Exercises on the Kazhdan-Lusztig polynomial of a matroid (WARTHOG 2015)

Fix a field k. An arrangement $\mathcal{A}$ over k consists of a finite set I and a linear subspace $V \subset k^I$ that is not contained in any coordinate hyperplane. We have the **complement** $U_{\mathcal{A}} := V \cap (k^\times)^I$, and we define the **reciprocal plane** $X_{\mathcal{A}}$ to be the closure of $U_{\mathcal{A}}^{-1}$ in k^I. For any flat F of $\mathcal{A}$, we have the **restriction** $\mathcal{A}^F$ (obtained by intersecting V with $k^{I \smallsetminus F}$) and the **localization** $\mathcal{A}_F$ (obtained by projecting V onto k^F).

1. For any $x \in X_{\mathcal{A}} \subset k^I$, let $F_x = \{i \mid x_i \neq 0\}$.
   (a) Show that F_x is a flat of $\mathcal{A}$.
   (b) Show that, for any flat F of $\mathcal{A}$, $\{x \in X_{\mathcal{A}} \mid F_x = F\} \cong U_{\mathcal{A}_F}$.
2. I stated in the lectures that the stratum $U_{\mathcal{A}_F} \subset X_{\mathcal{A}}$ has $X_{\mathcal{A}^F}$ as a normal slice. This problem is about proving this statement in the simplest nontrivial case.

   Let $I = \{1,2,3,4\}$ and $V = \{v \in k^4 \mid v_1 + v_2 + v_3 + v_4 = 0\}$. Then
   $$X_{\mathcal{A}} = \{(x_1,x_2,x_3,x_4) \in k^4 \mid x_2x_3x_4 + x_1x_3x_4 + x_1x_2x_4 + x_1x_2x_3 = 0\}.$$
   Let $F = \{1,2,3\}$, so that
   $$U_{\mathcal{A}_F} = \{(0,0,0,x_4) \mid x_4 \neq 0\} \subset X_{\mathcal{A}}$$
   and
   $$X_{\mathcal{A}^F} = \{(x_1,x_2,x_3) \in k^3 \mid x_2x_3 + x_1x_3 + x_1x_2 = 0\}.$$
   Let M be the open subset of $X_{\mathcal{A}}$ defined by the nonvanishing of x_4 and $1 + \frac{x_1}{x_4}$.

   Let N be the open subset of $U_{\mathcal{A}^F} \times X_{\mathcal{A}^F}$ defined by the nonvanishing of $1 - \frac{x_1}{x_4}$.

   Show that the maps
   $$\varphi : M \rightleftarrows N : \psi$$
   given by the formulas
   $$\varphi(x_1,x_2,x_3,x_4) = \left(\frac{x_1}{1+\frac{x_1}{x_4}}, x_2, x_3, x_4\right)$$
   and
   $$\psi(x_1,x_2,x_3,x_4) = \left(\frac{x_1}{1-\frac{x_1}{x_4}}, x_2, x_3, x_4\right)$$
   are mutually inverse isomorphisms, taking $U_{\mathcal{A}^F} \subset M$ to $U_{\mathcal{A}^F} \times \{0\} \subset N$.

3. Prove that there is a unique way to assign to each matroid M a polynomial $P_M(q) \in \mathbb{Z}[q]$ such that the following conditions are satisfied:

   (a) If $\operatorname{rk} M = 0$, then $P_M(q) = 1$.

   (b) If $\operatorname{rk} M > 0$, then $\deg P_M(q) < \frac{1}{2} \operatorname{rk} M$.

   (c) For every M, $q^{\operatorname{rk} M} P_M(q^{-1}) = \sum_F \chi_{M_F}(q) P_{M^F}(q)$.

   Hint: Uniqueness is trivial once you have existence. For existence, note that the recursion says

   $$q^{\operatorname{rk} M} P_M(q^{-1}) - P_M(q) = R_M(q) := \sum_{\emptyset \neq F} \chi_{M_F}(q) P_{M^F}(q).$$

   To solve the problem, you need to show that $q^{\operatorname{rk} M} R_M(q^{-1}) = -R_M(q)$.

4. Show that $P_{M \oplus M'}(q) = P_M(q) P_{M'}(q)$.

5. Show that $P_M(0) = 1$ for any M.

6. Show that the coefficient of q in $P_M(q)$ is equal to the number of flats of rank $\operatorname{rk} M - 1$ minus the number of flats of rank 1.

7. It is a classical result that the number of flats of rank $\operatorname{rk} M - 1$ minus the number of flats of rank 1 is always nonnegative, with equality if and only if the lattice of flats of M is modular. This in turn is equivalent to the statement that there is another matroid M' whose lattice of flats is opposite to that of M. Show that, in this case, $P_M(q) = 1$. (Thus if the linear coefficient vanishes, all higher coefficients vanish.)

8. Find a general formula for the coefficient of q^2 in $P_M(q)$.