PROBLEMS FOR JUNE 17 – CATEGORIFICATION USING PREPROJECTIVE ALGEBRAS

1. Consider the trivial quiver, with a single vertex and no arrows. Let $[n]$ be the unique n-dimensional representation. Let χ be the cluster character, so this sends modules for the quiver path algebra to Laurent polynomials in one variable (call it x).

1.a Compute $\chi_{[n]}$ by hand.

1.b Verify that $\chi_{[n]} = \chi_{[1]}^n$

2. Let Q be the quiver $\bullet \rightrightarrows \bullet$.

2.a Let M be the representation with dimension vector $(2, 2)$, where one map is $\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ and the other map is $\left(\begin{smallmatrix}1 & 0\\ 0 & 2\end{smallmatrix}\right)$. Compute the corresponding cluster character χ_M.

2.b Repeat the previous example, with the same dimension vector $(2, 2)$, but with maps $\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$

3. Let R be the preprojective algebra of type A_3.

3.a Which three indecomposable representations of R correspond to frozen variables?

3.b Give three indecomposable representations of R which correspond to a cluster.

3.c By repeated mutation from that cluster, find the entire cluster complex for R.

Note: You should be able to reuse some of your previous work.

4. Let $S = \bigoplus_{i=1}^n \mathbb{C}$. If Q is a quiver with n vertices, show that the path algebra $\mathbb{C}[Q]$ is isomorphic to the tensor algebra of some S-bimodule. Conversely, show that the tensor algebra of any S-bimodule is isomorphic to the path algebra of a quiver with n vertices.

5. The point of this problem is to explain how "rigid" representations get their name. Let A be a k-algebra. Let $D = k[\epsilon]/\epsilon^2$ and let $A' = A[\epsilon]/\epsilon^2$. (The square brackets mean that ϵ is adjoined in A as a central element.) Let N be an A'-module.

5.a (Problem/Definition) Consider the map $N/\epsilon N \to \epsilon N$ given by multiplication by ϵ. Show that N is a flat D-module if and only if this map is an isomorphism or, if you haven't heard of flatness before, take this as a definition.

Let M be an A-module. A ***deformation*** of M is a flat A'-module N equipped with an isomorphism $N/\epsilon N \cong M$. Clearly, $M \otimes_k D$ is always a deformation of M; we call a deformation ***trivial*** if it is isomorphic to this one

5.b Show that M has nontrivial deformations if and only if $\mathrm{Ext}^1_A(M, M) \neq 0$.

So M is rigid if and only if it has no nontrivial deformations.

6. This problem returns from Tuesday's set.

6.a List the isomorphism classes of indecomposable representations for the preprojective algebra of type A_3. (Hint: There are 12 of them.)

6.b For which pairs of representations is there nontrivial Ext^1? Draw the graph with vertices for the indecomposable representations and with edges where there are nontrivial extensions. (Hint: There are 3 isolated vertices.) Note that this is the quiver that appeared near the end of Sarah's talk.

7.a What are the isomorphism classes of indecomposable representations of the path algebra of $\bullet \to \bullet \to \bullet$?

7.b Which of them have nontrivial extensions between them?

7.c Repeat parts (a) and (b) for $\bullet \to \bullet \leftarrow \bullet$.