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On rational $K[\pi, 1]$ spaces and Koszul algebras

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Abstract

The main result of the paper is that a formal topological space X is a rational $K[\pi, 1]$ space if and only if the graded algebra $H^*(X, Q)$ is Koszul. This implies the lower central series (LCS) formula for a formal rational $K[\pi, 1]$ space X:

$$P(X, -t) = \prod_{n \geq 1} (1 - t^n)^{\phi_n}.$$

Here $\phi_n = \text{rank}(\Gamma_n/\Gamma_{n+1})$, where $\{\Gamma_n\}_{n \geq 1}$ is the lower central series of the fundamental group $\pi_1(X)$, and $P(X, t)$ is the Poincaré polynomial of X. These results are applied to the complements of complex hyperplane arrangements that are known to be formal spaces. In particular, it is proved that the LCS formula implies the rational $K[\pi, 1]$ property for arrangements in C^3.

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1. Introduction

Let X be a connected topological space with finite Betti numbers. Then X is said to be a *rational* $K[\pi, 1]$ *space* if its Q-completion $\mathbf{Q}_\infty(X)$ constructed in [8] is aspheric, that is $\pi_n \mathbf{Q}_\infty(X) = 0$ for all $n > 1$.

Let A be a connected graded algebra of finite type over a field F. Then A is said to be a *Koszul algebra* if $\text{Ext}_A^{p,q}(F, F) = 0$, for every $p \neq q$ (see Definition 2.1 for details).

Our main result (Theorem 5.1) establishes an equivalence between the topological rational $K[\pi, 1]$ property of X and the algebraic Koszul property of $H^*(X, \mathbf{Q})$, in the

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case of a *formal* space X. The important class of formal spaces was introduced in [30]. One says that X is formal if $\mathbf{Q}_\infty(X)$ depends only on the cohomology algebra $H^*(X, \mathbf{Q})$, see Section 5 for details. We then derive (in Proposition 5.2) that the Koszulness of $H^*(X, \mathbf{Q})$ implies that $\mathbf{Q}_\infty(X)$ is aspheric, for an arbitrary space X. We also derive (in Corollary 5.3) the following *lower central series (LCS) formula* for a formal rational $K[\pi, 1]$ space X:

$$P(X, -t) = \prod_{n \geq 1} (1 - t^n)^{\phi_n}.$$

Here $\phi_n = \mathrm{rank}(\Gamma_n/\Gamma_{n+1})$, where $\{\Gamma_n\}_{n \geq 1}$ is the lower central series of the fundamental group $\pi_1(X)$ and $P(X, t)$ is the Poincaré polynomial of X. The main technical tool in our proofs is Sullivan's notion of *minimal model* for commutative differential graded algebras, see Section 3.

The rational $K[\pi, 1]$ formal spaces, especially those arising in low-dimensional topology, were studied in [24, 5]. More recently, the rational $K[\pi, 1]$ property has come in spotlight in connection with the higher logarithms, see [17]. There is an extensive literature on the Koszul property, see for example [3, 4, 28, 23] for connections with quantum groups. The rational $K[\pi, 1]$ spaces and Koszul algebras first appeared together implicitly in [24] and explicitly in [6].

The complements of complex hyperplane arrangements provide a rich supply of formal spaces [9] and have received a lot of attention. In particular, the rational $K[\pi, 1]$ property for these spaces was studied in [13, 15, 20–22]. In this context, we are able to prove (in Proposition 5.4) that the above LCS formula implies the rational $K[\pi, 1]$ property, for arrangements in $\mathbf{C}^3$.

The connection with the Koszul property is the following. It is known that the Koszulness of A implies *the Koszul duality formula* (see Theorem 2.3(iii)), a useful numerical test involving Hilbert series (see Example 5.7). Note however that this formula does not imply in general the Koszul property, see [27, 28]. Finally, it is not hard to see that the LCS formula for a formal space X is actually equivalent to the Koszul duality formula for $A = H^*(X, \mathbf{Q})$, if A is quadratic, i.e. generated in degree 1 with relations in degree 2.

It is therefore interesting to know whether the LCS formula implies the rational $K[\pi, 1]$ property in the particular case of arrangement complements, as asked by Falk and Randell in [15]. In this particular case Corollary 5.3 was proved by Falk [13] and Kohno [22]. It was proved in full generality in [24]; here we give a different simple proof, based on the Koszul duality formula. More results related to Koszulness and LCS-type formulae are known for the so-called *fiber-type* arrangements introduced in [14], see [29], and for their recent generalization, the *hypersolvable* arrangements of [19] (see Remarks 5.6).

The paper is organized as follows. In Section 2, we recall the definition and a few basic results about Koszul algebras. In Section 3, we recall the definition of a 1-minimal model and give a concrete realization convenient for our purposes. In Section 4, we prove the main algebraic result (Proposition 4.4). In Section 5, we derive our main results 5.1–5.4 about topological spaces and discuss an example of a hyperplane complement.

2. Koszul algebras: Preliminaries

In this section we state the definitions and certain basic results about Koszul algebras, see [3, 4, 23]. Let F be a field and let $U = \bigoplus_{n\geq 0} U_n$ be a positively graded F-algebra. We will always assume that $\dim U_n < \infty$ for all n. U is called connected if $U_0 = F$.

Considering just graded left U-modules we denote by Ext_U the derived functor of the *graded* homomorphism functor. For two such modules M and N the linear space $\mathrm{Ext}_U(M,N)$ is bigraded, i.e. $\mathrm{Ext}_U(M,N) = \bigoplus_{p,q\geq 0} \mathrm{Ext}_U^{p,q}(M,N)$ where p is the homological (resolution) degree and q is the pure (internal) degree coming from the gradings of U, M and N.

Definition 2.1. A positively graded connected F-algebra U is *Koszul* if

$$\mathrm{Ext}_U^{p,q}(F,F) = 0 \tag{2.1}$$

for every $p \neq q$, where $F = {}_UF$ is the trivial graded left U-module (equal to $U/U_{>0}$).

If V is a finite-dimensional vector space over F let $T = T(V)$ denote the full F-tensor algebra on V. The algebra T is provided with the standard grading $\bigoplus_n T_n$ where $T_0 = F$ and $T_1 = V$. If U is a connected graded F-algebra then there exists a canonical graded algebra homomorphism $T(U_1) \to U$. The algebra U is called *quadratic* if this homomorphism is surjective and its kernel I is generated, as an ideal of T, by its degree 2 part I_2. We call I the ideal of U.

Definition 2.2. Let U be a quadratic algebra and let I be its ideal. Let $I_2^!$ be the annihilator in $U_1^* \otimes U_1^* = (U_1 \otimes U_1)^*$ of the linear space I_2 and let $I^!$ be the ideal of $T(U_1^*)$ generated by $I_2^!$. Then the quadratic algebra $U^! = T(U_1^*)/I^!$ is called the quadratic dual of U.

Observe that $(U^!)^! = U$.

For any graded F-linear space $V = \bigoplus_n V_n$ with $\dim V_n < \infty$ for all n we denote the Hilbert series of V by $H(V,t)$ $(= \sum_n \dim_F(V_n)t^n)$. The following statements (for example, see [4]) will be used often without reference.

Theorem 2.3. (i) *If U is Koszul then it is quadratic. For any quadratic algebra the condition* (2.1) *holds for $p < 3$ and $q \neq p$.*

(ii) *A quadratic algebra U is Koszul if and only if $U^!$ is Koszul.*

(iii) *If U is Koszul then $H(U,t) \cdot H(U^!,-t) = 1$.*

Quite recently, Positselski [27] and Roos [28] have independently constructed examples showing that the converse of Theorem 2.3(iii) is false.

3. 1-minimal models of DGA's

In this section we suppose that F is a field of characteristic 0 and consider algebras over F. Let $M = \bigoplus_{n \geq 0} M_n$ be a connected DGA. That is M is a connected graded F-algebra which is graded-commutative (i.e. $xy = (-1)^{mn} yx$, for $x \in M_m$ and $y \in M_n$), equipped with a *differential* $d : M \to M$, which is homogenous of degree 1, satisfies $d^2 = 0$ and acts on M as a graded derivation (i.e. $d(x \cdot y) = dx \cdot y + (-1)^n x \cdot dy$, if $x \in M_n$). The DGA homomorphisms are required to be homogenous of degree zero, multiplicative, and to commute with the differentials. Two DGAs are said to be *quasi-isomorphic* if they can be connected by a sequence of DGA maps inducing homology isomorphisms.

Assuming that M is generated by M_1, the canonical (increasing) filtration of M is defined by $M(0) = F$ and $M(n)$ being the subalgebra of M generated by $M(n-1)$ and $d^{-1}(M(n-1)) \cap M_1$. Recall [7] that M is called *minimal* if $M = \wedge M_1$ (i.e. M is freely generated in degree one as a graded commutative algebra) and $\bigcup_n M(n) = M$. For every DGA A with $H^0(A) = F$ there exists a minimal algebra M as above and a DGA homomorphism $f : M \to A$ such that $f^* : H^*(M) \to H^*(A)$ is an isomorphism in dimension 1 and a monomorphism in dimension 2. The DGA M is unique up to isomorphism [16, Theorem 12.1]. M is called a *1-minimal model* of A.

For the rest of the paper we denote by $A = \bigoplus_{n \geq 0} A_n$ a graded commutative connected algebra. The 1-minimal model of A (provided with $d = 0$) can be realized as dual to an important Lie algebra attached to A. Let $L = \mathbf{L}(A_1^*)$ be the free Lie algebra on A_1^*, graded by bracket length. Notice that L_2 can be identified with $A_1^* \wedge A_1^*$. The multiplication of A generates a comultiplication $A_2^* \to A_1^* \wedge A_1^*$. Define the *holonomy Lie algebra* $G = G(A)$ of A as the factor of L by the Lie ideal generated by the image of the comultiplication. G is a graded Lie algebra and we write $G = \bigoplus_{n \geq 1} G_n$. We also consider the filtration of G by the Lie ideals $\Gamma_n G$ where $\Gamma_n G$ is the nth term of the lower central series of G, i.e. $\Gamma_1 G = G$ and $\Gamma_{n+1} G = [\Gamma_n G, G]$. Notice that the grading of G is closely related to the filtration, namely $\Gamma_n G = \bigoplus_{k \geq n} G_k$. In particular if we denote by $G(n)$ the nilpotent Lie algebra $G/\Gamma_{n+1} G$ then $G(n) \approx \bigoplus_{i=1}^n G_i$, as F-linear spaces.

For every n we define the DGA freely generated in degree one $M(n) =: \wedge(G(n))^*$. The differential d is dual to the Lie bracket $G(n) \wedge G(n) \to G(n)$ on $(G(n))^*$ and is extended to $M(n)$ by the graded derivation property. The natural projections $G(n+1) \to G(n)$ generate the embeddings $M(n) \subset M(n+1)$ of DGA's and we put $M = \bigcup_{n \geq 1} M(n)$. Notice that $M(1) \approx \wedge A_1$ as a graded algebra, with differential $d = 0$, whence there is a natural DGA homomorphism $M(1) \to A$.

In order to analyze M deeper we put $V_n = G_n^*$, for every $n \geq 1$. Of course $V_1 = A_1$ and $d = 0$ on V_1. More importantly, $M(n) = \wedge(\bigoplus_{i=1}^n V_i)$ and

$$d(V_n) \subset \sum_{i+j=n} V_i \wedge V_j \ , \tag{3.1}$$

for $n > 1$, where the symbol $\wedge$ applied to different spaces means the tensor product. Now, extend f to a DGA homomorphism $f : M \to A$ by setting $f = 0$ on $\bigoplus_{n>1} V_n$.

Proposition 3.1 (Markl and Papadima [24, Lemma 1.8(i)]). *Provided with the above homomorphism f, M is a 1-minimal model of A.*

Note that there is a shift of the degrees: what we call here V_n was called V_{n-1} in [24]. This rescaling has features convenient for our purposes. Put

$$C^p(q) = \bigoplus_{i_1+\cdots+i_p=q} V_{i_1} \wedge \cdots \wedge V_{i_p}, \ C(q) = \bigoplus_p C^p(q) \ ,$$

and notice that $M = \bigoplus_q C(q)$ (as a linear space). The following easily seen properties were noticed in the particular case of arrangements in [13], for a completely different construction of the 1-minimal model M.

Proposition 3.2. (i) *The space $C(q)$ is a subcomplex of M for every q, whence $H^*(M) = \bigoplus_q H^*(C(q))$.*

(ii) *The subspace $B =: \bigoplus_q H^q(C(q))$ is the subalgebra of $H^*(M)$ generated by $H^1(M) = A_1$. Moreover, this algebra is quadratic.*

(iii) *The homomorphism $f^* : H^*(M) \to A$ is 0 on $\bigoplus_{p \neq q} H^p(C(q))$.*

(iv) *If A is quadratic then $B = A$ and f^* is the projection of $H^*(M)$ to B along $\bigoplus_{p \neq q} H^p(C(q))$.*

Proof. (i) The invariance of $C(q)$ with respect to d follows immediately from (3.1) and implies the first statement. (ii) Denote by J the ideal of $M(1) = \wedge V_1 = \bigoplus_q C^q(q)$ generated by $d(V_2)$. Then $d(C^{q-1}(q)) = J_q$ whence $B = \bigoplus_q H^q(C(q)) = \wedge V_1/J$. (iii) It follows immediately from the construction of f. (iv) If A is quadratic then $A = \wedge A_1/I$ where I is a homogeneous ideal of $\wedge A_1$ generated by I_2. It follows from the construction of M that $V_1 = A_1$ and $d(V_2) = I_2$. Thus $B = A$. Using (iii) and descending to cohomology we obtain the second statement. □

4. 1-minimal models and Koszul algebras

In this section, we prove the main algebraic result. We need to use the universal enveloping algebra U of the Lie algebra G. A more explicit description of U is as follows. Let T be the tensor algebra of the linear space A_1^* and $\tau : A_1^* \otimes A_1^* \to A_1^* \otimes A_1^*$ the linear map generated by $v \otimes w \mapsto w \otimes v$. Then U is the factor of T by the ideal I generated by $(\mathrm{Im}(1-\tau)) \cap \mathrm{Im}\Delta$, where Δ is the comultiplication $A_2^* \to A_1^* \otimes A_1^*$. This description gives the grading of U (which is induced from the standard grading of T) and shows that U is a quadratic algebra.

Lemma 4.1. *If the algebra A is quadratic then it is the quadratic dual of U, i.e. $A = U^!$.*

Proof. If A is quadratic it is the factor of $\wedge A_1$ by the ideal generated by the kernel of the product $A_1 \wedge A_1 \to A_2$. Equivalently, A is the factor of the tensor algebra of A_1 by the ideal generated by the kernel of the product $A_1 \otimes A_1 \to A_2$ and the elements $a + \tau^* a$, $a \in A_1 \otimes A_1$. The result follows since the degree 2 component of this ideal is the annihilator of the degree 2 component of I. □

Remark 4.2. Similarly, one can show that in the general case $U^!$ is the factor algebra of $\wedge A_1$ by the ideal generated by the kernel of the multiplication $A_1 \wedge A_1 \to A_2$.

Now, we study the Yoneda algebra $\mathrm{Ext} = \bigoplus_{p,q} \mathrm{Ext}_U^{p,q}(F,F)$.

Proposition 4.3. *The bigraded algebra* $\mathrm{Ext} = \bigoplus_{p,q} \mathrm{Ext}^{p,q}$ *coincides with the algebra* $H^*(M) = \bigoplus_p H^p(M)$ *where the second (pure) grading is induced on* $H^p(M)$ *by the grading of* M *by the subcomplexes* $C(q)$.

Proof. We have $\mathrm{Ext}^p = H^p(G,F) = H^p(\mathrm{Hom}(\wedge G, F))$ (see e.g. Ch. XIII of [10]). The space of n-cochains of the complex $\mathscr{K} = \mathrm{Hom}(\wedge G, F)$ consists of the skew symmetric multilinear functions of n variables from G with the differential δ defined by

$$(\delta f)(x_1,\ldots,x_{n+1}) = \sum_{1\le i<j\le n+1} (-1)^{i+j} f([x_i,x_j],x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_j,\ldots,x_{n+1}).$$

The functions of pure degree q are those ones that are nonzero only if $\sum_{i=1}^n \deg x_i = q$, where the initial grading on G is used. These functions form a finite-dimensional subcomplex $\mathscr{K}(q)$ of $\mathscr{K}$ and $\mathscr{K} = \bigoplus_q \mathscr{K}(q)$. Since $\mathscr{K}(q)$ is finite dimensional, we have

$$\mathscr{K}(q) = \left(\bigoplus_{i_1+i_2+\cdots+i_p=q} G_{i_1} \wedge G_{i_2} \wedge \cdots \wedge G_{i_p} \right)^* = C(q)$$

as graded linear spaces. Thus, one can identify $\mathscr{K}$ and M as linear spaces. The algebra multiplication on M induces then the standard shuffle multiplication of skew symmetric forms from $\mathscr{K}$. The restrictions to the algebra generators of the differentials δ and d clearly coincide, up to sign. Besides both differentials satisfy the graded Leibnitz condition with respect to the algebra multiplication on M and the shuffle multiplication on $\mathscr{K}$. Thus these differentials coincide up to sign, which implies the statement. □

Proposition 4.4. *The graded algebra homomorphism* $f^* : H^*(M) \to A$ *is an isomorphism if and only if* A *is Koszul.*

Proof. Suppose A is Koszul. Then it is quadratic and U is its quadratic dual whence U is also Koszul. Thus,

$$H^p(C(q)) = \mathrm{Ext}_U^{p,q}(F,F) = 0 \tag{4.1}$$

for $q \neq p$. Now Proposition 3.2(iv) implies that f^* is an isomorphism.

Conversely, if f^* is an isomorphism, then we have immediately

$$\bigoplus_{p\neq q} H^p(C(q)) = 0,$$

so U is Koszul. Proposition 3.2(iii) implies that $A \approx B$ and consequently A is quadratic, by Proposition 3.2(ii). Thus, A is the quadratic dual of U whence A is also Koszul. □

Remark 4.5. Proposition 3.2 also implies that both statements of Proposition 4.4 are equivalent to A being quadratic and $H^*(M)$ being generated in degree 1.

5. Formal topological spaces

In this section we interpret the results of the previous section for topological spaces. A space X (connected and with finite Betti numbers) is called *formal* [16, p. 158] if it has the homotopy type of a simplicial complex K and the DeRham DGA $A_{PL}(K)$ of $\mathbf{Q}$-polynomial forms on the complex K is quasi-isomorphic to $A = H^*(X, \mathbf{Q})$ endowed with zero differential. Note that this coincides with the definition given in the Introduction, as follows e.g. from [7, 12.2.] (An enlighting discussion of this definition can be found in [18, pp. 235–236]). Among the well-known examples of formal spaces are the compact Kähler manifolds [12] and the complements of complex hyperplane arrangements [9]. If X is formal then a 1-minimal model of A is also a 1-minimal model of $A_{PL}(K)$ and it is called a 1-minimal model of X. The holonomy Lie algebra $G = G(X)$ of A is then the $\mathbf{Q}$-version of the $\mathbf{R}$-algebra defined by Chen in [11] and it is called the holonomy Lie algebra of X.

Using Proposition 3.1, the holonomy algebra G of a formal space X has the following interpretation in terms of $\pi_1(X)$. Let Γ_n be the nth term of the lower central series of $\pi = \pi_1(X)$, i.e. $\Gamma_1 = \pi$ and inductively $\Gamma_n = [\Gamma_{n-1}, \pi]$. The graded abelian group $\mathrm{gr}^*_\Gamma(\pi) =: \bigoplus_{n\geq 1}(\Gamma_n/\Gamma_{n+1})$ has a natural graded Z-Lie algebra structure induced by the group commutator. Then one has a graded Lie algebra isomorphism $G_* \approx \mathrm{gr}^*_\Gamma(\pi_1(X)) \otimes \mathbf{Q}$, see e.g. [16, Theorem 12.2]. In particular, we have $\dim G_n = \phi_n$, where $\phi_n = \mathrm{rank}(\Gamma_n/\Gamma_{n+1})$. Applying the Poincaré–Birkhoff–Witt theorem we obtain the following expression for the Hilbert series of the universal enveloping algebra $U = U(X)$ of G:

$$H(U,t) = \prod_{n\geq 1} (1-t^n)^{-\phi_n} . \tag{5.1}$$

The space X is a rational $K[\pi, 1]$, in the sense of the definition given in the Introduction, if and only if its 1-minimal model M is a *minimal model* of X, i.e. $f^* : H^*(M) \to H^*(X, \mathbf{Q})$ is an isomorphism. See [7, 12.8(iii)] for a proof. Applying Proposition 4.4 we obtain the following theorem.

Theorem 5.1. *Suppose X is a formal topological space. Then X is a rational $K[\pi, 1]$ if and only if $H^*(X, \mathbf{Q})$ is a Koszul algebra.*

The above result may be used to get certain information also for non-formal spaces X.

Proposition 5.2. *Let X be an arbitrary connected space with finite Betti numbers. If $H^*(X,\mathbf{Q})$ is a Koszul algebra then X is a rational $K[\pi,1]$.*

Proof. As mentioned before, the $K[\pi,1]$ property of X may be checked via the minimal model of X. It is known that one may associate to X a DGA $\mathscr{M}$, called the minimal model of X. As a commutative graded algebra $\mathscr{M}$ is freely generated by a graded Q-vector space $W=\bigoplus_{n\geq 1} W^n$ (see e.g. [7, 7.7 and 7.8], for the complete definitions and the precise result). For our purposes, it will suffice to recall that X is a rational $K[\pi,1]$ if and only if $W^n=0$ for every $n>1$ [7, 12.8(iii)].

Set now $A=H^*(X,\mathbf{Q})$. Our result will follow at once from the fact that W is in general related to A as follows. Firstly, A has a so-called *bigraded model* $\mathscr{B}$, see [18, Proposition 3.4]. It is a DGA freely generated by a graded vector space $V=\bigoplus_{n\geq 1} V^n$, with several other defining properties. Secondly, one may associate to X a new (perturbed) differential D on $\mathscr{B}$ and an induced differential D_ζ on V, see [18, Theorem 4.4 and 4.14]. Finally, one knows [18, 4.14] that there is an isomorphism of graded vector spaces $W^* \approx H^*(V,D_\zeta)$ (see [30, Section 8 for a proof]).

If A is Koszul we know from Proposition 4.4 that the 1-minimal model M constructed in Section 3 is actually the minimal model of A. It follows then from [24], Lemma 1.8(i) that M is, in fact, the bigraded model of A. Since the free generators V of M are concentrated by construction in degree one, the same property obviously will hold also for W and therefore X is a rational $K[\pi,1]$. $\square$

The next corollary was obtained in [24, Lemma 4.5], by using the bigraded models of [18]. We present here a new proof, based on the Koszul duality formula.

Corollary 5.3. *If a formal space X is a rational $K[\pi,1]$ then the LCS formula holds for X (see the Introduction).*

Proof. By Theorem 5.1, $A=H^*(X,\mathbf{Q})$ is Koszul. Since U is the quadratic dual of A, Theorem 2.3(iii) implies that

$$P(X,-t)=H(U,t)^{-1},$$

where $P(X,t)$ is the Poincaré polynomial of X. Then (5.1) completes the proof. $\square$

The most intensively studied class of formal spaces consists of the complements of hyperplane arrangements. Let $\mathscr{A}$ be a finite set of linear hyperplanes in $\mathbf{C}^\ell$ and $X=\mathbf{C}^\ell\setminus\bigcup_{H\in\mathscr{A}} H$. Then $A=H^*(X,\mathbf{Q})$ can be embedded in the DGA of holomorphic differential forms on X (see [9]) whence X is formal. Besides A is determined by combinatorial data, namely by the intersection lattice of $\mathscr{A}$ (see [25]). The Hilbert series $H(A,t)=P(X,t)$ is a polynomial of degree ℓ.

The problems related to the topic of this paper were first studied in the arrangement setting in [2, 20–22, 14, 13]. Falk [13] studied conditions for X to be a rational

$K[\pi,1]$. He proved that X is a rational $K[\pi,1]$ for every arrangement of the so-called fiber-type. This is a topologically defined class that coincides with the combinatorially defined class of supersolvable arrangements (see [25] for definitions). Falk [13] and Kohno [22] proved Corollary 5.3 for complements of hyperplane arrangements.

It is still unknown if X can be a rational $K[\pi,1]$ for a not supersolvable arrangement or if the LCS formula can hold without X being a rational $K[\pi,1]$. The following result solves the latter problem negatively in a low-dimensional case.

Proposition 5.4. *If $\ell = 3$ and the lower central series formula holds for X then X is a rational $K[\pi,1]$.*

In order to prove the proposition we need to bring into consideration affine arrangements. Let $\mathscr{A}$ be an arrangement in $\mathbf{C}^\ell$ and $H \in \mathscr{A}$. Pick $\alpha \in (\mathbf{C}^\ell)^*$ with kernel H. Then intersecting every $K \in \mathscr{A}\setminus\{H\}$ with the affine hyperplane $\tilde{H} = \{v \in \mathbf{C}^\ell \,|\, \alpha(v) = 1\}$ we obtain a set $\mathscr{A}'$ of affine hyperplanes in $\tilde{H}$. Put $X' = \tilde{H}\setminus\bigcup_{K\in\mathscr{A}'} K$, $A' = H^*(X',\mathbf{Q})$ and $U' = U(X')$. Set also $\phi'_n = \operatorname{rank}(\Gamma_n\pi_1(X')/\Gamma_{n+1}\pi_1(X'))$, $n \geq 1$.

One knows that X' is formal [25, Theorem 5.90]. It is also well known that X is homeomorphic to $X' \times (\mathbf{C}^*)$, see [25, Proposition 5.1]. This enables us to drop the dimension in the statement of Proposition 5.4 to 2, in particular,

$$H(A',t) = 1 + mt + nt^2. \tag{5.2}$$

This also implies that $A \approx A' \otimes \wedge(e)$, with $\deg(e) = 1$.

Now, we can infer the following two equivalences. On the one hand, X is a rational $K[\pi,1]$ if and only if X' is a rational $K[\pi,1]$. Indeed, denoting by $f' : (\wedge V', d') \to (A', 0)$ the 1-minimal model of A', we may infer from the Künneth theorem that $f =: f' \otimes id : (\wedge V' \otimes \wedge(e), d' \otimes id) \to (A, 0)$ will be the 1-minimal model of A. Hence f^* is an isomorphism if and only if f'^* is an isomorphism, as asserted.

On the other, obviously $\phi_k = \phi'_k$, if $k > 1$, and $\phi_1 = \phi'_1 + 1$. Since $H(A,t) = H(A',t)(1+t)$, it follows that the LCS formula holds for X if and only if it holds for X'. According to (5.1) and (5.2) the latter is equivalent to

$$H(U',t) = (1 - mt + nt^2)^{-1}.$$

Consequently Proposition 5.4 will follow from the simple lemma below, via Theorem 5.1.

Lemma 5.5 (Polishchuk and Positselski [26, 3.4, Proposition 2]). *Let U' be a quadratic algebra with*

$$H(U',t) = (1 - mt + nt^2)^{-1}. \tag{5.3}$$

Then U' is Koszul.

Proof. Let I be the ideal of U'. Condition (5.3) immediately implies that $m = \dim U'_1$ and $n = \dim I_2$. The natural complex

$$0 \to U' \otimes I_2 \to U' \otimes U'_1 \to U' \to F \to 0$$

is exact except may be at $U' \otimes I_2$ for every quadratic algebra. Since all the maps in that complex are of degree 1 we have

$$\mathrm{Ext}^{p,q}_{U'}(F,F) = 0 \tag{5.4}$$

for $0 \le p \le 2$ and $q \ne p$. Condition (5.3) implies that the above complex is indeed exact whence it is a graded free resolution of F. Thus $\mathrm{Ext}^{i}_{U'}(F,F) = 0$ for $i \ge 3$, which together with (5.4) implies the result. □

Proof of the proposition. The Koszul duality formula 2.3(iii) together with (5.2) and (5.3) imply that A' and $(U')^!$ have the same Hilbert series. On the other hand, one knows that A' is generated in degree one [25, Theorem 5.90] and therefore A' is a quotient of $(U')^!$, see Remark 4.2. Consequently, $A' \approx (U')^!$ and A' is Koszul. □

Remark 5.6. The Koszul property for $A = H^*(X, \mathbf{Q})$ was first considered in [29], where it was proved that A is Koszul for fiber-type arrangements. A substantial generalization of the fiber-type class, namely the hypersolvable class, was recently introduced in [19]. Among other things it was shown there [19, Theorem E] that the associated quadratic algebra $U^!$ (see Remark 4.2) is always Koszul for a hypersolvable arrangement. Consequently, a generalization of the LCS formula [19, Theorem C(ii)] holds for this new class. At the same time [19, Theorem D(iv),(v)] the implication of Proposition 5.4 holds for hypersolvable arrangements. Moreover, one knows that the only rational $K[\pi,1]$ hypersolvable complements are the fiber-type ones [19, Theorem D(ii),(iv)].

Falk in [13] asked the question if the quadraticity of A is sufficient for X being a rational $K[\pi,1]$. In 1995, he ran the following example through a computer program and found out that it gives the negative answer to the above question. Our method applies nicely to the example.

Example 5.7. Consider the hyperplanes given in $\mathbf{C}^3 = \{(x,y,z)\}$ by the functionals $x, y, x+y, z, x-z, y-z, x+y-2z$. Then the algebra A is quadratic. (In fact, it is Kohno's parallel arrangement discussed in [15, 22]. It is mistakenly written in [22] that X is rational $K[\pi,1]$ for this arrangement.) One can compute that $H(A',t) = 1 + 6t + 10t^2$. If X is a rational $K[\pi,1]$ then by Corollary 5.3 and (5.1) the series $(1 - 6t + 10t^2)^{-1}$ equals $H(U',t)$ whence all its coefficients are nonnegative. This implies that the roots of $1-6t+10t^2$ are real as in the proof of [1, Lemma 3.4], which is a contradiction.

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