\pi_1(X), 1)$ satisfy conditions (i)–(iii), then $p(X) \geqslant 2$.

Remark 2.9. Set $Y = K(\pi_1(X), 1)$ and consider a classifying map, $j: X \to Y$. Assume both X and Y satisfy conditions (i)–(iii) from Section 2.1. These conditions readily imply that j induces a split surjection on cohomology, and a split injection on homology. Consequently, $j_{*r}: H_r(X) \to H_r(Y)$ is an isomorphism, for all $r \leqslant p(X)$, and the groups $H_*(Y, X) := \operatorname{coker}(j_*\colon H_*(X) \to H_*(Y))$ fit into split exact sequences

$$0 \to H_*(X) \xrightarrow{j_*} H_*(Y) \xrightarrow{\Pi_*} H_*(Y, X) \to 0. \tag{2.2}$$

The next two results provide a complete proof of Theorem 1.3 from the Introduction.

THEOREM 2.10. *Let X be a minimal space, such that $Y = K(\pi, 1)$ is also minimal. Assume that both X and Y satisfy condition* (iii) *from* 2.1. *Set $p = p(X)$. Then:*

(1) *$\tilde{X}$ is $(p-1)$-connected.*

(2) *If $p < \infty$, then*

$$H_{p+2}(Y) \otimes \mathbb{Z}\pi \xrightarrow{\Delta_p := (\Pi_{p+1} \otimes \mathrm{id}) \circ \partial_{p+2}} H_{p+1}(Y, X) \otimes \mathbb{Z}\pi \to \pi_p(X) \to 0 \tag{2.3}$$

is a finite presentation of $\pi_p(X)$ as $\mathbb{Z}\pi$-module. Moreover, the presentation is ε-minimal, i.e., $\Delta_p \otimes_{\mathbb{Z}\pi} \mathbb{Z} = 0$.

Proof. (1) Fix minimal CW-decompositions on X and Y. We may assume $j\colon X \to Y$ is cellular, and $j_{\#}\colon \pi_1(X) \to \pi_1(Y)$ identifies the respective fundamental groups.

We have to show that $\pi_q(\tilde{X}) = 0$, for $q < p$. Of course, $\pi_1(\tilde{X}) = 0$. Fix $1 < q < p$, and assume that $\pi_r(\tilde{X}) = 0$, for $r < q$. By the Hurewicz isomorphism theorem, $\pi_q(\tilde{X}) = H_q(\tilde{X})$. By minimality of X and Y, we have a commuting ladder between the (equivariant) chain complexes of $\tilde{X}$ and $\tilde{Y}$:

$$\begin{array}{llclclc}
C_*(\tilde{X}): & H_{q+1}(X) \otimes \mathbb{Z}\pi & \xrightarrow{d_{q+1}} & H_q(X) \otimes \mathbb{Z}\pi & \xrightarrow{d_q} & H_{q-1}(X) \otimes \mathbb{Z}\pi \\
& \downarrow{\scriptstyle j_* \otimes \mathrm{id}} & & \downarrow{\scriptstyle j_* \otimes \mathrm{id}} & & \downarrow{\scriptstyle j_* \otimes \mathrm{id}} \\
C_*(\tilde{Y}): & H_{q+1}(Y) \otimes \mathbb{Z}\pi & \xrightarrow{\partial_{q+1}} & H_q(Y) \otimes \mathbb{Z}\pi & \xrightarrow{\partial_q} & H_{q-1}(Y) \otimes \mathbb{Z}\pi
\end{array}$$

The three vertical arrows are isomorphisms, since $j_{*r} \otimes \mathrm{id} : H_r(X) \otimes \mathbb{Z}\pi \xrightarrow{\cong} H_r(Y) \otimes \mathbb{Z}\pi$ for $r \leqslant p(X)$. It follows that $H_q(\tilde{X}) = H_q(\tilde{Y})$. But $\tilde{Y}$ is acyclic, and so $\pi_q(\tilde{X}) = 0$.

(2) Consider the commuting diagram

$$\begin{array}{ccccccc}
& & H_{p+1}(X) \otimes \mathbb{Z}\pi & \xrightarrow{d_{p+1}} & H_p(X) \otimes \mathbb{Z}\pi & \xrightarrow{d_p} & H_{p-1}(X) \otimes \mathbb{Z}\pi \\
& & \downarrow{\scriptstyle j_{*(p+1)} \otimes \mathrm{id}} & & \downarrow{\scriptstyle j_{*p} \otimes \mathrm{id}} & & \downarrow{\scriptstyle j_{*(p-1)} \otimes \mathrm{id}} \\
H_{p+2}(Y) \otimes \mathbb{Z}\pi & \xrightarrow{\partial_{p+2}} & H_{p+1}(Y) \otimes \mathbb{Z}\pi & \xrightarrow{\partial_{p+1}} & H_p(Y) \otimes \mathbb{Z}\pi & \xrightarrow{\partial_p} & H_{p-1}(Y) \otimes \mathbb{Z}\pi
\end{array}$$

By Part (1) and Hurewicz, $\pi_p(X) = H_p(\widetilde{X})$. A diagram chase yields isomorphisms

$$H_p(\widetilde{X}) = \frac{\ker d_p}{\operatorname{im} d_{p+1}} \xrightarrow{j_{*p} \otimes \mathrm{id}} \frac{\ker \partial_p}{\operatorname{im}((j_{*p} \otimes \mathrm{id}) \circ d_{p+1})}$$
$$= \frac{\operatorname{im} \partial_{p+1}}{\operatorname{im}(\partial_{p+1} \circ (j_{*(p+1)} \otimes \mathrm{id}))} \xleftarrow{\partial_{p+1}} \frac{H_{p+1}(Y) \otimes \mathbb{Z}\pi}{\operatorname{im} \partial_{p+2} + \operatorname{im}(j_{*(p+1)} \otimes \mathrm{id})}$$
$$\xrightarrow{\Pi_{p+1} \otimes \mathrm{id}} \frac{H_{p+1}(Y, X) \otimes \mathbb{Z}\pi}{\operatorname{im} \Delta_p}$$

and we are done. ▮

Corollary 2.11. *With assumptions as above, and if $p = p(X) < \infty$, then the group of coinvariants of $\pi_p(X)$ under the action of $\pi = \pi_1(X)$ is given by*

$$(\pi_p(X))_\pi = H_{p+1}(Y, X).$$

In particular, $(\pi_p(X))_\pi \neq 0$.

Remark 2.12. If X has the homotopy type of a CW-complex of dimension $p = p(X)$, and Y is finite, then the presentation (2.3) for $\pi_p(X)$ may be continued to a free $\mathbb{Z}\pi$-resolution of length $d - p$, where $d = \dim Y$. (We shall encounter such a situation later on, in Theorem 5.4.) Indeed, let $(C_*(\widetilde{Y}), \partial_*)$ be the π-equivariant cellular chain complex of $\widetilde{Y}$. Note that $H_{p+1}(X) = 0$ (since $\dim X = p$), and so $H_{p+1}(Y, X) = H_{p+1}(Y)$ and $\Delta_p = \partial_{p+2}$. Hence, $\pi_p(X)$ has finite, free, ε-minimal, resolution

$$0 \to C_d(\widetilde{Y}) \xrightarrow{\partial_d} C_{d-1}(\widetilde{Y}) \to \cdots \to C_{p+2}(\widetilde{Y}) \xrightarrow{\partial_{p+2}} C_{p+1}(\widetilde{Y}) \to \pi_p(X) \to 0.$$

Remark 2.13. An especially simple situation where Theorem 2.10 applies is as follows. Let Y be a minimal $K(\pi, 1)$-complex satisfying condition (iii), and let $X \subset Y$ be a proper, connected subcomplex, such that $X^{(2)} = Y^{(2)}$. Since Y is minimal, X is also minimal, and (iii) also holds for X. Since X and Y share the same 2-skeleton, the inclusion $j: X \to Y$ is a classifying map, inducing an isomorphism on π_1. Since X is a subcomplex of Y, we have an exact sequence of cellular chain complexes, $0 \to C_*(X) \xrightarrow{j} C_*(Y) \xrightarrow{\mathrm{pr}} C_*(Y, X) \to 0$, and $\Pi_* = \mathrm{pr}_* : H_*(Y) \to H_*(Y, X)$. Moreover,

$$p(X) = \max\{q \mid \#\{r\text{-cells of } X\} = \#\{r\text{-cells of } Y\}, \forall r \leqslant q\},$$

and $p(X) < \infty$. Therefore, $\pi_k(\widetilde{X}) = 0$, for $k < p = p(X)$, and $\pi_p(X)$ has a finite $\mathbb{Z}\pi$-presentation, given in (2.3).

Remark 2.14. Most of the results in this section have only relevance for non-simply connected spaces. Indeed, if X has the homotopy type of a finite-type CW-complex, and $\pi_1(X)=0$, then X cannot satisfy condition (iii), unless X is contractible.

On the other hand, if X is 1-connected and satisfies conditions (i) and (ii), then X has the homotopy type of a minimal CW-complex K. The complex K may be obtained from a bouquet of spheres $\bigvee_{b_{p+1}(X)} S^{p+1}$, where $p=p(X)$, by attaching suitable cells. For details of the proof, see Anick [1], where the notion of minimality for simply connected spaces was first introduced.

3. FITTING IDEALS AND JUMPING LOCI

The $\mathbb{Z}\pi$-module $\pi_p(X)$ determined in Theorem 2.10 can be rather intractable. We now associate to $\pi_p(X)$ a more manageable module (over a commutative ring) and extract from it invariants that can be understood as jumping loci for homology with coefficients in rank 1 local systems.

3.1. *Fitting Ideals.* Let π be a group, with abelianization $\pi^{\mathrm{ab}} \cong \mathbb{Z}^n$, and let M be a finitely presented module over $\mathbb{Z}\pi$. Let $\widetilde{M} = M \otimes_{\mathbb{Z}\pi} \mathbb{Z}\mathbb{Z}^n$ be the module over $\mathbb{Z}\mathbb{Z}^n$ obtained by extending scalars via $\mathbb{Z}\pi \xrightarrow{\mathrm{ab}} \mathbb{Z}\mathbb{Z}^n$. For $k \geqslant 0$, let $F_k(\widetilde{M})$ be the corresponding kth *Fitting ideal*, generated by the codimension $k-1$ minors of a presentation matrix for $\widetilde{M}$. As is well-known, the Fitting ideals are independent of the choice of presentation; see, e.g., [9, p. 493]. Now fix a basis $\{x_1, \ldots, x_n\}$ for $\mathbb{Z}^n$. Then the group ring $\mathbb{C}\mathbb{Z}^n = \mathbb{Z}\mathbb{Z}^n \otimes \mathbb{C}$ may be identified with $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the coordinate ring of the complex algebraic torus $(\mathbb{C}^*)^n$. For each $k \geqslant 0$, the kth Fitting ideal of $\widetilde{M}$ defines a subvariety of this torus,

$$V_k(M) := \{t \in (\mathbb{C}^*)^n \mid g(t) = 0, \forall g \in F_k(\widetilde{M}) \otimes \mathbb{C}\}.$$

Alternatively, $V_k(M)$ can be described as the variety defined by the annihilator of $\bigwedge^k \widetilde{M}$. Indeed, $\mathrm{Rad}(F_k(\widetilde{M})) = \mathrm{Rad}(\mathrm{ann}(\bigwedge^k \widetilde{M}))$; see [9, pp. 511–513].

Let X be a path-connected space. Assume that $\pi = \pi_1(X)$ has abelianization $\mathbb{Z}^n$, and that $\pi_p(X)$ is a finitely presented $\mathbb{Z}\pi$-module.

DEFINITION 3.2. The *kth Fitting variety* of $\pi_p(X)$ is the subvariety of the complex algebraic n-torus, $V_k(\pi_p(X))$, defined as above.

A standard argument (which we include for the sake of completeness) shows that the Fitting varieties are invariants of the π_1-module π_p.

PROPOSITION 3.3. *Suppose* $f: \pi_1(X) \to \pi_1(X')$ *and* $g: \pi_p(X) \to \pi_p(X')$ *are compatible isomorphisms, i.e.,* $g(xm) = f(x)\, g(m)$, *for all* $x \in \pi_1(X)$ *and* $m \in \pi_p(X)$. *Then there is a monomial isomorphism* $\Phi : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$ *such that* $\Phi(V_k(\pi_p(X)) = V_k(\pi_p(X'))$.

Proof. The extension of $f_*: H_1(X) \to H_1(X')$ to group rings gives rise to an isomorphism between $\tilde{M} = \pi_p(X) \otimes_{\mathbb{Z}\pi_1(X)} \mathbb{Z}H_1(X)$ and $\tilde{M}' = \pi_p(X') \otimes_{\mathbb{Z}\pi_1(X')} \mathbb{Z}H_1(X')$, and thus maps $\operatorname{Rad}(\operatorname{ann}(\bigwedge^k \tilde{M}))$ bijectively to $\operatorname{Rad}(\operatorname{ann}(\bigwedge^k \tilde{M}'))$.

Now identify $H_1(X)$ and $H_1(X')$ with $\mathbb{Z}^n$, and let $\phi = (\phi_{i,j}): \mathbb{Z}^n \to \mathbb{Z}^n$ be the matrix of f_* under this identification. Let $\Phi: (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$ be the corresponding monomial isomorphism, given by $\Phi(t_i) = t_1^{\phi_{i,1}} \cdots t_n^{\phi_{i,n}}$. It is readily verified that Φ preserves the Fitting varieties. ∎

COROLLARY 3.4. *For each* $k \geqslant 0$, *the monomial isomorphism type of* $V_k(\pi_p(X))$ *is a homotopy type invariant for* X.

Now let X be a space satisfying the conditions of Theorem 2.10. Then $\pi^{\mathrm{ab}} \cong \mathbb{Z}^n$, where $n = b_1(X)$. Let $p = p(X)$ be the order of π_1-connectivity of X. The Fitting ideals of $\pi_p(X) \otimes_{\mathbb{Z}\pi} \mathbb{Z}\mathbb{Z}^n$, and the varieties defined by them, may be computed from the presentation matrix $\Delta_p \otimes_{\mathbb{Z}\pi} \mathbb{Z}\mathbb{Z}^n$, where Δ_p is the matrix given in (2.3). We shall give an explicit example of such a computation in Section 6.

3.5. *Characteristic Varieties.* There is another, well-known way to associate subvarieties of the complex algebraic torus to a space X satisfying conditions (i) and (ii) from Section 2.1. For positive integers k and p, set $V_k^p(X) = \{t \in (\mathbb{C}^*)^n \mid \dim_{\mathbb{C}} H_p(C_*(\tilde{X}) \otimes_{\mathbb{Z}\pi} \mathbb{C}_t) \geqslant k\}$, where $n = b_1(X)$, and $\mathbb{C}_t$ is the π-module $\mathbb{C}$, given by the representation $\pi \overset{\mathrm{ab}}{\to} \mathbb{Z}^n \to \mathbb{C}^*$, gotten by sending x_i to t_i. This is an algebraic subvariety of $(\mathbb{C}^*)^n$, called the (p, k)-*characteristic variety* of X. It is straightforward to extend this definition to the relative setting, as follows.

DEFINITION 3.6. Let (Y, X) be a CW-pair of spaces satisfying conditions (i) and (ii), and such that the inclusion $X \hookrightarrow Y$ induces an isomorphism on π_1. Set $n = b_1(X)$. For k, $p > 0$, the $(p+1, k)$-*characteristic variety* of (Y, X) is the subvariety of the complex algebraic n-torus defined by

$$V_k^{p+1}(Y, X) := \{t \in (\mathbb{C}^*)^n \mid \dim_{\mathbb{C}} H_{p+1}(C_*(\tilde{Y}, \tilde{X}) \otimes_{\mathbb{Z}\pi} \mathbb{C}_t) \geqslant k\}.$$

The characteristic variety $V_k^1(X)$ was interpreted by Hironaka [19] as the variety defined by $F_{k+1}(X)$, the ideal generated by the codimension k minors of the Alexander matrix of $\pi_1(X)$:

$$V_k^1(X) = V(F_{k+1}(X)).$$

The next result provides a higher-dimensional analogue of Hironaka's theorem, in a relative setting.

PROPOSITION 3.7. *Let Y be a minimal $K(\pi, 1)$-complex satisfying condition* (iii) *from Section* 2.1*, and let $X \subset Y$ be a proper, connected subcomplex, such that $X^{(2)} = Y^{(2)}$. Set $p = p(X)$. Then, for all $k \geqslant 0$,*

$$V_k(\pi_p(X)) = V_k^{p+1}(Y, X).$$

Proof. By Remark 2.13, $\pi_p(X)$ is a finitely presented $\mathbb{Z}\pi$-module, with presentation matrix Δ_p given in (2.3). Hence, $V_k(\pi_p(X)) = \{t \in (\mathbb{C}^*)^n \mid \dim \operatorname{coker} \Delta_p(t) \geqslant k\}$, where $\Delta_p(t)$ is the evaluation of the matrix of Laurent polynomials $\Delta_p \otimes_{\mathbb{Z}\pi} \mathbb{C}\mathbb{Z}^n$ at $x_i = t_i$.

Let $j\colon X \to Y$ be the inclusion. The lift to universal covers, $\tilde{j}\colon \tilde{X} \to \tilde{Y}$, gives rise to an exact sequence of π-equivariant chain complexes, a fragment of which is shown below:

$$\begin{array}{ccc} H_{p+2}(Y) \otimes \mathbb{Z}\pi & \xrightarrow{\Pi_{p+2} \otimes \mathrm{id}} & H_{p+2}(Y, X) \otimes \mathbb{Z}\pi \\ \downarrow{\scriptstyle \partial_{p+2}} & & \downarrow{\scriptstyle \bar{\partial}_{p+2}} \\ H_{p+1}(Y) \otimes \mathbb{Z}\pi & \xrightarrow{\Pi_{p+1} \otimes \mathrm{id}} & H_{p+1}(Y, X) \otimes \mathbb{Z}\pi. \end{array} \tag{3.1}$$

Now fix $t \in (\mathbb{C}^*)^n$. Tensoring (3.1) over $\mathbb{Z}\pi$ with $\mathbb{C}$, via the representation $t\colon \pi \to \mathbb{C}^*$, yields the commuting diagram

$$\begin{array}{ccc} H_{p+2}(Y) \otimes \mathbb{C} & \xrightarrow{\Pi_{p+2} \otimes \mathrm{id}} & H_{p+2}(Y, X) \otimes \mathbb{C} \\ \downarrow{\scriptstyle \partial_{p+2}(t)} & & \downarrow{\scriptstyle \bar{\partial}_{p+2}(t)} \\ H_{p+1}(Y) \otimes \mathbb{C} & \xrightarrow{\Pi_{p+1} \otimes \mathrm{id}} & H_{p+1}(Y, X) \otimes \mathbb{C}. \end{array}$$

(Note also that $H_p(Y, X) = 0$, by the definition of p.) Chasing this diagram, we see that $H_{p+1}(\tilde{Y}, \tilde{X}; \mathbb{C}_t) = \operatorname{coker} \bar{\partial}_{p+2}(t) = \operatorname{coker} \Delta_p(t)$. From Definition 3.6, we have $V_k^{p+1}(Y, X) = \{t \in (\mathbb{C}^*)^n \mid \dim H_{p+1}(\tilde{Y}, \tilde{X}; \mathbb{C}_t) \geqslant k\}$, and we are done. ∎

4. HIGHER HOMOTOPY GROUPS OF HYPERSOLVABLE ARRANGEMENTS

We now apply the machinery developed in Section 2 to the class of spaces we had in mind all along: complements of complex hyperplane arrangements. We begin with a brief review of basic notions and relevant general results.

4.1. *Minimal Cell Decompositions of Arrangements.* A *(complex) hyperplane arrangement* is a finite set, $\mathscr{A}$, of codimension-1 affine subspaces in a finite-dimensional complex vector space, V. The two main objects associated to an arrangement $\mathscr{A}$ are its *complement*, $X(\mathscr{A}) = V \setminus \bigcup_{H \in \mathscr{A}} H$, and its *intersection lattice*, $L(\mathscr{A}) = \{\bigcap_{H \in \mathscr{B}} H \mid \mathscr{B} \subseteq \mathscr{A}\}$. A general reference for the subject is the book by Orlik and Terao [25].

Since $X(\mathscr{A})$ is the complement of a complex hypersurface, it has the homotopy type of a finite CW-complex, and thus satisfies condition (i) from Section 2.1. Explicit regular CW-complexes (of dimension equal to $\dim_{\mathbb{C}} V$) onto which $X(\mathscr{A})$ deform-retracts were given by Salvetti [31] (in the complexified-real case), and by Björner and Ziegler [2] (in the general case). Neither of these complexes, though, is minimal.

In a first version of this paper [26], we proved that the complements of arbitrary complex hyperplane arrangements satisfy the minimality condition (2.1), up to $q = 2$, by combining results from [5, 22]. Since then, the minimality question for arrangement complements, raised in [26], has been solved in the affirmative by Dimca [8] and Randell [29] (independently). Using Morse theory, they proved the following.

THEOREM 4.2 (Dimca [8], Randell [29]). *Let $\mathscr{A}$ be a complex hyperplane arrangement, with complement $X(\mathscr{A})$. Then $X(\mathscr{A})$ is minimal, i.e., it admits a finite cell decomposition with number of q-cells equal to the qth Betti number, for all q.*

As noted in [8], complements of generic projective hypersurfaces fail to possess minimal cell structures. This indicates that minimality is a strong topological peculiarity of complements of complex arrangements.

4.3. *OS-Algebras.* As shown by Brieskorn [3], the complement of a complex hyperplane arrangement also satisfies condition (iii) from Section 2.1; i.e., its cohomology ring is generated in degree 1. Together with the above theorem, this opens the way for using our approach to generalize Hattori's results to a wider class of arrangements. But first, we need to recall an important result of Orlik and Solomon [24], which gives a combinatorial interpretation of Brieskorn's result.

Let $\mathscr{A} = \{H_1, \ldots, H_n\}$ be a *central* arrangement. By definition, the *OS-algebra* of $\mathscr{A}$ is

$$\text{(4.1)} \qquad \mathrm{A}^*(\mathscr{A}) = \bigwedge^* (e_1, \ldots, e_n) \Big/ \left(\partial e_{\mathscr{B}} \;\middle|\; \mathscr{B} \subset \mathscr{A} \text{ and } \operatorname{codim} \bigcap_{H \in \mathscr{B}} H < |\mathscr{B}| \right),$$

where $\bigwedge^* (e_1, \ldots, e_n)$ is the exterior algebra over $\mathbb{Z}$ on generators e_i in degree 1, and for $\mathscr{B} = \{H_{i_1}, \ldots, H_{i_r}\}$, $e_{\mathscr{B}} = e_{i_1} \cdots e_{i_r}$ and $\partial e_{\mathscr{B}} = \sum_q (-1)^{q-1} e_{i_1} \cdots \widehat{e}_{i_q} \cdots e_{i_r}$. There is then an isomorphism of graded algebras,

$$\text{(4.2)} \qquad \mathrm{OS}\colon H^*(X(\mathscr{A})) \cong \mathrm{A}^*(\mathscr{A}).$$

Under this identification, the basis $\{e_1, \ldots, e_n\}$ of $\mathrm{A}^1(\mathscr{A})$ is dual to the basis of $H_1(X(\mathscr{A}))$ given by the meridians of the hyperplanes; see [25]. With respect to a fixed ordering of the hyperplanes, a canonical basis for $\mathrm{A}^*(\mathscr{A})$ is the *no broken circuits* (or, **nbc**) basis; see [25].

There is another, closely related, graded algebra, $\bar{\mathrm{A}}^*(\mathscr{A})$, called the *quadratic Orlik-Solomon algebra*, defined as the quotient of $\bigwedge^* (e_1, \ldots, e_n)$ by relations of the form $\partial e_{\mathscr{B}}$, for all $\mathscr{B} \subset \mathscr{A}$ such that $\operatorname{codim} \bigcap_{H \in \mathscr{B}} H < |\mathscr{B}|$ and $|\mathscr{B}| = 3$; see [10, 33]. Clearly, the algebra $\mathrm{A}^*(\mathscr{A})$ is a quotient of $\bar{\mathrm{A}}^*(\mathscr{A})$, and the two algebras coincide up to degree 2. Denote by

$$\text{(4.3)} \qquad \pi^*_{\mathscr{A}}\colon \bar{\mathrm{A}}^*(\mathscr{A}) \to \mathrm{A}^*(\mathscr{A})$$

the canonical projection. Also denote by $P_{\mathscr{A}}(T)$ the Poincaré polynomial of $\mathrm{A}^*(\mathscr{A})$, and by $\bar{P}_{\mathscr{A}}(T)$ that of $\bar{\mathrm{A}}^*(\mathscr{A})$. It follows at once that $\bar{P}_{\mathscr{A}}(T) \succcurlyeq P_{\mathscr{A}}(T)$ (coefficientwise inequality).

4.4. *Supersolvable and Hypersolvable Arrangements.* Perhaps the best understood arrangements are the supersolvable (or, fiber-type) arrangements, introduced by Falk and Randell in [12]. A central arrangement $\mathscr{A}$ is called *supersolvable* if its intersection lattice is supersolvable, in the sense of Stanley [34]. For our purposes here, another (equivalent) combinatorial definition will be, however, more useful; see Definition 4.6. The standard example is the braid arrangement in $\mathbb{C}^\ell$, $\mathscr{B}_\ell = \{\ker(z_i - z_j)\}_{1 \leqslant i < j \leqslant \ell}$, with $L(\mathscr{B}_\ell) = \mathscr{P}_\ell$, the partition lattice, and $\pi_1(X(\mathscr{B}_\ell)) = P_\ell$, the pure braid group on ℓ strings. It follows from a theorem of Terao [35] and results in [12] that the complement of an arbitrary supersolvable arrangement is a $K(\pi, 1)$.

The class of hypersolvable arrangements actually motivated the framework for our Theorem 2.10. We start by reviewing the definition and basic properties of such arrangements.

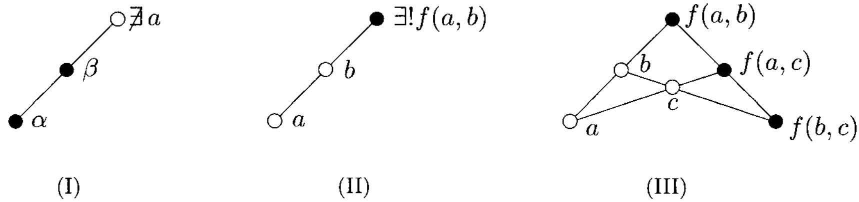

FIG. 1. Axioms for solvable extensions.

Let $\mathscr{A} = \{H_1, \ldots, H_n\}$ be a central arrangement in the complex vector space V. Denote also by $\mathscr{A} = \{\alpha_1, \ldots, \alpha_n\} \subset \mathbb{P}(V^*)$ its set of defining equations, viewed as points in the dual projective space. Let $\mathscr{B} \subset \mathscr{A}$ be a proper, non-empty sub-arrangement, and set $\overline{\mathscr{B}} := \mathscr{A} \setminus \mathscr{B}$. We say that $(\mathscr{A}, \mathscr{B})$ is a *solvable extension* (Fig. 1) if the following conditions are satisfied (see [20]):

(I) No point $a \in \overline{\mathscr{B}}$ sits on a projective line determined by $\alpha, \beta \in \mathscr{B}$.

(II) For every $a, b \in \overline{\mathscr{B}}$, $a \neq b$, there exists a point $\alpha \in \mathscr{B}$ on the line passing through a and b. (In the presence of condition (1), this point is uniquely determined, and will be denoted by $f(a, b)$.)

(III) For every distinct points $a, b, c \in \overline{\mathscr{B}}$, the three points $f(a, b)$, $f(a, c)$, and $f(b, c)$ are either equal or collinear.

Note that only two possibilities may occur: either $\operatorname{rank}(\mathscr{A}) = \operatorname{rank}(\mathscr{B}) + 1$ (*fibered case*), or $\operatorname{rank}(\mathscr{A}) = \operatorname{rank}(\mathscr{B})$ (*singular case*); see [20, Lemma 1.3(i)].

DEFINITION 4.5 [20]. The arrangement $\mathscr{A}$ is called *hypersolvable* if it has a *hypersolvable composition series*, i.e., an ascending chain of sub-arrangements, $\mathscr{A}_1 \subset \cdots \subset \mathscr{A}_i \subset \mathscr{A}_{i+1} \subset \cdots \subset \mathscr{A}_\ell = \mathscr{A}$, where $\operatorname{rank} \mathscr{A}_1 = 1$, and each extension $(\mathscr{A}_{i+1}, \mathscr{A}_i)$ is solvable.

The length of a composition series depends only on $\mathscr{A}$; it will be denoted by $\ell(\mathscr{A})$. Note that the property of being hypersolvable is purely combinatorial. In fact, given an arrangement $\mathscr{A}$, one can decide whether it is hypersolvable or not, only from the elements of rank one and two of $L(\mathscr{A})$, since the definitions only involve the collinearity relations in $\mathscr{A}$. The class of hypersolvable arrangements includes supersolvable arrangements, cones of generic arrangements (for which $\ell(\mathscr{A}) = |\mathscr{A}|$), and many others, see [20], and the examples in Sections 4.14, 5.6, and 6.8.

The connection between hypersolvable and supersolvable (or fiber-type) arrangements comes from the following fact, which is implicit in [20, Lemma 4.5] and is explicitly proved in [21, Proposition 1.3(i)]. If the solvable extension $(\mathscr{A}, \mathscr{B})$ is fibered, then there is a Serre fibration

$X(\mathscr{A}) \to X(\mathscr{B})$, with homotopy fiber $\mathbb{C}\setminus\{m \text{ points}\}$, where $m = |\overline{\mathscr{B}}|$. It follows from [21, Proposition 1.3] that the (topological) definition of fiber-type arrangements may be rephrased in hypersolvable terms, as follows.

DEFINITION 4.6 [12]. The arrangement $\mathscr{A}$ is *supersolvable* (or, *fiber-type*) if it has a *supersolvable composition series*, that is, a hypersolvable composition series as in Definition 4.5, for which all extensions are fibered.

We thus see that all fiber-type arrangements are hypersolvable. On the other hand, one knows from [20, Theorem D] that a hypersolvable arrangement $\mathscr{A}$ cannot be a $K(\pi, 1)$, unless $\mathscr{A}$ is fiber-type, which happens precisely when $\ell(\mathscr{A}) = \operatorname{rank}(\mathscr{A})$.

4.7. *Supersolvable Deformations.* Our basic tool for the topological study of hypersolvable arrangements is the following theorem, which puts together and organizes a number of known results.

THEOREM 4.8. *Let $\mathscr{A}$ be a hypersolvable arrangement, with composition series $\mathscr{A}_1 \subset \cdots \subset \mathscr{A}_\ell = \mathscr{A}$, and* exponents $d_i := |\mathscr{A}_i \setminus \mathscr{A}_{i-1}|$. *Then there exists a supersolvable arrangement $\hat{\mathscr{A}}$, called the* supersolvable deformation *of $\mathscr{A}$, such that:*

(1) *$\hat{\mathscr{A}}$ has a supersolvable composition series, $\hat{\mathscr{A}}_1 \subset \cdots \subset \hat{\mathscr{A}}_\ell = \hat{\mathscr{A}}$, with $|\hat{\mathscr{A}}_i| = |\mathscr{A}_i|$, for $1 \leqslant i \leqslant \ell$.*

(2) *$X(\hat{\mathscr{A}})$ sits atop a tower, $X(\hat{\mathscr{A}}_\ell) \xrightarrow{p_\ell} X(\hat{\mathscr{A}}_{\ell-1}) \to \cdots \to X(\hat{\mathscr{A}}_2) \xrightarrow{p_2} X(\hat{\mathscr{A}}_1) = \mathbb{C}^*$, of Serre fibrations, $p_i \colon X(\hat{\mathscr{A}}_i) \to X(\hat{\mathscr{A}}_{i-1})$, with fiber $\mathbb{C}\setminus\{d_i$ points$\}$, and monodromy $\rho_i \colon \pi_1(X(\hat{\mathscr{A}}_{i-1})) \to P_{d_i} \subset \operatorname{Aut}(F_{d_i})$.*

(3) *$X(\hat{\mathscr{A}})$ is a $K(\pi, 1)$ space. The fundamental group admits an iterated semidirect product decomposition, $\pi = F_{d_\ell} \rtimes_{\rho_{\ell-1}} \cdots \rtimes_{\rho_1} F_{d_1}$, which gives rise to an explicit minimal cell decomposition on $X(\hat{\mathscr{A}})$, and thus, to an explicit ε-minimal, free $\mathbb{Z}\pi$-resolution $0 \to C_\ell \xrightarrow{\partial_\ell} C_{\ell-1} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} \mathbb{Z} \to \mathbb{Z} \to 0$.*

(4) *The fundamental groups of $X(\mathscr{A})$ and $X(\hat{\mathscr{A}})$ are isomorphic. Moreover, we may choose a classifying map, $j \colon X(\mathscr{A}) \to X(\hat{\mathscr{A}})$, such that $j_* \colon H_1(X(\mathscr{A})) \to H_1(X(\hat{\mathscr{A}}))$ preserves the canonical bases given by the meridians.*

(5) *There is a canonical isomorphism,* $\overline{\mathrm{OS}}\colon H^*(X(\hat{\mathscr{A}})) \cong \bar{\mathrm{A}}^*(\mathscr{A})$. *Under this isomorphism, and the isomorphism* $\mathrm{OS}\colon H^*(X(\mathscr{A})) \cong \mathrm{A}^*(\mathscr{A})$, *the map $j^* \colon H^*(X(\hat{\mathscr{A}})) \to H^*(X(\mathscr{A}))$ corresponds to the canonical projection* $\pi^*_{\mathscr{A}} \colon \bar{\mathrm{A}}^*(\mathscr{A}) \to \mathrm{A}^*(\mathscr{A})$.

(6) *For each $q \geqslant 0$, there is a canonical identification $C_q =$* $\operatorname{Hom}(\bar{A}^q(\mathscr{A}), \mathbb{Z}) \otimes \mathbb{Z}\pi$.

(7) $\bar{P}_{\mathscr{A}}(T) = P_{\hat{\mathscr{A}}}(T) = \prod_{i=1}^{\ell} (1 + d_i T)$.

Proof. (1) The supersolvable arrangement $\hat{\mathscr{A}}$ is obtained from $\mathscr{A}$ by the deformation method introduced in [20], and refined in [21]. Part (1) follows from this deformation method, which proceeds inductively, using the given composition series of $\mathscr{A}$.

(2) Up to homotopy, we may view each $\hat{\mathscr{A}}_i$ as an arrangement in $\mathbb{C}^i$ and replace each map p_i by a bundle map, q_i, with the specified fiber (more precisely, by a linear fibration, admitting a section, see [12]). Moreover, the defining polynomials for $\hat{\mathscr{A}}_i$ may be written inductively as $f_1(z_1) = z_1$, and $f_i(z_1, \ldots, z_i) = f_{i-1}(z_1, \ldots, z_{i-1}) \cdot \prod_{k=1}^{d_i} (z_i - g_{i,k}(z_1, \ldots, z_{i-1}))$. Clearly, f_i / f_{i-1} is a completely solvable Weierstrass polynomial over $X(\hat{\mathscr{A}}_{i-1})$. Thus, by [5, Theorem 2.3], the monodromy of the bundle map q_i factors through the pure braid group P_{d_i}, acting on the free group F_{d_i} via the Artin representation.

(3) The fact that $X(\hat{\mathscr{A}})$ is a $K(\pi, 1)$ space, with fundamental group having the specified structure, follows from (2). The minimal cell structure on $X(\hat{\mathscr{A}})$ is determined by the iterated bundle structure (see [6, Sect. 1.3 and Proposition 3.3]). The corresponding π-equivariant chain complex of the universal cover, $(C_*(\tilde{X}(\hat{\mathscr{A}})), \partial_*)$ provides the required resolution (which can be computed explicitly by means of Fox calculus; see [6, Theorem 2.10]).

(4) The fact that $X(\mathscr{A})$ and $X(\hat{\mathscr{A}})$ have isomorphic fundamental groups was established in [20, Sect. 4.12] and [21, Proposition 3.6], by means of generic slice and isotopy arguments. Those arguments actually provide an isomorphism $\pi_1(X(\mathscr{A})) \xrightarrow{\simeq} \pi_1(X(\hat{\mathscr{A}}))$, whose abelianization respects the canonical bases.

(5) Since $\hat{\mathscr{A}}$ is supersolvable, $\mathrm{A}^*(\hat{\mathscr{A}}) \cong \bar{\mathrm{A}}^*(\hat{\mathscr{A}})$; see Falk [10] and Shelton and Yuzvinsky [33]. Moreover, Theorem 2.4 from [21] ensures that $\mathscr{A}$ and $\hat{\mathscr{A}}$ have the same collinearity relations, which implies that $\bar{\mathrm{A}}^*(\mathscr{A}) \cong \bar{\mathrm{A}}^*(\hat{\mathscr{A}})$. The canonical isomorphism in (5) is then given by

$$H^*(X(\hat{\mathscr{A}})) \cong \mathrm{A}^*(\hat{\mathscr{A}}) \cong \bar{\mathrm{A}}^*(\hat{\mathscr{A}}) \cong \bar{\mathrm{A}}^*(\mathscr{A}). \tag{4.4}$$

The identification of j^* with $\pi^*_{\mathscr{A}}$ follows from the fact that the basis $\{e_1, \ldots, e_n\}$ of A^1 is dual to the basis of H_1 given by the meridians.

(6) The identification for C_q is given by the isomorphism $\overline{\mathrm{OS}}$: $H^*(X(\hat{\mathscr{A}})) \cong \bar{\mathrm{A}}^*(\mathscr{A})$ and duality.

(7) The equality between the Poincaré polynomials of $\bar{\mathrm{A}}^*(\mathscr{A})$ and $\mathrm{A}^*(\hat{\mathscr{A}})$ follows from (5). The second equality follows from [12]. ∎

We record as a corollary the most important (for our purposes) consequence of the above theorem.

COROLLARY 4.9. *Let $\mathscr{A}$ be a hypersolvable arrangement, with supersolvable deformation $\hat{\mathscr{A}}$. Then $X(\hat{\mathscr{A}})$ is a $K(\pi, 1)$ space for $X(\mathscr{A})$. In particular, all hypersolvable complements, and their $K(\pi, 1)$ spaces, are minimal, with cohomology algebra generated in degree one.*

The following corollary shows that the order of π_1-connectivity of the complement of a hypersolvable arrangement is combinatorially determined (though π_1 itself is not *a priori* combinatorial).

COROLLARY 4.10. *Let $\mathscr{A}$ be a hypersolvable arrangement. Set $X = X(\mathscr{A})$, and $\pi = \pi_1(X)$. Let $p = p(X)$ be the order of π_1-connectivity of X. Then:*

(1) $p(X) = \sup\{k \mid P_{\mathscr{A}}(T) \equiv \bar{P}_{\mathscr{A}}(T) \bmod (T^{k+1})\}$.
(2) $p(X) \geqslant 2$.
(3) $p(X) = \infty \Leftrightarrow P_{\mathscr{A}}(T) = \bar{P}_{\mathscr{A}}(T) \Leftrightarrow \mathscr{A}$ *is supersolvable.*
(4) *If $p(X) < \infty$, then $\bar{P}_{\mathscr{A}}(T) - P_{\mathscr{A}}(T) \equiv c_{p+1}T^{p+1} \bmod (T^{p+2})$, where c_{p+1} is a* positive *integer.*

Proof. (1) Follows from Theorem 4.8, Parts (3)–(5).
(2) Follows from Remark 2.8 and Theorem 4.8, Parts (3), (4).
(3) Follows from (1) and [21, Proposition 3.4].
(4) Follows from (1) and the fact that $\bar{P}_{\mathscr{A}}(T) \succcurlyeq P_{\mathscr{A}}(T)$. ∎

4.11. *A Presentation for $\pi_p(X(\mathscr{A}))$.* We come now to the main result in this section. Together with Corollary 4.10(3), this result provides a complete proof of Theorem 1.4 from the Introduction.

Let $\mathscr{A}$ be a hypersolvable arrangement, with supersolvable deformation $\hat{\mathscr{A}}$, as in Theorem 4.8. Set $X = X(\mathscr{A})$, $\pi = \pi_1(X)$, and $Y = X(\hat{\mathscr{A}})$. Recall the split exact sequence $0 \to H_*(X) \xrightarrow{j_*} H_*(Y) \xrightarrow{\Pi_*} H_*(Y, X) \to 0$ from (2.2). By Theorem 4.8(5), we may identify the dual exact sequence with

$$0 \to \ker(\pi^*_{\mathscr{A}}) \xrightarrow{\iota^*} \bar{\mathrm{A}}^*(\mathscr{A}) \xrightarrow{\pi^*_{\mathscr{A}}} \mathrm{A}^*(\mathscr{A}) \to 0. \tag{4.5}$$

Consequently, the projection Π_* is the dual of the inclusion ι^*:

$$\Pi_* = (\iota^*)^\top. \tag{4.6}$$

THEOREM 4.12. *Let $\mathscr{A}$ be a hypersolvable arrangement, with complement $X = X(\mathscr{A})$ and fundamental group $\pi = \pi_1(X)$. Let (C_*, ∂_*) be the free $\mathbb{Z}\pi$-resolution of $\mathbb{Z}$ from* [6] *(as in Theorem* 4.8(3)*). Set $p = p(X)$. Then:*

(1) *X is aspherical* $\Leftrightarrow p = \infty$.

(2) *If $p < \infty$, then the first non-vanishing higher homotopy group of X is $\pi_p(X)$, which has the following (finite, ε-minimal) presentation as $\mathbb{Z}\pi$-module,*

(4.7)

$$\mathrm{Hom}(\bar{A}^{p+2}(\mathscr{A}), \mathbb{Z}) \otimes \mathbb{Z}\pi \xrightarrow{\Delta_p} \mathrm{Hom}(\ker(\pi_{\mathscr{A}}^{p+1}), \mathbb{Z}) \otimes \mathbb{Z}\pi \to \pi_p(X) \to 0,$$

where $\Delta_p = ((\iota^{p+1})^\top \otimes \mathrm{id}) \circ \partial_{p+2}$.

(3) *If $p < \infty$, then the group of π-coinvariants of $\pi_p(X)$ is free abelian, of rank*

$$c_{p+1} = \textit{coefficient of } T^{p+1} \quad \textit{in } \bar{P}_{\mathscr{A}}(T) - P_{\mathscr{A}}(T).$$

In particular, both p and the group $(\pi_p(X))_\pi$ are combinatorially determined.

Proof. (1) Follows from [20, Theorem D] and Corollary 4.10(3).

(2) Follows from Theorem 2.10 and Corollary 2.11, via Corollary 4.9, and the identifications from Theorem 4.8(6) and (4.6).

(3) Follows from Corollary 2.11 and Corollary 4.10, Parts (1) and (4). ∎

Remark 4.13. The presentation (4.7) involves two matrices: ∂_{p+2} and ι^{p+1}. The matrix ∂_{p+2} is not *a priori* combinatorial. Nevertheless, it depends only on the iterated semidirect product structure of $\pi = F_{d_\ell} \rtimes_{\rho_{\ell-1}} F_{d_{\ell-1}} \rtimes \cdots \rtimes_{\rho_1} F_{d_1}$. The matrix ι^{p+1}, on the other hand, is combinatorially determined.

4.14. *Comparison with Some Results of Randell.* A formula for the coinvariants of the first non-vanishing higher homotopy group, similar to our 4.12(3), was obtained by Randell, using different methods, in [28, Theorem 2 and Proposition 9], for the class of generic hyperplane sections (of rank $\geqslant 3$) of essential, aspherical arrangements. For an arrangement $\mathscr{A}$ in this class, $p(X(\mathscr{A})) = \mathrm{rank}(\mathscr{A}) - 1$, by results from [28]. Randell's class of arrangements and the class of hypersolvable arrangements have a similar behavior, from the point of view of the coinvariants of the first higher non-vanishing homotopy group. Nevertheless, the two classes are distinct, as the following examples show:

EXAMPLE 4.15. For $\ell \geqslant 5$, let $\mathscr{A}_\ell := \mathscr{B}_\ell \cup \{H\}$, where $\mathscr{B}_\ell = \{z_i - z_j = 0\}_{1 \leqslant i < j \leqslant \ell}$ and $H = \{z_1 + z_2 + z_3 - 3z_\ell = 0\}$. Each arrangement $\mathscr{A}_\ell$ is hypersolvable, of rank $\ell - 1$ and length ℓ. We claim that $p(X(\mathscr{A}_\ell)) = 2$. It follows that these arrangements cannot be (iterated) generic sections of essential, aspherical arrangements, since this would imply that $p(X(\mathscr{A}_\ell)) = \ell - 2$.

The claim may be verified by showing that rank $A^3(\mathscr{A}_\ell) <$ rank $\bar{A}^3(\mathscr{A}_\ell)$; see Corollary 4.10, Parts (1) and (2). Let $\mathscr{C} = \{H_1, H_2, H_3, H\}$, with $H_i = \{z_i - z_\ell = 0\}$, and let $\{e_1, e_2, e_3, e\}$ be the corresponding OS-generators. It is easy to check that rank $A^3(\mathscr{A}_\ell) \leqslant \operatorname{rank}(\bar{A}(\mathscr{B}_\ell) \otimes \bigwedge^* (e)/(\partial e_{\mathscr{C}}))^3$, and rank $\bar{A}^3(\mathscr{A}_\ell) = \operatorname{rank}(\bar{A}(\mathscr{B}_\ell) \otimes \bigwedge^* (e))^3$, directly from the definitions (see Section 4.3). Notice that $\{H_1, H_2, H_3\} \subset \mathscr{B}_\ell$ is a boolean subarrangement, hence $e_1 e_2 e_3$ is a non-zero element of $\bar{A}(\mathscr{B}_\ell)$ (use [25, Proposition 3.66]). We infer that $\partial e_{\mathscr{C}}$ is a non-zero element of $\bar{A}(\mathscr{B}_\ell) \otimes \bigwedge^* (e)$, whence the desired inequality.

EXAMPLE 4.16. Let $\mathscr{A}$ be an (iterated) generic section of an essential, aspherical arrangement $\mathscr{B}$ which is not hypersolvable. For example, take $\mathscr{B}$ to be the reflection arrangement of type D_n, with $n \geqslant 4$; see [20]. If $\operatorname{rank}(\mathscr{A}) \geqslant 3$, then necessarily $\mathscr{A}$ and $\mathscr{B}$ have the same collinearity relations, and therefore $\mathscr{A}$ cannot be hypersolvable.

There is, however, a certain overlap between the two classes. For instance, iterated generic sections (of rank $\geqslant 3$) of fiber-type arrangements are obviously hypersolvable. At the same time, every rank 3 hypersolvable arrangement of length 4 is a generic hyperplane section of a fiber-type arrangement of rank 4; see [21, Corollary 3.1].

5. ON THE STRUCTURE OF π_2 AS A $\mathbb{Z}\pi_1$-MODULE

We now analyze in more detail the structure of $\pi_2(X')$, viewed as a module over $\mathbb{Z}\pi_1(X')$, in the case when $X' = X(\mathscr{A})$ is the complement of an affine line arrangement whose cone $\mathscr{A} = \mathbf{c}\mathscr{A}'$ is hypersolvable.

5.1. *$K(\pi, 1)$ Tests.* Let $\mathscr{A}'$ be an arrangement of affine lines in $\mathbb{C}^2$. The complement $X' = X(\mathscr{A}')$ has the homotopy-type of a 2-complex, hence the only obstruction to X' being aspherical is the second homotopy group, $\pi_2(X')$. In [11], Falk gave several conditions (some sufficient, some necessary), for the vanishing of $\pi_2(X')$, providing a (partial) $K(\pi, 1)$-test for complexified line arrangements. This test is geometric in nature, involving Gersten–Stallings weight systems.

Another partial $K(\pi, 1)$-test, valid this time in all dimensions, but only for hypersolvable arrangements, was given in [20, Theorem D]. This test is purely combinatorial. Assuming the cone $\mathscr{A} = \mathbf{c}\mathscr{A}'$ is hypersolvable, it says that X' is aspherical if and only if $\ell(\mathscr{A}) = \operatorname{rank}(\mathscr{A})$.

None of these asphericity tests, though, gives a precise description of $\pi_2(X')$, viewed as a $\pi_1(X')$-module. Our machinery affords such a description, at least in the special case when $\mathscr{A}$ is hypersolvable.

5.2. *Affine Arrangements with Hypersolvable Cones.* We first describe the structure of the fundamental group of the complement of a deconed fiber-type arrangement.

LEMMA 5.3. *Let $\mathscr{A}$ be a supersolvable arrangement, with composition series $\mathscr{A}_1 \subset \cdots \subset \mathscr{A}_\ell$, and let $F_{d_\ell} \rtimes_{\rho_{\ell-1}} F_{d_{\ell-1}} \rtimes \cdots \rtimes_{\rho_2} F_{d_2} \rtimes_{\rho_1} F_1$ be the corresponding iterated semidirect product decomposition of $\pi_1(X(\mathscr{A}))$. If $\mathbf{d}\mathscr{A}$ is a decone of $\mathscr{A}$, then $\pi_1(X(\mathbf{d}\mathscr{A})) = F_{d_\ell} \rtimes_{\rho_{\ell-1}} \cdots \rtimes_{\rho_2} F_{d_2}$.*

Proof. Recall from the proof of Theorem 4.8(2) that $\mathscr{A}$ has defining polynomial of the form $f_{\mathscr{A}} = f_1 f_2 \cdots f_\ell$, where $f_1(z) = z_1$, and f_i/f_{i-1} is a completely solvable Weierstrass polynomial over $X(\mathscr{A}_{i-1})$. The decone $\mathbf{d}\mathscr{A}$, obtained by setting $z_1 = 1$, has defining polynomial $f_{\mathbf{d}\mathscr{A}}(z_2, \ldots, z_\ell) = f_2(1, z_2) \cdots f_\ell(1, z_2, \ldots, z_\ell)$. The result follows at once. ∎

Consider an arbitrary affine arrangement $\mathscr{A}'$. By the general results from Sections 4.1 and 4.3, we know that the complement $X(\mathscr{A}')$ is minimal, with cohomology generated in degree one. Now assume $\mathscr{A} = \mathbf{c}\mathscr{A}'$ is hypersolvable and denote by $\hat{\mathscr{A}}$ the supersolvable deformation of $\mathscr{A}$. We also know from Theorem 4.8 (and Corollary 4.9) that $X(\hat{\mathscr{A}}) = K(\pi_1(X(\mathscr{A})), 1)$ is minimal, with cohomology algebra generated in degree one.

Set $\mathbf{d}\hat{\mathscr{A}} = \hat{\mathscr{A}}'$. The arguments from [20, 21] show that $\pi_1(X(\mathscr{A}')) = \pi_1(X(\hat{\mathscr{A}}'))$. Consequently, $X(\hat{\mathscr{A}}') = K(\pi_1(X(\mathscr{A}')), 1)$ is also minimal (by [6], or by [8, 29]), with cohomology generated in degree 1. Moreover, the previous lemma implies that $\pi' = \pi_1(X(\mathscr{A}'))$ is an iterated semidirect product of free groups with all monodromy actions trivial in homology, and so, by [6], there is an explicit finite, free, ε-minimal $\mathbb{Z}\pi'$-resolution (C_*, ∂_*) of $\mathbb{Z}$ (see Example 2.4).

We are now ready to formulate the main result of this section.

THEOREM 5.4. *Let $\mathscr{A}'$ be an affine line arrangement in $\mathbb{C}^2$, such that $\mathscr{A} = \mathbf{c}\mathscr{A}'$ is hypersolvable. Set $\ell = \ell(\mathscr{A})$, $X' = X(\mathscr{A}')$, $\pi' = \pi_1(X')$, and $p = p(X')$. Then:*

(1) *X' is aspherical $\Leftrightarrow \ell \leqslant 3 \Leftrightarrow p \neq 2$.*

(2) *If $\ell > 3$, then $\pi_2(X')$ is non-trivial, and admits the following finite, free, ε-minimal $\mathbb{Z}\pi'$-resolution,*

$$0 \to C_{\ell-1} \xrightarrow{\partial_{\ell-1}} C_{\ell-2} \to \cdots \to C_4 \xrightarrow{\partial_4} C_3 \to \pi_2(X') \to 0, \tag{5.1}$$

where (C_*, ∂_*) is the free $\mathbb{Z}\pi'$-resolution of $\mathbb{Z}$ from [6].

Proof. (1) The space X' is aspherical if and only if $\ell \leqslant 3$, by [20, Theorem D] (since $\mathscr{A}$ is fiber-type, if $\ell \leqslant 3$). If $p \neq 2$, then necessarily $p > 2$ (by Remark 2.8), whence $\pi_2(X') = 0$ (by Theorem 2.10(1)), and so X' must be aspherical. Conversely, if $p = 2$, then $\pi_2(X')$ must be non-zero (use Corollary 2.11).

(2) If $\ell > 3$, we know from Part (1) that $p = 2$, and then everything follows from Theorem 2.10(2), via Remark 2.12, and the preceding discussion. ∎

Remark 5.5. The resolution (5.1) may have arbitrary length. Indeed, for each $\ell \geqslant 1$, there exists a hypersolvable arrangement $\mathscr{A}$ in $\mathbb{C}^3$ with $\ell(\mathscr{A}) = \ell$; see [20, Sect. 1].

5.6. *Structure of π_2 of a Hypersolvable Line Arrangement Complement.* The group of π'-coinvariants of $\pi_2(X')$ is very simple to describe: By Theorem 5.4, it is free abelian, of rank $b_3(X(\hat{\mathscr{A}}')) = b_3(\pi')$. On the other hand, the following result shows that $\pi_2(X')$, when non-trivial, has a fairly complicated structure as a $\mathbb{Z}\pi'$-module.

THEOREM 5.7. *Let $\mathscr{A}'$ be an affine line arrangement in $\mathbb{C}^2$ such that $\mathscr{A} = \mathbf{c}\mathscr{A}'$ is hypersolvable. Let ℓ be the length of $\mathscr{A}$, and $\{1 = d_1, d_2, ..., d_\ell\}$ the exponents. Set $X' = X(\mathscr{A}')$ and $\pi' = \pi_1(X')$. Assume $\ell > 3$ (so that $\pi_2(X') \neq 0$). Then:*

(1) *$\pi_2(X')$ is a projective $\mathbb{Z}\pi'$-module if and only if $\ell = 4$. In that case, $\pi_2(X')$ is free, with rank equal to $b_3(\pi') = d_2 d_3 d_4$.*

(2) *$\pi_2(X')$ is neither finitely generated as an abelian group, nor nilpotent as a $\mathbb{Z}\pi'$-module.*

Proof. (1) From resolution (5.1), we see that $\pi_2(X')$ is isomorphic to $\operatorname{coker}(\partial_4) = \operatorname{im}(\partial_3) \subset C_2$. If $\ell = 4$, then $\pi_2(X') = C_3$ is a free $\mathbb{Z}\pi'$-module, with rank $b_3(\pi')$ given by Theorem 4.8(7). If $\ell > 4$, then $\pi_2(X')$ is not projective, by the minimality of (5.1).

(2) Note first that the I-adic filtration of the group algebra $\mathbb{Q}\pi'$ is Hausdorff, in the sense that $\bigcap_{k \geqslant 0} I^k = 0$, where $I = \ker(\varepsilon\colon \mathbb{Q}\pi' \to \mathbb{Q})$ is the augmentation ideal. This follows from the fact that π' is an iterated semi-direct product of free groups, where all homology monodromy actions are trivial (cf. Lemma 5.3 and the discussion following it); therefore, π' is residually torsion-free nilpotent (see [13]), and so the I-adic filtration of $\mathbb{Q}\pi'$ must be Hausdorff (see [4]). It follows that the I-adic filtration of the free $\mathbb{Q}\pi'$-module $C_2 \otimes \mathbb{Q}$ is also Hausdorff.

Assume now that either $\pi_2(X')$ is finitely generated as an abelian group, or nilpotent as a $\mathbb{Z}\pi'$-module. It follows that $I^k \cdot \pi_2(X') \otimes \mathbb{Q} = 0$, for some

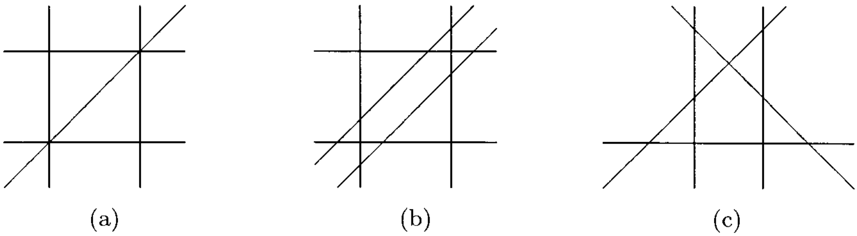

FIG. 2. Some line arrangements whose cones are hypersolvable.

$k \geqslant 0$, and thus $\pi_2(X') \otimes \mathbb{Q}$ must be a nilpotent, non-trivial $\mathbb{Q}\pi'$-module. This implies that $(g_1 - 1) \cdots (g_k - 1) \cdot b = 0$, for some $g_1, ..., g_k \in \pi' \setminus \{1\}$, and $b \in \mathbb{Q}\pi' \setminus \{0\}$. On the other hand, $\mathbb{Q}\pi'$ has no zero-divisors, since π' is residually torsion-free nilpotent (see [27]). This gives the desired contradiction, proving (2). ∎

EXAMPLE 5.8. Let $\mathscr{A}'$ be the affine line arrangement from Fig. 2a, with defining polynomial $f_{\mathscr{A}'} = z_1 z_2 (z_1 - 1)(z_2 - 1)(z_2 - z_1)$. Then $\mathscr{A} = \mathbf{c}\mathscr{A}'$ is an essential 3-slice of the braid arrangement $\mathscr{B}_4$. Hence, $\mathscr{A}$ is supersolvable, with length $\ell = 3$, and exponents $\{1, 2, 3\}$. We then have $\pi' = F_3 \rtimes F_2$, and $X' = K(\pi', 1)$.

EXAMPLE 5.9. Let $\mathscr{A}'$ be the arrangement from Fig. 2b, with defining polynomial $f_{\mathscr{A}'} = (z_1 - 1)(z_1 + 1)(2z_1 - 2z_2 - 1)(2z_1 - 2z_2 + 1)(3z_1 - 6z_2 - 1)$ $(3z_1 - 6z_2 + 1)$. Then $\mathscr{A} = \mathbf{c}\mathscr{A}'$ is the arrangement from Fan [15, Sect. 3.I]. It is readily seen that $\mathscr{A}$ is hypersolvable, with $\ell = 4$, and $\exp(\mathscr{A}) = \{1, 2, 2, 2\}$. We then have $\pi' = F_2 \times F_2 \times F_2$, and $\pi_2(X') = (\mathbb{Z}\pi')^8$. Notice that $V_1(\pi_2(X')) = (\mathbb{C}^*)^6$.

EXAMPLE 5.10. Let $\mathscr{A}'$ be the arrangement from Fig. 2c, with defining polynomial $f_{\mathscr{A}'} = z_1 z_2 (z_1 - 1)(z_2 - z_1 - 1)(z_2 + z_1 - 2)$. Then $\mathscr{A} = \mathbf{c}\mathscr{A}'$ is hypersolvable, with $\ell = 5$, and $\exp(\mathscr{A}) = \{1, 1, 1, 1, 2\}$. We then have $\pi' = \mathbb{Z}^3 \times F_2 = \langle x_1, x_2, x_3 \mid [x_i, x_j] \rangle \times \langle x_4, x_5 \rangle$, and $\pi_2(X')$ fits into the following exact sequence of $\mathbb{Z}\pi'$-modules:

$$0 \to (\mathbb{Z}\pi')^2 \xrightarrow{\left(\begin{smallmatrix} 1-x_4 & 1-x_3 & 0 & x_2-1 & 0 & 1-x_1 & 0 \\ 1-x_5 & 0 & 1-x_3 & 0 & x_2-1 & 0 & 1-x_1 \end{smallmatrix}\right)} (\mathbb{Z}\pi')^7 \to \pi_2(X') \to 0.$$

Notice that $V_1(\pi_2(X')) = \{t \in (\mathbb{C}^*)^5 \mid t_1 = t_2 = t_3 = 1\}$ is a 2-dimensional subtorus. That $M = \pi_2(X')$ is not nilpotent can be seen directly, as follows. Let $\tilde{M} = \pi_2(X') \otimes_{\mathbb{Z}\pi'} \mathbb{Z}\mathbb{Z}^5$, and let $\operatorname{gr} \tilde{M}$ be the associated graded module (with respect to the I-adic filtration). From the above presentation, we may easily compute its Hilbert series: $\mathrm{Hilb}(\mathrm{gr}\,\widetilde{M}, t) = (7-2t)/(1-t)^5$. Since this series is not a polynomial, $\widetilde{M}$ is not nilpotent, and so M isn't, either.

6. GRAPHIC ARRANGEMENTS

In this section, we apply our methods to graphic arrangements. We start by giving a graph-theoretic characterization of hypersolvable arrangements within this class. We then show how our algorithm for computing π_2 of the complement becomes purely combinatorial, in the case of graphic arrangements associated to graphs without 3-cycles.

6.1. *Graphs and Arrangements.* Let $G = (\mathscr{V}, \mathscr{E})$ be a non-empty subgraph of the complete graph on a finite set of vertices $\mathscr{V}$. Assume that there are no isolated vertices in the graph, so that the set of edges $\mathscr{E}$ determines G. All graphs considered in this section will be of this type.

Let $\mathscr{V} = \{1, ..., m\}$. The *graphic arrangement* associated to $G = (\mathscr{V}, \mathscr{E})$ is the arrangement in $\mathbb{C}^m$ given by $\mathscr{A}_G = \{\ker(z_i - z_j) \mid \{i, j\} \in \mathscr{E}\}$, see [25]. For each edge $e = \{i, j\}$, we will denote by $H_e := \ker(z_i - z_j)$ the corresponding hyperplane of $\mathscr{A}_G$.

Clearly, an arrangement is graphic if and only if it is a sub-arrangement of a braid arrangement. For example, if G is the complete graph on m vertices, then $\mathscr{A}_G = \mathscr{B}_m$, the braid arrangement in $\mathbb{C}^m$. If G is a diagram of type A_m, then $\mathscr{A}_G$ is a Boolean arrangement. If G is an m-cycle, then $\mathscr{A}_G$ is a generic arrangement.

Many of the usual invariants associated to $\mathscr{A}_G$ can be computed directly from G. For example, $P_{\mathscr{A}_G}(T) = (-T)^m \chi_G(-T^{-1})$, where $\chi_G(T)$ is the chromatic polynomial of G, see [25]. Also, an **nbc**-basis for $\mathrm{A}^*(\mathscr{A}_G)$ corresponds to an **nbc**-basis for G, as follows. Fix an ordering on the edges, $\mathscr{E}_G = \{e_1 < \cdots < e_n\}$, and denote by α_i the defining equation of H_{e_i}. Then, $\{\alpha_{i_1}, ..., \alpha_{i_r}\}$ is minimally dependent if and only if $\{e_{i_1}, ..., e_{i_r}\}$ is an r-cycle of G. Deleting the highest edge from this cycle yields a broken circuit. The resulting **nbc**-basis for $\mathrm{A}^*(\mathscr{A}_G)$ is given by

(6.1)

$$\{e_K \mid K \text{ is a subgraph of } G \text{ which does not contain any broken circuit of } G\},$$

where $e_K := e_{i_1} \cdots e_{i_s} \in \bigwedge^s (e_1, ..., e_n)$, if $\mathscr{E}_K = \{e_{i_1}, ..., e_{i_s}\}$.

6.2. *Supersolvable and Hypersolvable Graphs.* The following results, due to Stanley [34] and Fulkerson and Gross [16], tell us how to (easily) recognize supersolvable arrangements within the class of graphic arrangements.

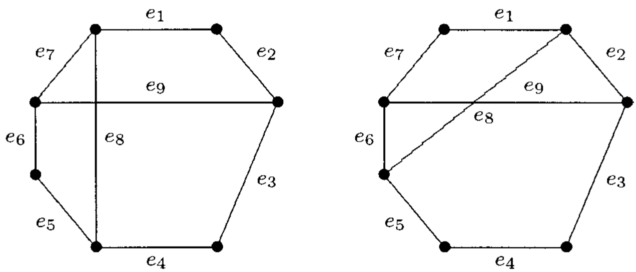

FIG. 3. The graphs G_1 and G_2.

THEOREM 6.3. *Let $\mathscr{A}_G$ be a graphic arrangement. Then:*

(Stanley [34]) *$\mathscr{A}_G$ is supersolvable if and only if the graph G is supersolvable; i.e., it has a* supersolvable composition series *of induced subgraphs, $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_\ell = G$, such that:*

(a) *for each $1 \leqslant i \leqslant \ell$, there is a single vertex in $G_i \setminus G_{i-1}$, say, v_i;*

(b) *the subgraph of G_i induced by v_i and its neighbors in G_i is complete.*

(Fulkerson and Gross [16]) *G has a supersolvable composition series if and only if every cycle in G of length greater than 3 has a chord.*

As a simple example, consider the two graphic arrangements given by the graphs in Fig. 3. Neither graph has a 3-cycle; each graph has two 4-cycles, with no chords. Hence, the two arrangements are not supersolvable.

Stanley's supersolvable test from Theorem 6.3 has a hypersolvable analogue. To state it, we first need a definition.

DEFINITION 6.4. A pair of graphs, (G, K), is called a *solvable extension* if K is a subgraph of G, with $\varnothing \neq \mathscr{E}_K \subsetneq \mathscr{E}_G$, and:

(1) There is no 3-cycle in G having two edges from $\mathscr{E}_K$ and one edge from $\mathscr{E}_G \setminus \mathscr{E}_K$.

(2) Either $\mathscr{E}_G \setminus \mathscr{E}_K = \{e\}$, and both endpoints of e are not in $\mathscr{V}_K$, or there exist distinct vertices, $\{v_1, ..., v_k, v\} \subset \mathscr{V}_G$, with $\{v_1, ..., v_k\} \subset \mathscr{V}_K$, such that:

(a) K contains the complete graph on $\{v_1, ..., v_k\}$, and

(b) $\mathscr{E}_G \setminus \mathscr{E}_K = \{\{v, v_s\} \mid 1 \leqslant s \leqslant k\}$.

LEMMA 6.5. *An extension of graphs, (G, K), is solvable if and only if the corresponding extension of graphic arrangements, $(\mathscr{A}_G, \mathscr{A}_K)$, is solvable.*

Proof. Let $e_1 = \{i_1, j_1\}$, $e_2 = \{i_2, j_2\}$, $e_3 = \{i_3, j_3\}$ be three distinct edges of G. Notice that the corresponding defining equations, $\{z_{i_r} - z_{j_r} \mid 1 \leqslant r \leqslant 3\}$, viewed as points in $\mathbb{P}(\mathbb{C}^{m*})$, are collinear if and only if

$\{e_r \mid 1 \leqslant r \leqslant 3\}$ are the edges of a 3-cycle. Using this remark, it is a straightforward exercise to translate conditions (I)–(III) from Section 4.4 into conditions (1) and (2) from Definition 6.4. ∎

DEFINITION 6.6. A graph G is called *hypersolvable* if it has a *hypersolvable composition series*, i.e., a chain of subgraphs, $G_1 \subset \cdots \subset G_i \subset G_{i+1} \subset \cdots \subset G_\ell = G$, such that G_1 has a single edge, and (G_{i+1}, G_i) is a solvable extension, for $i = 1, \ldots, \ell - 1$.

The class of hypersolvable graphs contains the supersolvable (or chordal) graphs described in Theorem 6.3, and many others. For example, the graphs from Fig. 3 are both hypersolvable, with composition series $G_i = \{e_1, \ldots, e_i\}$, $1 \leqslant i \leqslant 9$, but not supersolvable.

PROPOSITION 6.7. *A graph G is hypersolvable if and only if the graphic arrangement $\mathscr{A}_G$ is hypersolvable.*

Proof. Clearly, $G_1 \subset \cdots \subset G_\ell$ is a composition series for G if and only if $\mathscr{A}_{G_1} \subset \cdots \subset \mathscr{A}_{G_\ell}$ is a composition series for $\mathscr{A}_G$. ∎

6.8. *Graphs with No 3-Cycles.* We now analyze in more detail a very simple example: arrangements corresponding to graphs without 3-cycles, and their second homotopy group. It turns out that all such arrangements are hypersolvable, and that the algorithm for computing π_2 of their complement is purely combinatorial.

PROPOSITION 6.9. *Let G be a graph with no 3-cycles, and with edges $\{e_1, \ldots, e_n\}$. Then:*

(1) *The graph G is hypersolvable, with composition series $G_i = \{e_1, \ldots, e_i\}$, $1 \leqslant i \leqslant n$.*

(2) *The arrangement $\mathscr{A} = \mathscr{A}_G$ is hypersolvable, with length n and exponents $\{1, \ldots, 1\}$.*

(3) $\bar{\mathrm{A}}^*(\mathscr{A}) = \bigwedge^* (e_1, \ldots, e_n)$.

(4) $\pi_1(X(\mathscr{A})) = \mathbb{Z}^n$.

Proof. There are no collinearity relations among the defining equations of $\mathscr{A}$, since G has no 3-cycles. Part (1) then follows from Definitions 6.4 and 6.6, Part (2) from Proposition 6.7 and (1), and Part (3) from the definition of the quadratic OS-algebra.

Now set $m = \#\{\text{vertices of } G\}$. If $m \leqslant 3$, then Part (4) is trivially verified. If $m > 3$, we may take a generic 3-plane P in $\mathbb{C}^m$ with the property that $\pi_1(X(\mathscr{A})) = \pi_1(X(\mathscr{A} \cap P))$ (by [17]), and such that $\mathscr{A}$ and $\mathscr{A} \cap P$ have the same collinearity relations. The decone of the arrangement $\mathscr{A} \cap P$ is thus generic, and so $\pi_1(X(\mathscr{A} \cap P)) = \mathbb{Z}^n$ (by [18]). ∎

THEOREM 6.10. *Let G be a graph with no 3-cycles. Let $\mathscr{E} = \{e_1, \ldots, e_n\}$ be the set of edges of G, and $\mathscr{S}$ the set of 4-cycles, consisting of cycles $S_{abcd} = \{e_a, e_b, e_c, e_d\}$ with $a < b < c < d$. Set $\mathscr{A} = \mathscr{A}_G$ and $X = X(\mathscr{A})$, and identify $\pi_1(X)$ with $\mathbb{Z}^n$, as in Proposition* 6.9(4).

- *If $\mathscr{S} = \varnothing$, then $\pi_2(X) = 0$.*
- *If $\mathscr{S} \neq \varnothing$, then $\pi_2(X)$ has the following ε-minimal presentation as $\mathbb{Z}\mathbb{Z}^n$-module:*

$$\bigwedge^4 (\mathbb{Z}^n) \otimes \mathbb{Z}\mathbb{Z}^n \xrightarrow{\Delta_2 = (\Pi_3 \otimes \mathrm{id}) \circ d_4} \mathbb{Z}[\mathscr{S}] \otimes \mathbb{Z}\mathbb{Z}^n \to \pi_2(X) \to 0, \tag{6.2}$$

where:

- $d_4 \colon \bigwedge^4 (\mathbb{Z}^n) \otimes \mathbb{Z}\mathbb{Z}^n \to \bigwedge^3 (\mathbb{Z}^n) \otimes \mathbb{Z}\mathbb{Z}^n$ *is the Koszul differential from Example* 2.3,
- $\mathbb{Z}[\mathscr{S}]$ *is the free abelian group generated by* $\mathscr{S}$,
- $\Pi_3^\top : \mathbb{Z}[\mathscr{S}] \to \bigwedge^3 (\mathbb{Z}^n)$ *is the dual of* Π_3, *given by*

$$\Pi_3^\top (S_{abcd}) = e_b e_c e_d - e_a e_c e_d + e_a e_b e_d - e_a e_b e_c. \tag{6.3}$$

Consequently, the $\mathbb{Z}\mathbb{Z}^n$-module $\pi_2(X)$ is combinatorially determined (directly from the graph G), and its group of coinvariants equals $\mathbb{Z}[\mathscr{S}]$. In particular, $\pi_2(X) \neq 0$.

Proof. The identification $\pi_1(X) = \mathbb{Z}^n$ is given by Proposition 6.9(4). A quick inspection of the construction of the **nbc**-basis in degree 3 reveals that: $\mathscr{S} = \varnothing \Leftrightarrow b_3(X) = b_3(T^n)$. Since $\mathscr{A}_G$ is hypersolvable (cf. Proposition 6.9(2)), we may apply Theorem 4.12. All assertions follow directly from that theorem, except formula (6.3), which we now verify.

From (4.5), (4.6), and Proposition 6.9(3), we have a split exact sequence

$$0 \to \ker(\pi^3_{\mathscr{A}}) \xrightarrow{\Pi_3^\top} \bigwedge^3 (\mathbb{Z}^n) \xrightarrow{\pi^3_{\mathscr{A}}} \mathrm{A}^3(\mathscr{A}) \to 0.$$

Now recall from Section 4.3 the construction of the OS-algebra of $\mathscr{A}$, together with the graphic counterpart from Section 6.1. It follows that $\ker(\pi^3_{\mathscr{A}})$ is free abelian, with basis $\{\partial e_S \mid S \in \mathscr{S}\}$, and we are done. ∎

EXAMPLE 6.11. Consider the hypersolvable graphs G_1 and G_2 in Fig. 3, and let $\mathscr{A}_1$ and $\mathscr{A}_2$ be the corresponding graphic arrangements in $\mathbb{C}^7$, with complements X_1 and X_2. Each graph has no 3-cycles, but exactly two 4-cycles. Hence, both complements have $\pi_1 = \mathbb{Z}^9$ (with generators x_i

corresponding to edges e_i), $b_2 = \binom{9}{2} = 36$, and $b_3 = \binom{9}{3} - 2 = 82$. The spaces X_1 and X_2 have distinct homotopy types, as can be seen by computing their b_4's. We now show that, in fact, X_1 and X_2 have distinct homotopy 2-types, by analyzing their π_2's.

The $\mathbb{Z}\mathbb{Z}^9$-modules $\pi_2(X_1)$ and $\pi_2(X_2)$ have presentation matrices

$$\Delta^{(1)} = \begin{pmatrix} x_3-1 & x_4-1 & x_5-1 & x_6-1 & x_8-1 & x_1+x_2+x_7+x_9-4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix.$$
$$\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ x_3-1 & x_4-1 & x_5-1 & x_6-1 & x_8-1 & x_5+x_6+x_7+x_8-4 \end{matrix}\Bigg)^{\top},$$

$$\Delta^{(2)} = \Bigg(\begin{matrix} 1-x_6 & x_3-1 & x_4-1 & x_5-1 & x_8-1 & x_1+x_2+x_7+x_9-4 \\ x_2-1 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$
$$\begin{matrix} 0 & 0 & 0 & 0 & 0 \\ x_3-1 & x_4-1 & x_5-1 & x_9-1 & x_1+x_6+x_7+x_8-4 \end{matrix}\Bigg)^{\top},$$

respectively. Both modules have coinvariants equal to $\mathbb{Z}^2$. On the other hand, computing 2×2 minors of the matrices $\Delta^{(1)}$ and $\Delta^{(2)}$ gives the Fitting varieties

$$\begin{aligned} V_1(\pi_2(X_1)) &= \{t_1+t_2+t_7+t_9=4,\ t_3=t_4=t_5=t_6=t_8=1\} \\ &\quad \cup \{t_3=t_4=t_5=t_6=t_7=t_8=1\}, \\ V_1(\pi_2(X_2)) &= \{t_1+t_2+t_7+t_9=4,\ t_3=t_4=t_5=t_6=t_8=1\} \\ &\quad \cup \{t_1+t_6+t_7+t_8=4,\ t_2=t_3=t_4=t_5=t_9=1\} \\ &\quad \cup \{t_1+t_2+t_7=3,\ t_2=t_6,\ t_3=t_4=t_5=t_8=t_9=1\}. \end{aligned}$$

Since the two varieties have a different number of components, Proposition 3.3 shows that $\pi_2(X_1) \not\cong \pi_2(X_2)$, when viewed as $\mathbb{Z}\mathbb{Z}^9$-modules. Alternatively, one may distinguish the two modules by means of the Hilbert series of their associated graded modules:

$$\mathrm{Hilb}(\mathrm{gr}\,\pi_2(X_1)) = \frac{2}{(1-t)^3}, \qquad \mathrm{Hilb}(\mathrm{gr}\,\pi_2(X_2)) = \frac{(2-t)(1+t)}{(1-t)^3}.$$

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