Combinatorial and motivic structures in unstable cohomology

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Combinatorial structures and tropical moduli

$$\begin{aligned} M_g^{\text{trop}} &= \text{ moduli space of stable tropical curves of genus } g \\ &= \bigsqcup_G \mathbb{R}_{>0}^{E(G)}/\mathrm{Aut}(G) \\ &= \varinjlim \mathbb{R}_{\geq 0}^{E(G)} \end{aligned}$$

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2015

Theorem (Abramovich, Caporaso, P–)

The tropical moduli space M_g^{trop} is the skeleton of $\mathcal{M}_g^{\mathrm{an}}$ associated to the toroidal embedding $\mathcal{M}_g \subset \overline{\mathcal{M}}_g$

Weight zero cohomology

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Corollary

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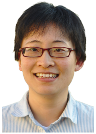

2021

Theorem (Chan, Galatius, P–)

$$\bigoplus_g W_0 H_c^{2g}(\mathcal{M}_g) \cong \mathfrak{grt} \supset \mathrm{Lie}\langle \sigma_3, \sigma_5, \sigma_7, \sigma_9, \ldots \rangle$$

Commutative graph complex

$$\mathcal{M} := \bigcup_{g \geq 2} \mathcal{M}_g \qquad M^{\mathrm{trop}} := \bigcup_{g \geq 2} M_g^{\mathrm{trop}}$$

$$C_c^*(M^{\mathrm{trop}}) = \bigoplus_G \Big(\det E(G) \Big)^{\mathrm{Aut}(G)}$$

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Theorem (Willwacher 2015, Brown 2012)

$$H^*(\mathrm{GC}_0) \cong \mathfrak{grt} \supset \mathrm{Lie}\langle \sigma_3, \sigma_5, \sigma_7, \sigma_9, \ldots \rangle$$

Corollary

$\dim H_c^{2g}(\mathcal{M}_g)$ grows at least exponentially with g

Higher weight cohomology

2024

Theorem (Bergström, Faber, P–)

$$H^k(\overline{\mathcal{M}}_{g,n}) = 0 \quad \text{for odd } k < 11$$

Confirms predictions from arithmetic (Chenevier, Lannes 2019)

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Study: $\mathrm{gr}_2 H_c^*(\mathcal{M})$ and $\mathrm{gr}_{11} H_c^*(\mathcal{M})$ (motivic structures L and S_{12})

Weight 2

Let $\mathcal{M}^n := \bigcup_g \mathcal{M}_{g,n}$, $\mathbb{V} := W_0 H_c^*(\mathcal{M}^1)$, $\mathbb{W} := W_0 H_c^*(\mathcal{M}^2)$

Theorem (P–, Willwacher 2024)

$$\mathrm{gr}_2^W H_c^*(\mathcal{M}) = \left(\Lambda^2 \mathbb{V}\right)[0,-3] \oplus \left(\Lambda^2 \mathbb{V}\right)[-1,-4] \oplus \\ \left((\mathbb{W} \otimes \mathrm{sgn})[-1,-3] \oplus (\mathbb{W} \otimes \mathrm{sgn})[-2,-4]\right)^{\mathbb{S}_2}$$

Proof uses graph complexes plus knowledge of $H^2(\overline{\mathcal{M}}^n) \circlearrowleft \mathbb{S}_n$

The weight spectral sequence as a graph complex

$$\mathcal{M}_G := \prod_{v \in V(G)} \mathcal{M}_{g_v, n_v}, \qquad \overline{\mathcal{M}}_G := \prod_{v \in V(G)} \overline{\mathcal{M}}_{g_v, n_v}$$

Observations:

$$\overline{\mathcal{M}} = \bigsqcup_G \mathcal{M}_G / \mathrm{Aut}(G), \qquad \overline{\widetilde{\mathcal{M}_G / \mathrm{Aut}(G)}} \cong \overline{\mathcal{M}}_G / \mathrm{Aut}(G)$$

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Deligne's weight spectral sequence, kth row of E_1:

$$H^k(\overline{\mathcal{M}}) \to \bigoplus_{|E(G)|=1} \left(H^k(\overline{\mathcal{M}}_G)\right)^{\mathrm{Aut}(G)} \to \bigoplus_{|E(G)|=2} \left(H^k(\overline{\mathcal{M}}_G) \otimes \det E(G)\right)^{\mathrm{Aut}(G)} \to$$

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Special case, $k = 0$: $\qquad E_1^{*,0} \cong C_c^*(M^{\mathrm{trop}})$

Weight 11

2023

Theorem (Canning, Larson, P–)

$$H^{11}(\overline{\mathcal{M}}_{g,n}) \cong \begin{cases} V_\lambda \otimes \mathsf{S}_{12} & \text{for } g = 1, n \geq 11, \text{ and } \lambda = (n-10, 1^{10}) \\ 0 & \textit{otherwise} \end{cases}$$

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Generating function for $\sum_g \chi_{11}(\mathcal{M}_g)$

- Derived using [BFP24], [CLP23] and the graph complex $E_1^{*,11}$
- Grows in absolute value like $\Theta(g^g)$

Higher weights in $H_c^*(\mathcal{M}_{g,n})$

- Many further results in collaboration with CLW, studying one motivic structure at a time
- Inductive arguments driven by recursive combinatorial aspects of compactified moduli spaces and graph complexes

Theorem (CLPW 2024)

- $\#\mathcal{M}_g(\mathbb{F}_q)$ is a polynomial function of q if and only if $g \leq 8$
- $H_c^*(\mathcal{M}_{g,n})$ is of Tate type if and only if $3g + 2n \leq 24$

Weight zero cohomology for $\mathcal{A}_g$

A_g^{trop} := moduli space of principally polarized tropical abelian varieties of genus g

- Parametrizes skeletons of p.p.a.v.s over nonarchimedean k
- Skeleton of any toroidal compactification $\mathcal{A}_g \subset \mathcal{A}_g^{\Sigma}$:

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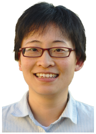


Theorem (Brown, Chan, Galatius, P– 2024)

$W_0 H_c^*(\mathcal{A})$ is a bigraded, connected, co-commutative Hopf algebra, and $\dim H_c^{2g+k}(\mathcal{A}_g)$ grows at least exponentially with g for all but finitely many $k \geq 0$

Stratification spectral sequence

- Dense open subset $\mathring{A}_g^{\text{trop}} \cong P_g/\mathrm{GL}_g(\mathbb{Z}) \subset A_g^{\text{trop}}$
- Stratification $A_g^{\text{trop}} = \bigsqcup_{j\leq g} \mathring{A}_j^{\text{trop}}$
- Spectral sequence:

$$E_1^{j,k} = \begin{cases} H_c^k(P_j/\mathrm{GL}_j(\mathbb{Z})) & \text{for } j \leq g \\ 0 & \text{otherwise} \end{cases} \Rightarrow H_c^*(A_g^{\text{trop}})$$

Theorem (Brandt, Bruce, Chan, Melo, Moreland, Wolfe 2024)

Degenerates at E_2, zero outside of $E_2^{g,*}$

Consequence of “inflation” (Elbaz-Vincent, Gangl, Soulé 2013)

The acyclic limit and relations to K-theory

$$A_\infty^{\text{trop}} := \bigsqcup_g \mathring{A}_g^{\text{trop}}, \qquad \mathcal{A} := \bigsqcup_g \mathcal{A}_g$$

$$E_1^{j,k} = H_c^k(P_j/\mathrm{GL}_j(\mathbb{Z})) \Rightarrow 0 \quad \text{(vanishes at } E_2\text{).}$$

Corollary

$$E_1^{*,*} \cong W_0 H_c^*(\mathcal{A}) \otimes \mathbb{Q}[\epsilon]/(\epsilon^2)$$

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Quillen's spectral sequence

$$^{Q}E_1^{j,k} = H_c^k(P_j/\mathrm{GL}_j(\mathbb{Z})) \Rightarrow H^* BK(\mathbb{Z}) \quad \text{(Hopf algebra!)}$$

A filtered coproduct on the Waldhausen S-construction

S-construction: Rank-filtered model for $BK(\mathbb{Z})$ built from the Waldhausen category of projective $\mathbb{Z}$-modules

- Diagonal $\Delta : BK(\mathbb{Z}) \to BK(\mathbb{Z}) \times BK(\mathbb{Z})$ is *not* filtered

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Theorem (Brown, Chan, Galatius, P– 2024)

There is a filtered map $\mathrm{es}\colon BK(\mathbb{Z}) \to BK(\mathbb{Z}) \times BK(\mathbb{Z})$, homotopic to Δ, making ${}^{Q}E_*^{*,*}$ a spectral sequence of Hopf algebras

Edgewise subdivision

Theorem (BCGP 2024)

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- ${}^{Q}E_1^{1,0}$ is primitive and generates a Hopf ideal I
- ${}^{Q}E_1^{*,*} \cong W_0 H_c^*(\mathcal{A}) \otimes \mathbb{Q}[\epsilon]/(\epsilon^2)$ identifies $I = (\epsilon)$

Corollary

Quotient $W_0 H_c^*(\mathcal{A})$ is a Hopf algebra!

Thank you