Quantizations of conical symplectic resolutions I: local and global structure

Tom Braden[1]
Department of Mathematics and Statistics, University of Massachusetts, Amherst, MA 01003
Nicholas Proudfoot[2]
Department of Mathematics, University of Oregon, Eugene, OR 97403
Ben Webster[3]
Department of Mathematics, University of Virginia, Charlottesville, VA 22904

Abstract. We re-examine some topics in representation theory of Lie algebras and Springer theory in a more general context, viewing the universal enveloping algebra as an example of the section ring of a quantization of a conical symplectic resolution. While some modification from this classical context is necessary, many familiar features survive. These include a version of the Beilinson-Bernstein localization theorem, a theory of Harish-Chandra bimodules and their relationship to convolution operators on cohomology, and a discrete group action on the derived category of representations, generalizing the braid group action on category $\mathcal{O}$ via twisting functors.

Our primary goal is to apply these results to other quantized symplectic resolutions, including quiver varieties and hypertoric varieties. This provides a new context for known results about Lie algebras, Cherednik algebras, finite W-algebras, and hypertoric enveloping algebras, while also pointing to the study of new algebras arising from more general resolutions.

1 Introduction

> *The dazzling success of algebraic geometry... has so much reorientated the field that one particular protagonist has suggested, no doubt with much justification, that enveloping algebras should now be relegated to a subdivision of the theory of rings of differential operators.*
>
> –Anthony Joseph, *On the classification of primitive ideals in the enveloping algebra of a semisimple Lie algebra* [Jos83]

In this paper, we argue against the relegation suggested above, in favor of a different geometric context. While viewing universal enveloping algebras as differential operators is

[1]Supported by NSA grants H98230-08-1-0097 and H98230-11-1-0180.
[2]Supported by NSF grant DMS-0950383.
[3]Supported by NSA grant H98230-10-1-0199.

unquestionably a powerful technique, the differential operators on flag varieties are odd men out in the world of differential operators as a whole. For example, the only known examples of projective varieties that are D-affine are homogeneous spaces for semi-simple complex Lie groups, and it is conjectured that no other examples exist. On the other hand, in this paper we consider a world where this special case is very much at home: quantizations of symplectic resolutions of affine singularities.

Differential operators on a smooth projective variety X form a deformation quantization of the cotangent bundle T^*X. If X is a homogeneous space for a semi-simple complex Lie group G, its cotangent bundle is a resolution of the closure of a nilpotent orbit in $\mathfrak{g}^*$ (or an affine variety finite over this one). If X is the flag variety, this is known as the **Springer resolution**. This is yet another sense in which these spaces are misfits; homogeneous spaces for semi-simple complex Lie groups are conjecturally the only examples of projective varieties whose cotangent bundles resolve affine singularities. For most projective varieties X, T^*X does not have enough global functions.

There are, however, many other examples of symplectic algebraic varieties that resolve affine cones. While the Springer resolution is the most famous, other examples include the minimal resolution of a Kleinian singularity, the Hilbert scheme of points on such a resolution, Nakajima quiver varieties, and hypertoric varieties. One can study deformation quantizations of these varieties, and many of them have the same affinity property enjoyed by the Springer resolution. This paper is a study of these deformation quantizations and their representation theory.

Several examples have been studied extensively by other authors. Universal enveloping algebras have been considered from an enormous number of angles for decades, and other examples such as spherical Cherednik algebras and finite W-algebras have been active fields of research for many years. The hypertoric case has recently been studied by Bellamy and Kuwabara [BK12] and by the authors of this paper, jointly with Licata [BLPW12]. On the other hand, very few works attempt to view all these examples in a single coherent theory. Kashiwara and Rouquier began to develop such a theory [KR08], and our paper might be regarded as a continuation of their work. A recent preprint of McGerty and Nevins [MN14] addresses similar issues, with results that are complementary to ours.

In Section 2, we discuss the algebraic geometry of conical symplectic resolutions; this is essentially all material already in the literature, but we collect it here for the convenience of the reader. Particularly important for us are deformations which appear in the work of Kaledin and Verbitsky; these show that any symplectic resolution flatly deforms to a smooth affine variety, which is key to many properties of its quantization. One ingredient we will use systematically is the conical structure: a choice of $\mathbb{C}^*$-action which makes the base into a cone and acts with positive weight on the symplectic form.

In Section 3, we discuss equivariant quantizations of a conical symplectic resolution $\mathfrak{M}$, which are classified by $H^2(\mathfrak{M};\mathbb{C})$ [BK04, Los12]. We prove some basic results about the ring A of $\mathbb{S}$-invariant global sections, a filtered algebra whose associated graded is isomorphic to $\mathbb{C}[\mathfrak{M}]$. We also study the behavior of quantizations under (quantum) Hamiltonian reduction, proving a quantum version of the Duistermaat-Heckman theorem (Proposition 3.16).

In Section 4 we introduce the appropriate category $\mathcal{D}$-mod of modules over a quantization $\mathcal{D}$, which one may regard as the quantum analogue of the category of coherent sheaves (in particular, there is a finiteness assumption built into the definition). In the case where $\mathfrak{M}$ is a cotangent bundle, we show that this category is equivalent to the category of finitely generated twisted D-modules on the base, where the twist is determined by the period of the quantization. The rest of the section is dedicated to the study of the sections and localization functors that relate the category of modules over a quantization to the category of modules over the section ring A. We establish in Theorem 4.17 that these functors induce derived equivalences for generic periods.

Theorem A *Let $\mathfrak{M}$ be a conical symplectic resolution, and fix two classes $\eta, \lambda \in H^2(\mathfrak{M};\mathbb{C})$ such that η is the Chern class of an ample line bundle, or the strict transform of an ample line bundle on any other conical symplectic resolution of $\mathfrak{M}_0$. For all but finitely many complex numbers k, the quantization of $\mathfrak{M}$ with period $\lambda + k\eta$ is derived affine; that is, the derived functors of global sections and localization are inverse equivalences.*

In order to obtain an equivalence of abelian (rather than derived) categories that works for all (rather than only generic) periods, we replace the section ring A with a $\mathbb{Z}$-algebra, which mimics in a non-commutative setting the homogeneous coordinate ring of a projective variety. Given a quantized symplectic resolution along with a very ample line bundle, we construct a $\mathbb{Z}$-algebra Z and prove the following result (Theorem 5.8).

Theorem B *Let $\mathfrak{M}$ be a conical symplectic resolution, let $\mathcal{L}$ be a very ample line bundle on $\mathfrak{M}$, and let Z be the associated $\mathbb{Z}$-algebra. Then the category* $\mathcal{D}$-mod *is equivalent to the category of finitely generated modules over Z modulo the subcategory of bounded modules.*

Theorem B has three nice consequences. First, we use it to prove the following abelian analogue of Theorem A (Corollary 5.17).

Corollary B.1 *Let $\mathfrak{M}$ be a conical symplectic resolution, and fix two classes $\eta, \lambda \in H^2(\mathfrak{M};\mathbb{C})$ such that η is the Chern class of an ample line bundle. For all but finitely many positive integers k, the quantization of $\mathfrak{M}$ with period $\lambda + k\eta$ is affine; that is, the (abelian) functors of global sections and localization are inverse equivalences.*

Next, we prove a version of Serre's GAGA theorem [Ser56]. More precisely, we consider the analytic quantization $\mathcal{D}^{\mathrm{an}}$ with the same period as $\mathcal{D}$, define the appropriate module category $\mathcal{D}^{\mathrm{an}}$-mod, and prove that it is equivalent to $\mathcal{D}$-mod (Theorem 5.22). The existing literature is fairly evenly divided between working in the algebraic and analytic categories, and this corollary is an indispensable tool that allows us to import previous results from both sides.

Corollary B.2 *If $\mathfrak{M}$ is a conical symplectic resolution, then the analytification functor from $\mathcal{D}$-mod to $\mathcal{D}^{\mathrm{an}}$-mod is an equivalence of categories.*

Finally, we use Theorem B to prove a categorical version of Kirwan surjectivity, relating the category of equivariant modules on a quantization to the category of modules on the Hamiltonian reduction. We consider a restriction functor defined by Kashiwara and Rouquier, and we use our $\mathbb{Z}$-algebra formalism to construct left and right adjoints, thus proving that the restriction functor is essentially surjective (Theorem 5.31). In particular, this result establishes that our category $\mathcal{D}$-mod is the same as the analogous category considered by McGerty and Nevins (Remark 5.32). For a precise statement of the hypotheses of the following result, see the beginning of Section 5.5.

Corollary B.3 *If $\mathfrak{M}$ is obtained via symplectic reduction from an action of a reductive group G on $\mathfrak{X}$, then every object of $\mathcal{D}$-mod extends to a twisted G-equivariant module over a quantization of $\mathfrak{X}$.*

Let $\mathfrak{M}_0 := \operatorname{Spec} \mathbb{C}[\mathfrak{M}]$ be the cone resolved by $\mathfrak{M}$, and consider the **Steinberg variety** $\mathfrak{Z} := \mathfrak{M} \times_{\mathfrak{M}_0} \mathfrak{M}$. The cohomology $H^{2\dim \mathfrak{M}}_{\mathfrak{Z}}(\mathfrak{M} \times \mathfrak{M})$ with supports in $\mathfrak{Z}$, which by Poincaré duality can be identified with the Borel-Moore homology group $H^{BM}_{2\dim \mathfrak{M}}(\mathfrak{Z})$, has a natural algebra structure via convolution [CG97, §2.7]. Furthermore, if $\mathfrak{L} \subset \mathfrak{M}$ is a Lagrangian subscheme that is equal to the preimage of its image in $\mathfrak{L}_0 \subset \mathfrak{M}_0$, then the convolution algebra acts on $H^{\dim \mathfrak{M}}_{\mathfrak{L}}(\mathfrak{M})$. In the special case where $\mathfrak{M} = T^*(G/B)$ and $\mathfrak{L}$ is the conormal variety to the Schubert stratification of G/B, the convolution algebra is isomorphic to the group algebra of the Weyl group, and $H^{\dim \mathfrak{M}}_{\mathfrak{L}}(\mathfrak{M})$ is isomorphic to the regular representation. More generally, there is a natural algebra homomorphism from the group algebra $\mathbb{C}[W]$ of the Namikawa Weyl group of $\mathfrak{M}$ to the convolution algebra $H^{2\dim \mathfrak{M}}_{\mathfrak{Z}}(\mathfrak{M} \times \mathfrak{M})$.

Section 6 is devoted to categorifying the picture described in the paragraph above. The convolution algebra is replaced by the monoidal category of Harish-Chandra bimodules, which comes in both an algebraic and a geometric version. The module $H^{\dim \mathfrak{M}}_{\mathfrak{L}}(\mathfrak{M})$ is replaced by a subcategory $\mathcal{C}^{\mathfrak{L}} \subset \mathcal{D}$-mod (respectively $C^{\mathfrak{L}_0} \subset A$-mod) which is a module category for the category of geometric (respectively algebraic) Harish-Chandra bimodules. Following Kashiwara and Schapira [KS12], we define the characteristic cycle of a geometric Harish-Chandra bimodule, which lies in $\mathfrak{Z}$, and the characteristic cycle of an object of $\mathcal{C}^{\mathfrak{L}}$, which lies in $\mathfrak{L}$. Using the machinery developed in [KS12], we prove that these cycles are compatible with convolution.

Theorem C *The characteristic cycle map intertwines convolution of geometric Harish-Chandra bimodules with convolution in the Borel-Moore homology of the Steinberg variety (Proposition*

6.15); it also intertwines the action of Harish-Chandra bimodules on $\mathcal{C}^{\mathfrak{L}}$ with the action of $H^{2\dim\mathfrak{M}}_{\mathfrak{Z}}(\mathfrak{M}\times\mathfrak{M})$ on $H^{\dim\mathfrak{M}}_{\mathfrak{L}}(\mathfrak{M})$ (Proposition 6.16).

There is particularly nice collection of algebraic Harish-Chandra bimodules which appear naturally from changing the period of the quantization. Let A_λ be the section ring of the quantization with period $\lambda \in H^2(\mathfrak{M};\mathbb{C})$. Derived tensor products with these special bimodules give derived equivalences between the derived categories of modules over A_λ for various different λ. These equivalences are far from being unique; instead, they induce a large group of autoequivalences of $D(A_\lambda\text{-mod})$ for each fixed λ, called **twisting functors**. There is a hyperplane arrangement in $H^2(\mathfrak{M};\mathbb{R})$ whose chambers are the Mori chambers of $\mathfrak{M}$; let $E \subset H^2(\mathfrak{M};\mathbb{C})$ be the complement of the complexification of this arrangement. The Namikawa Weyl group W acts on $H^2(\mathfrak{M};\mathbb{C})$ preserving E.

Theorem D *There is a weak action of $\pi_1(E/W,[\lambda])$ on $D(A_\lambda\text{-mod})$ by twisting functors (Theorem 6.35); this action preserves the subcategory $D(C^{\mathfrak{L}_0})$ (Remark 6.37). The subgroup $\pi_1(E,\lambda)$ preserves the characteristic cycle of a module, thus $W \cong \pi_1(E/W,[\lambda])/\pi_1(E,\lambda)$ acts on $H^{\dim\mathfrak{M}}_{\mathfrak{L}}(\mathfrak{M})$ (Proposition 6.39). This action agrees with the action induced by the natural map from $\mathbb{C}[W]$ to the convolution algebra (Remark 6.40).*

In the case where $\mathfrak{M}$ is the Springer resolution for G, the space E is the complement of the complexified Coxeter arrangement, W is the classical Weyl group, and $\pi_1(E/W)$ is the generalized braid group. If $\mathfrak{L} \subset \mathfrak{M}$ is taken to be the conormal variety to the Schubert stratification and the period of the quantization is regular, then $C^{\mathfrak{L}_0}$ is equivalent to a regular block of category $\mathcal{O}$ (Example 6.12). In this case, the action of the generalized braid group coincides with Arkhipov's twisting action (Proposition 6.38), which categorifies the regular representation of W.

Acknowledgments: The authors would like to thank Roman Bezrukavnikov, Dmitry Kaledin, Ivan Losev, and especially Anthony Licata for useful conversations. Additional thanks are due to Kevin McGerty and Thomas Nevins for bringing their work to the authors' attention. We are very grateful to the anonymous referee for many insightful comments and suggestions. Finally, the authors are grateful to the Mathematisches Forschungsinstitut Oberwolfach for its hospitality and excellent working conditions during the initial stages of work on this paper.

2 Conical symplectic resolutions

Let $\mathfrak{M}$ be a smooth, symplectic, complex algebraic variety. By this we mean that $\mathfrak{M}$ is equipped with a closed, nondegenerate, algebraic 2-form ω. Suppose further that $\mathfrak{M}$ is equipped with an action of the multiplicative group $\mathbb{S} := \mathbb{C}^\times$ such that $s^*\omega = s^n\omega$ for some integer $n \geq 1$. We also assume that $\mathbb{S}$ acts on the coordinate ring $\mathbb{C}[\mathfrak{M}]$ with only non-negative weights, and that the trivial weight space $\mathbb{C}[\mathfrak{M}]^{\mathbb{S}}$ is 1-dimensional, consisting only of the constant functions. Geometrically, this means that the affinization $\mathfrak{M}_0 := \operatorname{Spec} \mathbb{C}[\mathfrak{M}]$ is a cone, and the $\mathbb{S}$-action contracts $\mathfrak{M}_0$ to the cone point $o \in \mathfrak{M}_0$. Finally, we assume that the canonical map $\nu \colon \mathfrak{M} \to \mathfrak{M}_0$ is a projective resolution of singularities. (That is, it must be projective and an isomorphism over the smooth locus of $\mathfrak{M}_0$.) We will refer to this collection of data as a **conical symplectic resolution of weight n**.

Examples of conical symplectic resolutions include the following:

- $\mathfrak{M}$ is a crepant resolution of $\mathfrak{M}_0 = \mathbb{C}^2/\Gamma$, where Γ is a finite subgroup of $\mathrm{SL}(2;\mathbb{C})$. The action of $\mathbb{S}$ is induced by the inverse of the diagonal action on $\mathbb{C}^2$, and $n = 2$.

- $\mathfrak{M}$ is the Hilbert scheme of a fixed number of points on the crepant resolution of $\mathbb{C}^2/\Gamma$, and $\mathfrak{M}_0$ is the symmetric variety of unordered collections of points on the singular space. Once again, $\mathbb{S}$ acts by the inverse diagonal action on $\mathbb{C}^2$, and $n = 2$.

- $\mathfrak{M} = T^*(G/P)$ for a reductive algebraic group G and a parabolic subgroup P, and $\mathfrak{M}_0$ is the affinization of this variety. (If $G = \mathrm{SL}(r;\mathbb{C})$, then $\mathfrak{M}_0$ is isomorphic to the closure of a nilpotent orbit in the Lie algebra of G.) The action of $\mathbb{S}$ is the inverse scaling action on the cotangent fibers, and $n = 1$.

- $\mathfrak{M}$ is a hypertoric variety associated to a simple, unimodular, hyperplane arrangement in a rational vector space [BD00, Pro08], and $\mathfrak{M}_0$ is the hypertoric variety associated to the centralization of this arrangement. If the arrangement is coloop-free, then it possible to define an $\mathbb{S}$-action with $n = 1$ [HP04]; it is always possible to define an action with $n = 2$ [BK12, BLPW12].

- $\mathfrak{M}$ and $\mathfrak{M}_0$ are Nakajima quiver varieties [Nak94, Nak98]. If the quiver is acyclic, then there is a natural action with $n = 1$ [Nak94, §5]; it is always possible to define an action with $n = 2$ [Nak01, §2.7].

- $\mathfrak{M}_0$ is a transverse slice to one Schubert variety Gr^{μ} in an affine Grassmannian inside another Gr^{λ}. When λ is a sum of minuscule coweights, this variety has a natural conical symplectic resolution constructed from a convolution variety; in most other cases, it seems to possess no such resolution. This example is discussed in greater generality in [KWWY14].

Remark 2.1 The fifth class of examples overlaps significantly with each of the first four. The first two examples are special cases of quiver varieties, where the underlying graph of the quiver is the extended Dynkin diagram corresponding to Γ. When the group G of the third example is $\mathrm{SL}(r;\mathbb{C})$, then $T^*(G/P)$ is a quiver variety. Finally, a hypertoric variety associated to a cographical arrangement is a quiver variety.

Example 2.2 Almost all of the examples above arise as symplectic quotients of vector spaces. This applies to the first, second, fourth, and fifth classes of examples, as well as the third class when $G = \mathrm{SL}(r;\mathbb{C})$. More precisely, let G be a reductive algebraic group and V a faithful linear representation of G. Then G acts on the cotangent bundle $T^*V \cong V \times V^*$ with moment map

$$\mu\colon V \times V^* \to \mathfrak{g}^*$$

given by the formula $\mu(z,w)(x) := w(x \cdot z)$ for all $x \in \mathfrak{g}$, $z \in V$, and $w \in V^*$. Choose a character θ of G, and let $\mathfrak{M}$ be the associated GIT quotient of $\mu^{-1}(0)$. If G acts freely on the semistable locus of T^*V, then $\mathfrak{M}$ is symplectic and smooth. Its affinization $\mathfrak{M}_0$ is a normal affine variety, and the map $\nu\colon \mathfrak{M} \to \mathfrak{M}_0$ is automatically projective; if it is furthermore birational, then it is a symplectic resolution of singularities. We also have a natural map from $\mathfrak{M}_0$ to the the categorical quotient of $\mu^{-1}(0)$ with no stability condition imposed, which is not always an isomorphism, but will be in many interesting cases. The variety $\mathfrak{M}$ inherits a conical action of $\mathbb{S}$ of weight 2 from the inverse scaling action on $V \times V^*$. If V has no G-invariant functions, then we may take $\mathbb{S}$ to act only on V^* and obtain a conical action of weight 1.

Remark 2.3 All of these examples admit complete hyperkähler metrics, and in fact we know of no examples that do not admit complete hyperkähler metrics. (Such examples do exist if we drop the hypothesis that $\mathfrak{M}$ is projective over $\mathfrak{M}_0$; these examples will appear in subsequent work by the second author and Arbo.) The unit circle in $\mathbb{S}$ acts by hyperkähler isometries, but is Hamiltonian only with respect to the real symplectic form. Our assumptions about the $\mathbb{S}$-weights of $\mathbb{C}[\mathfrak{M}]$ translate to the statement that the real moment map for the circle action is proper and bounded below.

Proposition 2.4 *For all $i > 0$, $H^i(\mathfrak{M}; \mathfrak{S}_{\mathfrak{M}}) = 0$, where $\mathfrak{S}$ is the structure sheaf*[4] *of $\mathfrak{M}$.*

Proof: This follows from the Grauert-Riemenschneider theorem; see, for example, [Kal, 2.1]. □

Proposition 2.5 *All odd cohomology groups of $\mathfrak{M}$ vanish, and for all non-negative integers p we have $H^{2p}(\mathfrak{M};\mathbb{C}) = H^{p,p}(\mathfrak{M};\mathbb{C})$. In particular, the class of the symplectic form, which lies in $H^{2,0}(\mathfrak{M};\mathbb{C})$, is trivial.*

Proof: The analogous result with $\mathfrak{M}$ replaced by a fiber of ν is proven in [Kal09, 1.9], thus it suffices to prove that $\nu^{-1}(0)$ is homotopy equivalent to $\mathfrak{M}$. To see this, let $\Phi\colon \mathfrak{M}_0 \to \mathbb{R}$ be a real algebraic function which takes non-negative values and which is $\mathbb{S}$-equivariant for an action of the form $z \cdot t = |z|^k \cdot t$ of $\mathbb{S}$ on $\mathbb{R}$, where k is some positive integer. Such a function can

[4] Throughout this paper we will use the symbol $\mathfrak{S}$ for the structure sheaf of a variety. We avoid the usual symbol $\mathcal{O}$ because this symbol will needed for the analogue of BGG category $\mathcal{O}$ in the sequel to this paper [BLPW16].

be found of the form $\Phi = \sum_{i=1}^{r} |f_i|^{d_i}$, where f_i are homogeneous generators of $\mathbb{C}[\mathfrak{M}]$, with the grading induced by the action of $\mathbb{S}$. The argument from [Dur83, 1.6] shows that the inclusions $\nu^{-1}(0) \hookrightarrow (\Phi \circ \nu)^{-1}[0,t] \hookrightarrow \mathfrak{M}$ induce isomorphisms of homotopy groups, and so are homotopy equivalences. $\square$

Remark 2.6 The subvariety $\nu^{-1}(0) \subset \mathfrak{M}$ is often called the **core** or **compact core**, see for example [AB02, §4] or [Pro04, §2.2]. If $\mathfrak{M}$ is the cotangent bundle of a projective variety X, then the core of $\mathfrak{M}$ is simply the zero section. If $\mathfrak{M}$ is a crepant resolution of $\mathbb{C}^2/\Gamma$, then the core of $\mathfrak{M}$ is a union of projective lines in the shape of the Dynkin diagram for Γ. If $\mathfrak{M}$ is the Hilbert scheme of points on such a resolution, then the core of $\mathfrak{M}$ consists of configurations supported on the core of the resolution. If $\mathfrak{M}$ is the hypertoric variety associated to a real hyperplane arrangement, then the core of $\mathfrak{M}$ is a union of toric varieties corresponding to the bounded chambers of the arrangement [BD00, 6.5].

2.1 Deformations

We next collect some results of Namikawa and Kaledin on deformations of conical symplectic resolutions. The following proposition is due to Namikawa (see Lemma 12, Proposition 13, and Lemma 22 of [Nam08]).

Proposition 2.7 (Namikawa) *The variety $\mathfrak{M}$ has a universal Poisson deformation $\pi\colon \mathscr{M} \to H^2(\mathfrak{M};\mathbb{C})$ which is flat. The variety $\mathscr{M}$ admits an action of $\mathbb{S}$ extending the action on $\mathfrak{M} \cong \pi^{-1}(0)$, and π is $\mathbb{S}$-equivariant with respect to the weight $-n$ action on $H^2(\mathfrak{M};\mathbb{C})$.*

Remark 2.8 A formal version of this result appears in the work of Kaledin and Verbitsky [KV02]; the work of Kaledin on twistor families contains a very similar result, but not quite in the form we need.

Example 2.9 Suppose that $\mathfrak{M}$ arises from the quotient construction of Example 2.2. Let $\chi(\mathfrak{g})$ denote the vector space of characters $\mathfrak{g} \to \mathbb{C}$, and consider the Kirwan map $\mathsf{K} : \chi(\mathfrak{g}) \to H^2(\mathfrak{M};\mathbb{C})$ that takes an integral character to the Euler class of the induced line bundle on $\mathfrak{M}$. If the Kirwan map is an isomorphism (this is known when $\mathfrak{M}$ is a hypertoric variety, and conjectured in all cases), then $\mathscr{M}$ is isomorphic to the GIT quotient of $\mu^{-1}((\mathfrak{g}^*)^G)$, with the map to $H^2(\mathfrak{M};\mathbb{C}) \cong \chi(\mathfrak{g}) \cong (\mathfrak{g}^*)^G$ given by μ.

Given any class $\eta \in H^2(\mathfrak{M};\mathbb{C})$, let $\mathscr{M}_\eta := \mathscr{M} \times_{H^2(\mathfrak{M};\mathbb{C})} \mathbb{A}^1$, where $\mathbb{A}^1$ maps to $H^2(\mathfrak{M};\mathbb{C})$ via the linear map that takes 1 to η. Of particular interest is the case where η is the Euler class of a line bundle $\mathcal{L}$ on $\mathfrak{M}$. In this case, the following result follows from the work of Kaledin [Kal06, 1.4-1.6].

Proposition 2.10 (Kaledin) *There exists a unique $\mathbb{S}$-equivariant Poisson line bundle $\mathscr{L}$ on $\mathscr{M}_\eta$ extending the bundle $\mathcal{L}$ on $\mathfrak{M}$ such that the Poisson action of the coordinate function $t \in \mathbb{C}[\mathbb{A}^1]$ on the space of sections of $\mathscr{L}$ is the identity.*

Remark 2.11 Kaledin refers to the pair $(\mathscr{M}_\eta, \mathscr{L})$ as a **twistor family**. The second half of the proposition can be stated more geometrically as the condition that the complement $\mathscr{L}^\times$ of the zero section in the total space of $\mathscr{L}$ (the relative spectrum of the algebra sheaf $\bigoplus_{m\in\mathbb{Z}} \mathscr{L}^m$) carries a symplectic structure coinducing the Poisson structure on $\mathscr{M}_\eta$ such that the Hamiltonian vector field $\{t, -\}$ is the infinitesimal rotation of the fibers. In particular, $\mathfrak{M}$ is the symplectic reduction of $\mathscr{L}^\times$ by this Hamiltonian vector field.

Kaledin also tells us that $\mathscr{M}_\eta$ is symplectic over $\mathbb{A}^1$, and he computes the class of the relative symplectic form as follows [Kal06, 1.7].

Proposition 2.12 (Kaledin) *The Poisson structure on $\mathscr{M}_\eta$ is nondegenerate over $\mathbb{A}^1$, and the relative symplectic form $\omega_{\mathscr{M}_\eta} \in \Omega^2(\mathscr{M}_\eta/\mathbb{A}^1)$ satisfies*

$$[\omega_{\mathscr{M}_\eta}] = t\eta \in H^2_{DR}(\mathscr{M}_\eta/\mathbb{A}^1) \cong H^2(\mathfrak{M};\mathbb{C})[t].$$

Remark 2.13 Proposition 2.12 may be easily extended to say that $\mathscr{M}$ has a nondegenerate Poisson structure over $H^2(\mathfrak{M};\mathbb{C})$ with relative symplectic form

$$[\omega_{\mathscr{M}}] = I \in H^2_{DR}(\mathscr{M}/H^2(\mathfrak{M};\mathbb{C})),$$

where we identify the latter cohomology group with the space of polynomial maps from $H^2(\mathfrak{M};\mathbb{C})$ to itself, and I is the identity map.

Note that the $\mathbb{S}$-action may be used to identify all of the nonzero fibers of $\mathfrak{M}_\eta$ with a single symplectic variety $\mathscr{M}_\eta(\infty) := (\mathscr{M}_\eta \smallsetminus \mathfrak{M}) \,/\, \mathbb{S}$. The following result of Kaledin [Kal08, 2.5] will be crucial to our proof of Proposition 5.16.

Proposition 2.14 (Kaledin) *If $\mathcal{L}$ is ample, then $\mathscr{M}_\eta(\infty)$ is affine.*

2.2 The Weyl group

Next, we put some results of Namikawa [Nam10] into a form which is convenient for our purposes. Let $\{\Sigma_j\}$ be the codimension 2 connected components of the smooth part of the singular locus of $\mathfrak{M}_0$. At any point $\sigma_j \in \Sigma_j$, there exists a normal slice to Σ_j at σ_j which is isomorphic to a Kleinian singularity, thus the preimage $\nu^{-1}(\sigma_j) \subset \mathfrak{M}$ is a union of projective lines in the shape of a simply-laced finite-type Dynkin diagram D_j. The monodromy representation of the fundamental group $\pi_1(\Sigma_j)$ defines an action on D_j by diagram automorphisms. Let W_j be the centralizer of $\pi_1(\Sigma_j)$ in the Coxeter group associated to D_j, and let $W := \prod W_j$. We will call

W the **Weyl group** of $\mathfrak{M}$ (see Remark 2.15 for motivation). Namikawa constructs an action of W on $H^2(\mathfrak{M};\mathbb{R})$; he proves that the natural restriction map

$$H^2(\mathfrak{M};\mathbb{R}) \to \bigoplus_j H^2\big(\nu^{-1}(\sigma_j);\mathbb{R}\big)^{\pi_1(\Sigma_j)} \tag{1}$$

is W-equivariant and that W acts trivially on the kernel [Nam10, 1.1].[5]

Remark 2.15 Let G be the reductive algebraic group associated to a simply-laced finite-type Dynkin diagram D, and let B be a Borel subgroup. If $\mathfrak{M} = T^*(G/B)$, then $\mathfrak{M}_0$ is isomorphic to the nilpotent cone in $\mathfrak{g} := \mathrm{Lie}(G)$. The singular locus of $\mathfrak{M}_0$ is irreducible, and its smooth locus is called the subregular nilpotent orbit. The normal slice to the subregular orbit is isomorphic to the Kleinian singularity associated to D and W is isomorphic to the Weyl group of G. The action of W on $H^2(\mathfrak{M};\mathbb{C})$ is isomorphic to the action on the dual of a Cartan subalgebra of $\mathfrak{g}$ and the restriction map (1) is an isomorphism.

Let $\mathscr{N} := \operatorname{Spec}\mathbb{C}[\mathscr{M}]$.[6] Then the map $\pi : \mathscr{M} \to H^2(\mathfrak{M};\mathbb{C})$ factors canonically through $\mathscr{N}$. Namikawa [Nam11, Nam10, Nam] proves that the action of W on $H^2(\mathfrak{M};\mathbb{C})$ lifts to a symplectic action on $\mathscr{N}$, and that the quotient map $\mathscr{N}/W \to H^2(\mathfrak{M};\mathbb{C})/W$ is the universal Poisson deformation of the central fiber $\mathfrak{M}_0$.

Remark 2.16 The quotient $H^2(\mathfrak{M};\mathbb{C})/W$ is itself a vector space, which may be identified by a theorem of Namikawa [Nam10, 1.1] with the Poisson cohomology group $HP^2(\mathfrak{M}_0;\mathbb{C})$ as defined in [Nam08, §2].

2.3 Birational geometry

Let $P := \mathrm{Pic}(\mathfrak{M})$ be the Picard group of $\mathfrak{M}$. Proposition 2.5 tells us that

$$P_\mathbb{R} := P \otimes_\mathbb{Z} \mathbb{R} \cong H^2(\mathfrak{M};\mathbb{R});$$

in particular, P has finite rank. A class $\eta \in P$ is called **movable** if the associated line bundle is globally generated away from a codimension 2 subvariety of $\mathfrak{M}$. Let $\mathrm{Mov} \subset P_\mathbb{R}$ be the movable cone (the convex hull of the images of movable classes), and let $\overline{\mathrm{Mov}}$ be its closure.

Proposition 2.17 *The cone $\overline{\mathrm{Mov}} \subset P_\mathbb{R}$ is a fundamental domain for W.*

Proof: Consider the restriction map (1). Since W acts trivially on the kernel, any fundamental domain for the action on the target pulls back to a fundamental domain for the action on the

[5]This statement is equivalent to Namikawa's statement that the map ι, which he defines in the proof of his theorem, is an isomorphism.

[6]It would be natural to use the notation $\mathscr{M}_0$ rather than $\mathscr{N}$, but unfortunately that notation has already been used in the previous section to mean something else.

source. The space $H^2\big(\nu^{-1}(\sigma_j);\mathbb{R}\big)^{\pi_1(\Sigma_j)}$ may be identified W_j-equivariantly with the real part of the dual of the Cartan subalgebra of the Lie algebra determined by the Dynkin diagram D_j. The standard fundamental domain is the positive Weyl chamber, which may be characterized as the set of classes that are non-negative on the fundamental classes of the components of $\nu^{-1}(\sigma_i)$.

We have thus reduced the proposition to showing that a class $\eta \in P$ is movable if and only if $\eta \cdot E \geq 0$ for every curve $E \subset \mathfrak{M}$ such that E is a component of $\nu^{-1}(\sigma_j)$ for some j. Suppose first that $\eta \cdot E < 0$ for some such curve E. Since $E \cong \mathbb{P}^1$, this implies that every section of the line bundle associated to η vanishes on E, and therefore on the component of $\nu^{-1}(\Sigma_j)$ containing E. Since this component has codimension 1 in $\mathfrak{M}$, η cannot be movable. On the other hand, suppose that $\eta \cdot E \geq 0$ for every such curve. This implies that the associated line bundle is globally generated over $\nu^{-1}(\Sigma_j)$ for every j. It is obviously globally generated over the preimage of the smooth locus of $\mathfrak{M}_0$, since $\mathfrak{M}_0$ is affine. It is therefore globally generated over an open set whose complement has codimension 2, thus η is movable. □

We will wish to consider not just a single conical symplectic resolution, but rather a collection of varieties $\mathfrak{M}_1, \ldots, \mathfrak{M}_\ell$, all conical symplectic resolution of the same cone $\mathfrak{M}_0$; for any two of these, there is a birational map $f_{ij} \colon \mathfrak{M}_i \dashrightarrow \mathfrak{M}_j$, given by composing the resolution of $\mathfrak{M}_0$ by $\mathfrak{M}_i$ with the inverse of the resolution by $\mathfrak{M}_j$.

Proposition 2.18 *Each $\mathfrak{M}_i$ contains an open subvariety U_i with* $\operatorname{codim}(\mathfrak{M}_i \setminus U_i) \geq 2$ *such that f_{ij} induces an isomorphism $U_i \cong U_j$ for all j, and thus a canonical isomorphism between Picard groups of the different resolutions.*

Proof: Since the spaces in question are symplectic and therefore Calabi-Yau, there exist open subsets $U_i^j \subset \mathfrak{M}_i$ and $U_j^i \subset \mathfrak{M}_j$ with complements of codimension ≥ 2 such that f_{ij} induces an isomorphism from U_i^j to U_j^i; see, for example [Kaw02, 4.2]. Briefly, one can take any resolution $Q \to \mathfrak{M}_0$ (no longer symplectic!) which factors through $\mathfrak{M}_i$ and $\mathfrak{M}_j$ and pull out all irreducible components of the canonical divisor of Q; the remainder of Q maps isomorphically to subsets of $\mathfrak{M}_i$ and $\mathfrak{M}_j$ with complements of codimension ≥ 2; there is a canonical largest such set, so we can take U_i^j to be that one. We then let $U_i := \bigcap_{j=1}^{\ell} U_i^j$. □

Note that any class $\eta \in P$ which is movable for $\mathfrak{M}_i$ is also movable for $\mathfrak{M}_j$, thus we have a well-defined movable cone $\mathrm{Mov} \subset P_{\mathbb{R}}$. The following result of Namikawa [Nam] can be roughly summarized by the statement that $\mathfrak{M}$ is a **relative Mori dream space** over $\mathfrak{M}_0$ [AW, 2.4].

Theorem 2.19 (Namikawa) *There are finitely many isomorphism classes of conical symplectic resolutions of $\mathfrak{M}_0$. Furthermore, there exists a finite collection $\mathcal{H}$ of hyperplanes in $P_{\mathbb{R}}$, preserved by the action of W, with the following properties:*

- *For each conical symplectic resolution $\mathfrak{M}$, the ample cone of $\mathfrak{M}$ is a chamber of $\mathcal{H}$ (and different resolutions have different ample cones).*

- *The union of the closures of these ample cones is equal to $\overline{\mathrm{Mov}}$.*

- *The union $\bigcup_{H \in \mathcal{H}} H_{\mathbb{C}} \subset P_{\mathbb{C}} \cong H^2(\mathfrak{M}; \mathbb{C})$ is precisely equal to the locus over which the map $\mathscr{M} \to \mathscr{N}$ fails to be an isomorphism. Equivalently, it is the locus over which the fibers are not affine.*

Remark 2.20 Note that, by Proposition 2.17 and Theorem 2.19, the chambers of $\mathcal{H}$ are in bijection with the set of pairs $(\mathfrak{M}, w)$, where $\mathfrak{M}$ is a conical symplectic resolution of $\mathfrak{M}_0$ and w is an element of W. This bijection sends the pair $(\mathfrak{M}, w)$ to the w translate of the ample cone of $\mathfrak{M}$.

Remark 2.21 If $\mathfrak{M}$ is a quotient as in Example 2.2 and the Kirwan map of Example 2.9 is an isomorphism in degree 2, then the chambers of $\mathcal{H}$ are exactly the top dimensional cones in the GIT fan in $\chi(G)_{\mathbb{R}} \cong P_{\mathbb{R}}$.

3 Quantizations

Throughout the remainder of the paper, we will always use S to denote a scheme of finite type over $\mathbb{C}$ and $\mathfrak{X}$ to denote a smooth finite type S-scheme, projective over an affine scheme[7] $\mathfrak{X}_0$, equipped with a symplectic form $\omega_{\mathfrak{X}} \in \Omega^2(\mathfrak{X}/S)$. After Section 3.1, we will also assume throughout that $\mathfrak{X}$ and S carry compatible actions of $\mathbb{S}$ such that:

- The function algebra $\mathbb{C}[\mathfrak{X}]$ has no elements of negative $\mathbb{S}$-weight.

- The symplectic form satisfies $s^*\omega_{\mathfrak{X}} = s^n \omega_{\mathfrak{X}}$ for some positive integer n. Equivalently, the induced Poisson bracket $\{-,-\}$ on $\mathfrak{S}_{\mathfrak{X}}$ is homogeneous of weight $-n$.

- We have $H^1(\mathfrak{X}; \mathfrak{S}_{\mathfrak{X}})^{\mathbb{S}} \cong H^2(\mathfrak{X}; \mathfrak{S}_{\mathfrak{X}})^{\mathbb{S}} = 0$.

The cases that will be of primary interest to us arise in connection with a conical symplectic resolution $\mathfrak{M}$:

- $\mathfrak{X} = \mathfrak{M}$ and S is a point

- $\mathfrak{X} = \mathscr{M}_\eta$ and $S = \mathbb{A}^1$

- $\mathfrak{X} = \mathscr{M}$ and $S = H^2(\mathfrak{M}; \mathbb{C})$.

[7] In [Kal06], Kaledin uses the terminology "algebraically convex", but in other papers this term allows the map to only be *proper*; we emphasize that projectivity is essential.

Here $\mathscr{M}_\eta$ is the twistor deformation and $\mathscr{M}$ is the universal deformation, as in Section 2.1. We'll use these notations consistently throughout the paper. Each of these examples satisfies our assumptions for $\mathfrak{X}$; the only assumption which needs explanation is the cohomology vanishing, which holds for all three as a consequence of Grauert-Riemenschneider.

3.1 The period map

A **quantization** of $\mathfrak{X}$ consists of

- sheaf $\mathcal{Q}$ of flat $\pi^{-1}\mathfrak{S}_S[[h]]$-algebras on $\mathfrak{X}$, complete in the h-adic topology
- an isomorphism from $\mathcal{Q}/h\mathcal{Q}$ to the structure sheaf $\mathfrak{S}_\mathfrak{X}$ of $\mathfrak{X}$

satisfying the condition that, if f and g are functions over some open set and $\tilde{f}$ and $\tilde{g}$ are lifts to $\mathcal{Q}$, the image in $\mathfrak{S}_\mathfrak{X} \cong \mathcal{Q}/h\mathcal{Q} \cong h\mathcal{Q}/h^2\mathcal{Q}$ of the element $[\tilde{f}, \tilde{g}] \in h\mathcal{Q}$ is equal to the Poisson bracket $\{f, g\}$. Note that while we have assumed that $\mathfrak{X}$ is smooth over S and that the base field is $\mathbb{C}$, the notion of a quantization makes sense for any Poisson variety.

If $H^1(\mathfrak{X}; \mathfrak{S}_\mathfrak{X}) \cong H^2(\mathfrak{X}; \mathfrak{S}_\mathfrak{X}) = 0$ then, Bezrukavnikov and Kaledin [BK04, 1.8] show that the set of quantizations of $\mathfrak{X}$ is in natural bijection via the **period map** with the vector space[8]

$$[\omega_\mathfrak{X}] + h \cdot H^2_{DR}(\mathfrak{X}/S; \mathbb{C})[[h]].$$

More concretely, by Propositions 2.5 and 2.12 and Remark 2.13,

- the period map for $\mathfrak{M}$ takes values in $h \cdot H^2(\mathfrak{M}; \mathbb{C})[[h]]$
- the period map for $\mathscr{M}_\eta$ takes values in $t\eta + h \cdot H^2(\mathfrak{M}; \mathbb{C})[t][[h]]$
- the period map for $\mathscr{M}$ takes values in $I + h \cdot \big(H^2(\mathfrak{M}; \mathbb{C}) \otimes \mathbb{C}[H^2(\mathfrak{M}; \mathbb{C})]\big)[[h]]$.

The (unique) quantization with period $[\omega_\mathfrak{X}]$ is called the **canonical quantization** of $\mathfrak{X}$.

Let $\mathscr{Q}$ be a quantization of $\mathscr{M}_\eta$. There is an obvious way to recover a quantization of $\mathfrak{M}$ from $\mathscr{Q}$: if we divide by the ideal sheaf of $\pi^{-1}(0)$, we obtain a sheaf supported on $\pi^{-1}(0) \cong \mathfrak{M}$, and this sheaf is clearly a quantization. However, this is not the only quotient of $\mathscr{Q}$ which is supported on $\pi^{-1}(0)$. Fix an element $P(h) \in h \cdot \mathbb{C}[[h]]$. The map from $\mathbb{C}[t]$ to $\mathbb{C}[[h]]$ taking t to $P(h)$ induces a map from $\Delta := \operatorname{Spec} \mathbb{C}[[h]]$ to $\mathbb{A}^1$ sending the closed point to 0, and therefore a section σ_P of the projection $\mathbb{A}^1 \times \Delta \to \Delta$ which sends the closed point of Δ to 0. Dividing $\mathscr{Q}$ by the ideal sheaf in $\mathfrak{S}_{\mathbb{A}^1}[[h]]$ of the image of σ_P also gives a quantization of $\mathfrak{M}$. Following Bezrukavnikov and Kaledin, we denote this quantization by $\sigma_P^*\mathscr{Q}$. Note that the first construction in this paragraph corresponds to the choice $P = 0$.

[8] Following the conventions of [BK04], we will mean here the h-adic completion of $H^2_{DR}(\mathfrak{X}/S; \mathbb{C}) \otimes \mathbb{C}[[h]]$. This applies whenever we use the notation $V[[h]]$ for some vector space V.

More generally, for any quantization $\mathscr{Q}$ of $\mathfrak{X}/S$, let $\sigma\colon \Delta \to \Delta \times S$ be any section of the projection $\Delta \times S \to \Delta$. If $*$ is the unique closed point of Δ and $\sigma(*) = (*, s)$, then we may define $\sigma^*\mathscr{Q}$ to be the quotient of $\mathscr{Q}$ by the ideal sheaf of this section, thought of as a sheaf on $\pi^{-1}(s)$.

Let $\mathscr{Q}_R$ be the quantization of $\mathfrak{X}$ with period given by $R(h) \in H^2_{DR}(\mathfrak{X}/S;\mathbb{C})[[h]]$. If $\mathfrak{X} = \mathscr{M}_\eta$, we can think of this as a two variable function $R(t,h) \in t\eta + h \cdot H^2(\mathfrak{M};\mathbb{C})[t][[h]]$. The following proposition is an easy modification of [BK04, 6.4]; it follows immediately from the naturality of periods under pullback.

Proposition 3.1 (Bezrukavnikov and Kaledin) *The period of the quantization $\sigma^*\mathscr{Q}_R$ is $\sigma^*R(h) \in H^2(\pi^{-1}(s);\mathbb{C})[[h]]$. In particular, if $\mathfrak{X} = \mathscr{M}_\eta$ and $S = \mathbb{A}^1$, then $\sigma_P^*\mathscr{Q}_R$ has period $R(P(h),h) \in h \cdot H^2(\mathfrak{M};\mathbb{C})[[h]]$.*

Let us collect one more fact about quantizations which will be important for us. If $\mathcal{Q}$ is a quantization of $\mathfrak{X}/S$, we let $\mathcal{Q}^{\mathrm{op}}$ be the opposite algebra of $\mathcal{Q}$, thought of as a $\mathbb{C}[[h]]$-algebra with the action twisted by the automorphism $h \mapsto -h$; this convention is necessary to assure that $\mathcal{Q}^{\mathrm{op}}$ again quantizes the same Poisson structure.

Proposition 3.2 *If $P(h) \in [\omega_{\mathfrak{X}}] + h \cdot H^2_{DR}(\mathfrak{X}/S;\mathbb{C})[[h]]$ is the period of $\mathcal{Q}$, then the period of $\mathcal{Q}^{\mathrm{op}}$ is $P(-h)$.*

Proof: A proof of this fact is given in the proof of [Los12, 2.3.2], but the result is not stated as a theorem. As defined in [BK04, 4.1], the period map is the localization of a universal class $c \in H^2((\operatorname{Aut} D, \operatorname{Der} D), h\mathbb{C}[[h]])$ in the cohomology of the Harish-Chandra pair $(\operatorname{Aut} D, \operatorname{Der} D)$, where D is the Weyl algebra. The existence of a particular anti-automorphism sending $h \mapsto -h$ and $c(h) \mapsto c(-h)$ given in [Los12] shows that the period transforms the same way. □

Remark 3.3 In this Remark, contrary to our usage elsewhere, we will not assume *a priori* that the symbols $\mathfrak{M}$ and $\mathfrak{X}$ denote smooth varieties. Not every symplectic variety (in the sense of Beauville [Bea00]) admits a symplectic resolution; for example, closures of non-Richardson nilpotent orbits do not [Fu03]. On the other hand, every symplectic variety has a crepant partial resolution $\mathfrak{M}$ which is terminal and $\mathbb{Q}$-factorial; this is again a symplectic variety, since it is dominated by some resolution of $\mathfrak{M}_0$. The fact that this variety is $\mathbb{Q}$-factorial means that it cannot be resolved further without introducing discrepancy: a crepant partial resolution of $\mathfrak{M}$ would have to be isomorphic to $\mathfrak{M}$ in codimension 1 so their group of Weil divisors would be the same; thus an ample line bundle on the resolution would have to correspond to a Weil divisor on $\mathfrak{M}$, some power of which is a Cartier divisor, showing that the resolution is in fact $\mathfrak{M}$.

While the theory of periods we have discussed thus far cannot be directly applied to $\mathfrak{M}$, it can be applied to the smooth locus $\mathring{\mathfrak{M}}$. More generally, let $\mathfrak{X}/S$ be a convex symplectic (not necessarily smooth) variety with terminal singularities [Nam08, §1], and $\mathring{\mathfrak{X}}$ its smooth locus. As noted by Namikawa in the proof of [Nam08, Lemma 12], $H^1(\mathring{\mathfrak{X}}; \mathfrak{S}_{\mathring{\mathfrak{X}}}) = H^2(\mathring{\mathfrak{X}}; \mathfrak{S}_{\mathring{\mathfrak{X}}}) = 0$, so $\mathring{\mathfrak{X}}$ satisfies our running assumptions. By [BK04, 1.8], the quantizations of $\mathring{\mathfrak{X}}$ are in bijection with $[\omega_{\mathfrak{X}}] + h \cdot H^2_{DR}(\mathring{\mathfrak{X}}/S; \mathbb{C})[[h]]$. Let $i\colon \mathring{\mathfrak{X}} \to \mathfrak{X}$ be the inclusion map.

Proposition 3.4 *If $\mathring{\mathcal{Q}}$ is a quantization of $\mathring{\mathfrak{X}}$, then $i_*\mathring{\mathcal{Q}}$ is a quantization of $\mathfrak{X}$. If $\mathcal{Q}$ is a quantization of $\mathfrak{X}$, then $i^{-1}\mathcal{Q}$ is a quantization of $\mathring{\mathfrak{X}}$. These two operations induce inverse bijections between isomorphism classes of quantizations of $\mathring{\mathfrak{X}}$ and $\mathfrak{X}$.*

Proof: The fact that $i_*\mathring{\mathcal{Q}}$ is a quantization follows from normality of symplectic varieties; the fact that $i^{-1}\mathcal{Q}$ is a quantization is trivial, as is the isomorphism $i^{-1}i_*\mathring{\mathcal{Q}} \cong \mathring{\mathcal{Q}}$. In the other direction, the natural map $i_*i^{-1}\mathcal{Q} \to \mathcal{Q}$ is an isomorphism mod h, and thus is an isomorphism by Nakayama's lemma. □

In most sections of this paper (with the exception of Section 6.2), we could allow our conical symplectic resolutions to be terminal and $\mathbb{Q}$-factorial rather than smooth. For ease of exposition, however, we will continue to assume smoothness.

3.2 $\mathbb{S}$-structures

From this point forward, we will assume that $\mathfrak{X}$ and S carry compatible actions of $\mathbb{S}$ such that:

- The function algebra $\mathbb{C}[\mathfrak{X}]$ has no elements of negative $\mathbb{S}$-weight.
- The symplectic form satisfies $s^*\omega_{\mathfrak{X}} = s^n\omega_{\mathfrak{X}}$ for some positive integer n. Equivalently, the induced Poisson bracket $\{-,-\}$ on $\mathfrak{S}_{\mathfrak{X}}$ is homogeneous of weight $-n$.
- We have $H^1(\mathfrak{X}; \mathfrak{S}_{\mathfrak{X}})^{\mathbb{S}} \cong H^2(\mathfrak{X}; \mathfrak{S}_{\mathfrak{X}})^{\mathbb{S}} = 0$.

In this section we define the notion of an $\mathbb{S}$-structure on a quantization of $\mathfrak{X}/S$, and we consider the question of which quantizations carry $\mathbb{S}$-structures.

Let $a\colon \mathbb{S} \times \mathfrak{X} \to \mathfrak{X}$ be the action map, let $p\colon \mathbb{S} \times \mathfrak{X} \to \mathfrak{X}$ be the projection onto $\mathfrak{X}$, and let $e\colon \mathbb{S} \times \mathfrak{X} \to \mathbb{S}$ be the projection onto $\mathbb{S}$. If $\mathcal{Q}$ is a quantization of $(\mathfrak{X}, \omega_{\mathfrak{X}})$, then the naive pullback $a^*\mathcal{Q} := a^{-1}\mathcal{Q} \otimes_{\mathbb{C}[[h]]} e^{-1}\mathfrak{S}_{\mathbb{S}}[[h]]$ is a quantization of $\mathfrak{X} \times \mathbb{S}$ over $\mathbb{S}$ with the relative symplectic form $a^*\omega_{\mathfrak{X}} = z^n p^*\omega_{\mathfrak{X}}$, where z is the coordinate function on $\mathbb{S}$. Since forms are contravariant and bivectors covariant, the corresponding Poisson brackets are related by $\{-,-\}_a = z^{-n}\{-,-\}_p$. As long as the Poisson bracket on $\mathfrak{X}$ is nontrivial, the sheaves $a^*\mathcal{Q}$ and $p^*\mathcal{Q}$ are quantizations of different Poisson brackets on $\mathfrak{X} \times \mathbb{S}$, thus they are never isomorphic.

This difference between the two Poisson brackets can be resolved by twisting the action of h. More precisely, let $a^*_{\mathrm{tw}}\mathcal{Q} := a^{-1}\mathcal{Q} \otimes_{\mathbb{C}[[h]]} e^{-1}\mathfrak{S}_{\mathbb{S}}[[h]]$, where this time the action of $\mathbb{C}[[h]]$

on $e^{-1}\mathfrak{S}_{\mathbb{S}}[[h]]$ is given by sending h to $z^n h$. Put differently, $a_{\text{tw}}^*\mathcal{Q}$ and $a^*\mathcal{Q}$ are isomorphic as sheaves of vector spaces, but the endomorphism given by multiplication by h in $a_{\text{tw}}^*\mathcal{Q}$ corresponds to the endomorphism given by multiplication by $z^{-n}h$ in $a^*\mathcal{Q}$. Then $a_{\text{tw}}^*\mathcal{Q}$ is a quantization of the Poisson bracket $z^n\{-,-\}_a = \{-,-\}_p$, that is, corresponding to the relative symplectic form $p^*\omega_{\mathfrak{X}}$.

An $\mathbb{S}$-**structure** on $\mathcal{Q}$ is an isomorphism $a_{\text{tw}}^*\mathcal{Q} \cong p^*\mathcal{Q}$ as $(\mathrm{id}_{\mathbb{S}} \times \pi)^{-1}\mathfrak{S}_{\mathbb{S}\times S}[[h]]$-algebras, satisfying the natural cocycle condition. That is, the above isomorphism induces an isomorphism $s^*\mathcal{Q} \cong \mathcal{Q}$ for every $s \in \mathbb{S}$, and we require that for any three elements of $\mathbb{S}$ with $s \cdot s' \cdot s'' = 1$, the composition of the three isomorphisms is the identity. In [Los12], this is called a "grading" on the quantization. We will often refer to a quantization endowed with an $\mathbb{S}$-structure as an $\mathbb{S}$-**equivariant** quantization.

As a general principle, quantizations have $\mathbb{S}$-structures whenever their period does not obstruct this possibility. More precisely, Losev [Los12, 2.3.3] proves the following result.[9]

Proposition 3.5 (Losev) *A quantization of $\mathfrak{X}$ admits an $\mathbb{S}$-structure if and only if its period lies in the vector space $[\omega_{\mathfrak{X}}] + h \cdot H^2_{DR}(\mathfrak{X}/S;\mathbb{C}) \subset [\omega_{\mathfrak{X}}] + h \cdot H^2_{DR}(\mathfrak{X}/S;\mathbb{C})[[h]]$, in which case its $\mathbb{S}$-structure is unique.*

As noted in [BK04, §6.1], as long as we have the assumptions $H^1(\mathfrak{X};\mathfrak{S}_{\mathfrak{X}})^{\mathbb{S}} = H^2(\mathfrak{X};\mathfrak{S}_{\mathfrak{X}})^{\mathbb{S}} = 0$, the variety $\mathfrak{X}$ is $\mathbb{S}$-equivariantly admissible. Even if there are vectors of non-zero weight in $H^1(\mathfrak{X};\mathfrak{S}_{\mathfrak{X}})$ or $H^2(\mathfrak{X};\mathfrak{S}_{\mathfrak{X}})$, we can still apply the theory of [BK04] to $\mathbb{S}$-equivariant quantizations; in particular, every period in $[\omega_{\mathfrak{X}}] + h \cdot H^2_{DR}(\mathfrak{X}/S;\mathbb{C})$ has a corresponding unique $\mathbb{S}$-equivariant quantization.

3.3 The section ring

Let $\mathcal{Q}$ be an $\mathbb{S}$-equivariant quantization of $\mathfrak{X}$. Define

$$\mathcal{D}(0) := \mathcal{Q}[h^{1/n}], \qquad \mathcal{D} := \mathcal{Q}[h^{-1/n}], \qquad \text{and} \qquad \mathcal{D}(m) := h^{-m/n}\mathcal{D}(0) \subset \mathcal{D} \text{ for all } m \in \mathbb{Z}.$$

We will frequently abuse notation by referring to $\mathcal{D}$ as a quantization of $\mathfrak{X}$.

Let $A := \Gamma_{\mathbb{S}}(\mathcal{D})$ be the ring of $\mathbb{S}$-invariant sections of $\mathcal{D}$. This ring inherits a $\mathbb{Z}$-filtration

$$\ldots \subset A(-1) \subset A(0) \subset A(1) \subset \ldots \subset A$$

given by putting

$$A(m) := \Gamma_{\mathbb{S}}\big(\mathcal{D}(m)\big).$$

The associated graded of A may be canonically identified with $\mathbb{C}[\mathfrak{X}]$ as a $\mathbb{Z}$-graded ring via the

[9]Losev assumes that $n = 2$, but his proof works for arbitrary n.

maps

$$A(m) = \Gamma_{\mathbb{S}}\big(\mathcal{D}(m)\big) \xrightarrow{\cdot h^{m/n}} \Gamma\big(\mathcal{D}(0)\big) \twoheadrightarrow \Gamma\big(\mathcal{D}(0)/\mathcal{D}(-1)\big) \cong \Gamma(\mathfrak{S}_{\mathfrak{X}}) = \mathbb{C}[\mathfrak{X}].$$

Many of the examples of conical symplectic resolutions we gave at the beginning of Section 2 admit quantizations for which the ring A is of independent interest. (In all of these examples S is a point.)

- Let $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ be a finite subgroup. Any quantization of the Hilbert scheme of m points on a crepant resolution of $\mathbb{C}^2/\Gamma$ has its invariant section ring A isomorphic to a spherical symplectic reflection algebra for the wreath product $S_m \wr \Gamma$, with parameters corresponding to the period of the quantization [EGGO07, 1.4.4], [Gor06, 1.4].

- Let G be a reductive Lie group and $B \subset G$ a Borel subgroup. Then each quantization of $T^*(G/B)$ has its invariant section ring A isomorphic to a central quotient of the universal enveloping algebra $U(\mathfrak{g})$. All central quotients arise this way, and two quantizations give the same central quotient if their periods are related by the action of the Weyl group [BB81, Lemma 3].

- Many quantizations of a resolution of a Slodowy slice to a nilpotent orbit in $\mathfrak{g}$ have invariant section ring A isomorphic to a central quotient of a finite W-algebra; the cases where every quantization has this property (which includes all slices in type A) are classified by Ambrosio, Carnovale, Esposito and Topley [ACET]. Again, all central quotients of the W-algebra arise this way, and two quantizations give the same central quotient if their periods are related by the action of the Weyl group [Pre02, 6.4].

- Any quantization of a hypertoric variety has its invariant section ring A isomorphic to a central quotient of the hypertoric enveloping algebra. Once more, all central quotients arise this way, and two quantizations give isomorphic central quotients if their periods are related by the action of the Weyl group [BK12, §5], [BLPW12, 5.9].

- In [KWWY14], it is conjectured that the algebra arising from the slices in the affine Grassmannian can be described as a quotient of a **shifted Yangian**, a variant of the usual Yangian of Drinfeld.

Consider the universal Poisson deformation $\pi\colon \mathscr{M} \to H^2(\mathfrak{M};\mathbb{C})$ of $\mathfrak{M}$. Let $\mathscr{D}$ be the canonical quantization of $\mathscr{M}$, and let $\mathscr{A} := \Gamma_{\mathbb{S}}(\mathscr{D})$ be its invariant section algebra. The $\pi^{-1}\mathfrak{S}_{H^2(\mathfrak{M};\mathbb{C})}$-structure on $\mathscr{D}$ induces a map

$$c\colon \mathbb{C}[H^2(\mathfrak{M};\mathbb{C})] \to \Gamma(\mathscr{M};\mathscr{D})$$

which is $\mathbb{S}$-equivariant for the weight n action on $\mathbb{C}[H^2(\mathfrak{M};\mathbb{C})]$. In particular, if $x \in H^2(\mathfrak{M};\mathbb{C})^*$ is a linear function on $H^2(\mathfrak{M};\mathbb{C})$, we have that $h^{-1}c(x) \in \mathscr{A}$.

Let $\lambda \in H^2(\mathfrak{M};\mathbb{C})$ be the period of $\mathcal{D}$. By Proposition 3.1, $\mathcal{D} = \sigma_{h\lambda}^*\mathscr{D}$, and this induces a restriction map from $\mathscr{A}$ to A.

Proposition 3.6 *The map from $\mathscr{A}$ to A is surjective with kernel generated by $h^{-1}c(x) - \lambda(x)$ for all $x \in H^2(\mathfrak{M};\mathbb{C})^*$.*

Proof: Let $\mathbb{C}_\lambda$ be the evaluation module at λ of $\mathbb{C}[H^2(\mathfrak{M};\mathbb{C})]$. The sheaf $\mathcal{D}$ can be rewritten as the cohomology of the tensor product of $\mathscr{D}$ with the Koszul resolution of $\mathbb{C}_\lambda$. Thus, the sheaf cohomology of $\mathcal{D}$ is the hypercohomology of this complex. Filtering this complex by degrees in the Koszul resolution, we obtain the spectral sequence

$$\mathrm{Tor}^i_{\mathbb{C}[H^2(\mathfrak{M};\mathbb{C})]}(H^j(\mathscr{D})^{\mathbb{S}}, \mathbb{C}_\lambda) \Rightarrow H^{j-i}(\mathcal{D})^{\mathbb{S}}$$

converging to the cohomology of $\mathcal{D}$. Since $\mathscr{D}$ has trivial higher cohomology, this spectral sequence collapses immediately, and we obtain the desired isomorphism. $\square$

Lemma 3.7 *Let $\mathfrak{X}$ be a smooth symplectic variety over a smooth base S. Let $i\colon U \hookrightarrow \mathfrak{X}$ be an open inclusion, and let d be the codimension of the complement of U.*

- *If $d \geq 2$, then for any quantization $\mathcal{Q}$ of $\mathfrak{X}$, the restriction $i^*\mathcal{Q}$ to U is a quantization of U with $\Gamma(U; i^*\mathcal{Q}) \cong \Gamma(\mathfrak{X};\mathcal{Q})$.*
- *If $d \geq 3$, then for any quantization $\mathcal{Q}'$ of U, the pushforward $i_*\mathcal{Q}'$ is a quantization of $\mathfrak{X}$ with $\Gamma(U;\mathcal{Q}') \cong \Gamma(\mathfrak{X}; i_*\mathcal{Q}')$.*

Proof: Let $j\colon \mathfrak{X} \setminus U \to \mathfrak{X}$ be the inclusion. As usual for complementary closed and open embeddings, we have an exact triangle $j_*j^!\mathfrak{S}_{\mathfrak{X}} \to \mathfrak{S}_{\mathfrak{X}} \to i_*i^*\mathfrak{S}_{\mathfrak{X}} \to j_*j^!\mathfrak{S}_{\mathfrak{X}}[1]$. The induced long exact sequence takes the form of a short exact sequence

$$0 \to \mathfrak{S}_{\mathfrak{X}} \to i_*\mathfrak{S}_U \to j_*\mathbb{R}^1 j^!\mathfrak{S}_{\mathfrak{X}} \to 0$$

along with isomorphisms $\mathbb{R}^k i_*\mathfrak{S}_U \cong j_*\mathbb{R}^{k+1}j^!(\mathfrak{S}_{\mathfrak{X}})$ for all $k > 0$. The local cohomology sheaf $\mathbb{R}^k j^!\mathfrak{S}_{\mathfrak{X}}$ vanishes for all $k < d$, so we may conclude that $i_*\mathfrak{S}_U \cong \mathfrak{S}_{\mathfrak{X}}$ if $d \geq 2$, and $\mathbb{R}^1 i_*\mathfrak{S}_U = 0$ if $d \geq 3$.

Assume that $d \geq 2$, and consider a quantization $\mathcal{Q}$ on $\mathfrak{X}$. It is clear that $i^*\mathcal{Q}$ is a quantization of U, so we need only show that the sections are unchanged. For each $m \geq 0$, the natural map $\mathcal{Q}/h^m\mathcal{Q} \to i_*i^*(\mathcal{Q}/h^m\mathcal{Q})$ is an isomorphism; this follows from induction and the five-lemma applied to the diagram:

$$\begin{tikzcd}
h\mathcal{Q}/h^m\mathcal{Q} \arrow[r] \arrow[d] & \mathcal{Q}/h^m\mathcal{Q} \arrow[r] \arrow[d] & \mathfrak{S}_{\mathfrak{X}} \arrow[d] \\
i_*i^*(h\mathcal{Q}/h^m\mathcal{Q}) \arrow[r] & i_*i^*(\mathcal{Q}/h^m\mathcal{Q}) \arrow[r] & i_*i^*(\mathfrak{S}_{\mathfrak{X}})
\end{tikzcd}$$

Since U is open, i_*i^* commutes with projective limits, so we have an isomorphism $i_*i^*\mathcal{Q} \cong \mathcal{Q}$. The isomorphism of sections of $\mathcal{Q}$ and $i^*\mathcal{Q}$ now follows by the functoriality of push-forward.

Now assume that $d \geq 3$, and let $\mathcal{Q}'$ be a quantization on U. The flatness of $i_*\mathcal{Q}'$ is automatic, so we need only show that $i_*\mathcal{Q}'/i_*(h\mathcal{Q}') \cong \mathfrak{S}_{\mathfrak{X}}$. The short exact sequence

$$h\mathcal{Q}'/h^m\mathcal{Q}' \to \mathcal{Q}'/h^m\mathcal{Q}' \to \mathfrak{S}_U$$

similarly shows inductively that $\mathbb{R}^1 i_*(\mathcal{Q}'/h^m\mathcal{Q}') = 0$ for all m. An argument as in [KR08, 2.12], using the Mittag-Leffler condition, shows that thus $\mathbb{R}^1 i_*\mathcal{Q}' = 0$.

Consider the long exact sequence

$$0 \to i_*(h\mathcal{Q}') \to i_*\mathcal{Q}' \to \mathfrak{S}_{\mathfrak{X}} \to \mathbb{R}^1 i_*(h\mathcal{Q}') \to \cdots .$$

Since $\mathbb{R}^1 i_*(h\mathcal{Q}') \cong \mathbb{R}^1 i_*\mathcal{Q}' = 0$, we get an isomorphism $i_*\mathcal{Q}'/i_*(h\mathcal{Q}') \cong \mathfrak{S}_{\mathfrak{X}}$, and so the $\mathbb{C}[[h]]$-module $i_*\mathcal{Q}'$ is a quantization. □

Now we turn to the case of a conical symplectic resolution $\mathfrak{M}$. In this case, the ring A depends only on the cone $\mathfrak{M}_0$, and not on the choice of resolution.

More precisely, let $\mathfrak{M}$ and $\mathfrak{M}'$ be two conical symplectic resolutions of the same affine cone. By Proposition 2.18, the groups $H^2(\mathfrak{M};\mathbb{C})$ and $H^2(\mathfrak{M}';\mathbb{C})$ are canonically isomorphic. Let $\mathcal{D}$ and $\mathcal{D}'$ be quantizations of $\mathfrak{M}$ and $\mathfrak{M}'$ with the same period, and $\mathscr{D}$ and $\mathscr{D}'$ the corresponding quantizations of the universal quantizations $\mathscr{M}$ and $\mathscr{M}'$.

Proposition 3.8 *There is a canonical isomorphism between the section rings $\mathscr{A} := \Gamma_{\mathbb{S}}(\mathscr{M};\mathscr{D})$ and $\mathscr{A}' := \Gamma_{\mathbb{S}}(\mathscr{M}';\mathscr{D}')$.*

Proof: We have a canonical rational map $\mathscr{M} \dashrightarrow \mathscr{M}'$. This induces an isomorphism between the fiber over a generic point in $H^2(\mathfrak{M};\mathbb{C}) \cong H^2(\mathfrak{M}';\mathbb{C})$, and gives a pair of crepant resolutions of each fiber. Thus, applying Proposition 2.18 to each fiber, we find that the exceptional locus of this map is codimension 2 in each fiber. Combining this with the fact that the generic fiber avoids the exceptional locus, we see that it has codimension 3. Let $U \subset \mathscr{M}, U' \subset \mathscr{M}'$ be the complements to the exceptional loci, so that $\mathscr{M} \dashrightarrow \mathscr{M}'$ induces an isomorphism $U \cong U'$.

Let $i\colon U \to \mathscr{M}$ and $i'\colon U' \to \mathscr{M}'$ be the inclusions of these sets. By Lemma 3.7, $\mathscr{D}'' := i'_* i^* \mathscr{D}$ is a quantization of $\mathscr{M}'$ with section ring

$$\Gamma_{\mathbb{S}}(\mathscr{M}'; \mathscr{D}'') \cong \Gamma_{\mathbb{S}}(U; \mathscr{D}) \cong \Gamma_{\mathbb{S}}(\mathscr{M}; i_* \mathscr{D}) \cong \mathscr{A}.$$

Since $(i')^* \mathscr{D}'' \cong i^* \mathscr{D}$ and $(i')^* \mathscr{D}'$ have the same period (by definition), the quantizations $\mathscr{D}'$ and $\mathscr{D}''$ must also have the same period and thus are isomorphic. Thus, we have that $\mathscr{A} \cong \Gamma_{\mathbb{S}}(\mathscr{M}'; \mathscr{D}'') \cong \Gamma_{\mathbb{S}}(\mathscr{M}'; \mathscr{D}') \cong \mathscr{A}'$. □

Propositions 3.6 and 3.8 have the following corollary.

Corollary 3.9 *There is a canonical isomorphism between the section rings $A := \Gamma_{\mathbb{S}}(\mathfrak{M}; \mathcal{D})$ and $A' := \Gamma_{\mathbb{S}}(\mathfrak{M}'; \mathcal{D}')$.*

We may now use Proposition 3.6 to show that the ring A does not change when the period of $\mathcal{D}$ changes by an element of the Weyl group; this unifies the isomorphisms mentioned in three of the four examples above. For any $\lambda \in H^2(\mathfrak{M}; \mathbb{C})$, let A_λ be the invariant section algebra of the quantization with period λ.

Proposition 3.10 *For any $\lambda \in H^2(\mathfrak{M}; \mathbb{C})$ and $w \in W$, we have an isomorphism $A_\lambda \cong A_{w\cdot\lambda}$. Furthermore, these isomorphisms may be chosen to be compatible with multiplication in the Weyl group.*

Proof: As in Section 2.1, let $\mathscr{N} := \operatorname{Spec} \mathbb{C}[\mathscr{M}]$ be the affinization of the universal deformation of $\mathfrak{M}$, and let $\mathring{\mathscr{M}} \subset \mathscr{M}$ be the locus on which the map to $\mathscr{N}$ is a local isomorphism. Since this map is a crepant resolution of singularities, it induces an isomorphism from $\mathring{\mathscr{M}}$ to the smooth locus of $\mathscr{N}$. Thus, $\mathring{\mathscr{M}}$ inherits a W-action from $\mathscr{N}$ and the canonical quantization $\mathscr{D}$ of $\mathscr{M}$ restricted to $\mathring{\mathscr{M}}$ is also W-equivariant. Note that $\mathscr{A} := \Gamma_{\mathbb{S}}(\mathscr{M}; \mathscr{D})$ is isomorphic to $\Gamma_{\mathbb{S}}(\mathring{\mathscr{M}}; \mathscr{D})$ by Lemma 3.7, since the codimension of the complement of $\mathring{\mathscr{M}}$ is at least 2. Thus $\mathscr{A}$ carries a natural W-action. The proposition now follows from Proposition 3.6 and the W-equivariance of $h^{-1}c$. □

3.4 Quantum Hamiltonian reduction

Let $\mathcal{Q}$ be an $\mathbb{S}$-equivariant quantization of $\mathfrak{X}$. Let G be a connected reductive algebraic group over $\mathbb{C}$, and assume that $\mathfrak{X}$ is equipped with a G-action commuting with the action of $\mathbb{S}$. We will assume that the action of G is Hamiltonian with moment map $\mu : \mathfrak{X} \to \mathfrak{g}^*$, and that μ is $\mathbb{S}$-equivariant with respect to the weight n scalar action on $\mathfrak{g}^*$. A **Hamiltonian *G*-action** on the pair $(\mathfrak{X}, \mathcal{Q})$ consists of

1. an action of G on $\mathfrak{X}$ as above

2. a G-equivariant structure on $\mathcal{Q}$ so that the algebra map $\mathcal{Q} \to \mathfrak{S}_{\mathfrak{X}}$ is equivariant

3. a G-equivariant filtered[10] $\mathbb{C}[S]$-algebra homomorphism $\eta\colon U(\mathfrak{g}) \to \Gamma_{\mathbb{S}}(\mathcal{Q}[h^{-1}]) \subset A$

such that for all $x \in \mathfrak{g}$, the adjoint action of $\eta(x)$ on $\mathcal{Q}$ agrees with the action of x induced by the G-structure on $\mathcal{Q}$. The map η is called a **quantized moment map** because the associated graded

$$\operatorname{gr}\eta\colon \mathbb{C}[\mathfrak{g}^*] \cong \operatorname{gr} U(\mathfrak{g}) \longrightarrow \operatorname{gr} A \cong \mathbb{C}[\mathfrak{X}]$$

induces a $G \times \mathbb{S}$-equivariant classical moment map $\mu\colon \mathfrak{X} \to \mathfrak{g}^*$, where $\mathbb{S}$ acts on $\mathfrak{g}^*$ with weight $-n$. We note that for any $x \in \mathfrak{g}$, we will have

$$\eta(x) \in \Gamma_{\mathbb{S}}(h^{-1}\mathcal{Q}) \subset \Gamma_{\mathbb{S}}(\mathcal{D}(n)) = A(n) \subset A.$$

The following proposition says that the condition of admitting a quantized moment map is no stronger than the condition of admitting a classical moment map. Recall that we use $\chi(\mathfrak{g})$ to denote the vector space of characters $\mathfrak{g} \to \mathbb{C}$.

Proposition 3.11 *For any $\mathbb{S}$-equivariant quantization $\mathcal{Q}$ of $\mathfrak{X}$, the pair $(\mathfrak{X}, \mathcal{Q})$ admits a Hamiltonian G-action that induces μ in the manner described above and the set of quantized moment maps compatible with a given G-action on $\mathcal{Q}$ is a torsor for $\chi(\mathfrak{g}) \otimes \mathbb{C}[\mathfrak{X}]^{\mathbb{S}\times G}$.*

Proof: We break this up to into proofs of the different components of this result. First, we prove that $\mathcal{Q}$ has a unique G-equivariant structure. Since G is connected, its action on the de Rham cohomology $H^2_{DR}(\mathfrak{X}/S;\mathbb{C})$ is trivial. As noted in the proof of [BK04, Prop. 6.2] and expanded on in more detail in [Los12, §2.3], this means that the quantization for every class in $H^2_{DR}(\mathfrak{X}/S;\mathbb{C})$ has a unique equivariant structure. As discussed in the sources above, this can be seen by thinking about inserting G-equivariance at each point in the original proof of [BK04, Thm. 1.8]. Since the discussions in [BK04] and [Los12] are quite brief, we will add some more details below.

This proof of [BK04, Thm. 1.8] proceeds by interpreting the problem of lifting a given quantization modulo h^n to be a quantization modulo h^{n+1} as finding an extension of the corresponding Harish-Chandra torsor over the pair $\langle(\operatorname{Aut} D)_n, (\operatorname{Der} D)_n\rangle$ to $\langle(\operatorname{Aut} D)_{n+1}, (\operatorname{Der} D)_{n+1}\rangle$. The set of such extensions is a torsor for the sheaf cohomology $H^1(\mathfrak{X}; \mathfrak{S}_{\mathfrak{X}}/\pi^{-1}\mathfrak{S}_S)$ of the quotient of functions on $\mathfrak{X}$ by the pullback of functions on S. This torsor is identified with the de Rham cohomology $H^1_{DR}(\mathfrak{X}/S;\mathcal{V})$ of a relative local system $\mathcal{V}$ on $\mathfrak{X}/S$, and thus can be identified with the set of extensions of relative local systems $\mathcal{V} \to \mathcal{W} \to \mathfrak{S}_{\mathfrak{X}}$. As usual, a $\mathcal{V}$-valued closed 1-form θ gives a natural relative local system, whose flat sections are identified with the set of sections of

[10] We filter $U(\mathfrak{g})$ so that the associated graded $\mathbb{C}[\mathfrak{g}^*]$ has $\mathfrak{g}$ sitting in degree n.

$\mathcal{V}$ satisfying $df = k\theta$ for $k \in \mathbb{C}$. Two such extensions $\mathcal{W}, \mathcal{W}'$ arising from 1-forms θ, θ' are isomorphic if and only $\theta - \theta' = dg$ for some g in the holomorphic sections of $\mathcal{V}$, with the isomorphism given by $f \mapsto f + g$. We have a short exact sequence $0 \to \pi^{-1}\mathfrak{S}_S \to \mathfrak{S}_{\mathfrak{X}} \to \mathfrak{S}_{\mathfrak{X}}/\pi^{-1}\mathfrak{S}_S \to 0$. The boundary map in the long exact sequence induced on sheaf cohomology gives an isomorphism $H^1(\mathfrak{X}; \mathfrak{S}_{\mathfrak{X}}/\pi^{-1}\mathfrak{S}_S) \cong H^2_{DR}(\mathfrak{X}/S; \mathbb{C}) \cong H^2(\mathfrak{X}; \pi^{-1}\mathfrak{S}_S)$.

In order to pass to the equivariant case, we only need to think about the same lifting problem equivariantly for G; thus, the lifts are a torsor for the groups of G-equivariant extensions of local systems $\mathcal{V} \to \mathcal{W} \to \mathfrak{S}_{\mathfrak{X}}$. Of course, any such extension corresponds to a 1-form θ whose cohomology class is G-invariant. Since G is reductive, we can assume θ itself is invariant by projecting it to invariant 1-forms, which doesn't change its class in cohomology. The extension of local systems discussed above has an induced G-equivariant structure, and using reductivity again, this structure is unique, because the projection of a not necessarily G-equivariant isomorphism of extensions to invariants is a G-equivariant isomorphism. This completes the proof that $\mathcal{Q}$ has a unique G-equivariant structure.

Now, we turn to the question of the existence of a quantized moment map for this G-equivariant structure. Since h has $\mathbb{S}$-weight $n > 0$, the lack of functions on $\mathfrak{X}$ of negative $\mathbb{S}$-weight shows that $\Gamma_{\mathbb{S}}(\mathfrak{X}; \mathcal{Q})$ is a commutative algebra, canonically isomorphic to the $\mathbb{S}$-invariants $\mathbb{C}[\mathfrak{X}]^{\mathbb{S}}$. Furthermore, we have a natural Lie algebra structure on $\Gamma_{\mathbb{S}}(\mathfrak{X}; h^{-1}\mathcal{Q})$ induced by the bracket, since sections of $\mathcal{Q}$ commute modulo h. We have a short exact sequence of Lie algebras

$$0 \to \Gamma_{\mathbb{S}}(\mathfrak{X}; \mathcal{Q}) \to \Gamma_{\mathbb{S}}(\mathfrak{X}; h^{-1}\mathcal{Q}) \to \mathbb{C}[\mathfrak{X}]_n \to 0, \tag{2}$$

where the Lie bracket on $\mathbb{C}[\mathfrak{X}]_n$ is the Poisson bracket. To show a quantized moment map exists, we wish to show that the moment map $\mu\colon \mathfrak{g} \to \mathbb{C}[\mathfrak{X}]_n$ lifts to a G-equivariant Lie algebra homomorphism $\mathfrak{g} \to \Gamma_{\mathbb{S}}(\mathfrak{X}; h^{-1}\mathcal{Q})$.

Note that the subsheaf $h\mathcal{Q}$ satisfies $\Gamma_{\mathbb{S}}(h\mathcal{Q}) = 0$ and $H^1(\mathfrak{X}; h\mathcal{Q}) = 0$, so the exact sequence (2) is isomorphic to

$$0 \to \Gamma_{\mathbb{S}}(\mathfrak{X}; \mathcal{Q}/h\mathcal{Q}) \cong \mathbb{C}[\mathfrak{X}]_0 \to \Gamma_{\mathbb{S}}(\mathfrak{X}; h^{-1}\mathcal{Q}/h\mathcal{Q}) \to \mathbb{C}[\mathfrak{X}]_n \to 0.$$

Thus we aim to show that there is a lift $\mathfrak{g} \to \Gamma_{\mathbb{S}}(\mathfrak{X}; h^{-1}\mathcal{Q}/h\mathcal{Q})$.

If such a lift exists, then the set of such lifts forms a torsor over $\chi(\mathfrak{g}) \otimes \mathbb{C}[\mathfrak{X}]^{\mathbb{S}\times G}$, the group of Lie algebra homomorphisms $\mathfrak{g} \to \mathbb{C}[\mathfrak{X}]^{\mathbb{S}\times G}$. Thus, to prove the result, it only remains that establish that a lift exists. The obstruction to the existence of a lift is a Lie algebra cohomology class $\nu \in H^1(\mathfrak{g}; \mathbb{C}[\mathfrak{X}]^{\mathbb{S}\times G})$. Since $\mathfrak{g} \cong \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$ and the latter Lie algebra is semi-simple, Whitehead's first lemma gives an isomorphism $H^1(\mathfrak{g}; \mathbb{C}[\mathfrak{X}]^{\mathbb{S}\times G}) \cong H^1(\mathfrak{z}(\mathfrak{g}); \mathbb{C}[\mathfrak{X}]^{\mathbb{S}\times G})$. So it suffices to check that a lift exists after replacing G by the connected component of the identity in its center. That is, we can assume without loss of generality that G is commutative.

First, we will prove the existence of a quantum moment map in the case where $\mathcal{Q}$ is the canonical quantization. In this case, we have an isomorphism $\eta\colon \mathcal{Q} \cong \mathcal{Q}^{\mathrm{op}}$ as quantizations, where $\mathcal{Q}^{\mathrm{op}}$ has its $\mathbb{C}[[h]]$-module structure twisted by $h \mapsto -h$. The isomorphism η induces a Lie algebra anti-automorphism of the exact sequence

$$0 \to \mathcal{Q}/h\mathcal{Q} \cong \mathfrak{S}_{\mathfrak{X}} \to h^{-1}\mathcal{Q}/h\mathcal{Q} \to h^{-1}\mathcal{Q}/\mathcal{Q} \cong \mathfrak{S}_{\mathfrak{X}} \to 0,$$

and it induces the identity on $\mathcal{Q}/h\mathcal{Q}$, and -1 on $h^{-1}\mathcal{Q}/\mathcal{Q}$. Thus, considering the eigenspaces of η splits this sequence. This induces an isomorphism of sheaves of Lie algebras

$$h^{-1}\mathcal{Q}/h\mathcal{Q} \cong (\mathfrak{S}_{\mathfrak{X}} \otimes h^{-1}\mathbb{C}[[h]]/h\mathbb{C}[[h]]) \tag{3}$$

where the RHS has the Lie algebra structure

$$[f_0 + h^{-1}f_{-1}, g_0 + h^{-1}g_{-1}] = \{f_0, g_{-1}\} + \{f_{-1}, g_0\} + h^{-1}\{f_{-1}, g_{-1}\}.$$

We can also deduce this result from [BK04, Lem. 3.6], which has effectively the same proof. In this case, $h^{-1}\mu$ gives us the desired splitting.

Finally, we prove that a quantum moment map exists for a quantization $\mathcal{Q}$ with general period λ. By standard results, we can G-equivariantly embed $\mathfrak{X}$ as a locally closed subvariety of the projectivization of a representation of G. Using the usual affine cover of projective space associated to a basis of weight vectors, we can cover $\mathfrak{X}$ with G-invariant open affines U_i, and by taking the cover fine enough, we can trivialize the period λ of the quantization on each U_i. Thus we have isomorphisms $\beta_i\colon \mathcal{Q}|_{U_i} \xrightarrow{\sim} \mathcal{Q}^{\mathrm{op}}|_{U_i}$. We can assume the β_i are G-equivariant, since G acts on the set of ring homomorphisms $\mathcal{Q}|_{U_i} \to \mathcal{Q}^{\mathrm{op}}|_{U_i}$, and the projection to the invariants of an isomorphism which is the identity modulo h (and thus G-equivariant modulo h) remains an isomorphism by Nakayama's lemma. The compositions $\gamma_{ij} = \beta_j^{-1} \circ \beta_i$ give a Čech 1-cocycle of automorphisms of $\mathcal{Q}$. We can view $\mathcal{Q}^{\mathrm{op}}$ as the regluing (as in [Los, §4.1]) of $\mathcal{Q}$ with respect to this cocycle; that is, the sheaf of algebras $\mathcal{Q}^{\mathrm{op}}$ is isomorphic to the sheaf

$$\mathcal{Q}^{\mathrm{glue}}(V) = \{(a_1, \dots a_n) \mid a_i \in \mathcal{Q}(V \cap U_i), \gamma_{ij}(a_j) = a_i \ \ \forall i, j\}$$

via the map $a \mapsto (\beta_1^{-1}(a), \dots \beta_n^{-1}(a))$.

By [Los, Prop. 4.1], we can assume that $\gamma_{ij} = \exp(\operatorname{ad} c_{ij})$ for $c_{ij} \in \mathcal{Q}(U_i \cap U_j)$. Again, we can assume that c_{ij} is G-invariant by projection to invariants, since γ_{ij} is G-invariant. By the cocycle equation, for any f in $\mathcal{Q}(U_i \cap U_j \cap U_k)/h^2$ we have

$$f = \gamma_{ij}\gamma_{jk}\gamma_{ki}(f) = f + h\{\bar{c}_{ij} + \bar{c}_{jk} + \bar{c}_{ki}, \bar{f}\}$$

where we use $\bar{g}$ to mean the reduction of a section of $\mathcal{Q}$ mod h. This shows that $\alpha_{ijk} =$

$\bar{c}_{ij} + \bar{c}_{jk} + \bar{c}_{ki}$ is constant on $U_i \cap U_j \cap U_k$, and thus defines a class $\alpha \in H^2(\mathfrak{X}; \mathbb{C})$ constructed via Čech cohomology.

Let us now compute the effect this has on the period of the regluing of $\mathcal{Q}$. In the sentences below we use the notation of [BK04, §4]. It's enough to calculate the reduced period map Per_1 since by $\mathbb{S}$-equivariance, all higher terms in the period vanish. Poisson bracket with the classes $\bar{c}_{ij}$ defines a Čech 1-cocycle of Hamiltonian vector fields, i.e. a class in $H^1(\mathfrak{X}; \mathcal{W})$, and the regluing is exactly the torsor structure of this group on the set of quantizations modulo h^2 (that is, $H^1_{\mathcal{M}}(\mathfrak{X}, \langle(\mathsf{Aut}D)_2, (\mathrm{Der}\, D)_2\rangle)$. The class α is the image of $\bar{c}_{ij}$ under the boundary map, so it follows from the compatibility of the period map with the $H^1(\mathfrak{X}; \mathcal{W})$-torsor structure that the period of the regluing is given by $\lambda + \alpha$. Since the period of $\mathcal{Q}^{\mathrm{op}}$ is $-\lambda$, we have $\alpha = -2\lambda$.

Let $\gamma'_{ij} = \exp(\frac{1}{2}\,\mathrm{ad}\, c_{ij})$. Consider the regluing $\mathcal{Q}'$ of $\mathcal{Q}$ with respect to γ'_{ij}; this has period $\lambda + \frac{1}{2}\alpha = 0$, and so it is the canonical quantization $\mathcal{Q}^{\mathrm{can}}$. Note that this defines a G-structure on $\mathcal{Q}^{\mathrm{can}}$ by taking the obvious G-structure on the regluing.

We have seen that $\mathcal{Q}^{\mathrm{can}}$ admits a quantum moment map $\mu' \colon \mathfrak{g} \to \Gamma_{\mathbb{S}}(\mathfrak{X}; h^{-1}\mathcal{Q}^{\mathrm{can}}/h\mathcal{Q}^{\mathrm{can}})$. Viewing $\mathcal{Q}^{\mathrm{can}}$ as a regluing, this means we have maps $\mu_i \colon \mathfrak{g} \to \Gamma_{\mathbb{S}}(U_i; h^{-1}\mathcal{Q}/h\mathcal{Q})$ that satisfy $\mu_j = \gamma'_{ji} \circ \mu_i = \exp(\frac{1}{2}\,\mathrm{ad}\, c_{ji})\mu_i$ on the overlap $U_i \cap U_j$. However, since c_{ji} is G-invariant, $[c_{ji}, \mu_i(X)] = 0$ for all $X \in \mathfrak{g}$. Thus $\exp(\frac{1}{2}\,\mathrm{ad}\, c_{ji})\mu_i = \mu_i$. This shows that μ_j and μ_i agree on $U_i \cap U_j$, so they define a Lie algebra map $\mu \colon \mathfrak{g} \to \Gamma_{\mathbb{S}}(\mathfrak{X}; h^{-1}\mathcal{Q}/h\mathcal{Q})$, which gives the desired lifting. This completes the proof. □

Assume that $(\mathfrak{X}, \mathcal{Q})$ carries a Hamiltonian G-action with quantized moment map $\eta \colon U(\mathfrak{g}) \to A$ and associated classical moment map $\mu \colon \mathfrak{X} \to \mathfrak{g}^*$. Fix a G-equivariant ample line bundle $\mathcal{L}$ on $\mathfrak{X}$, and let $\mathfrak{U} \subset \mathfrak{X}$ be the associated semistable locus. We will assume through the end of the section that the action of G on $\mathfrak{U}$ is free; in particular, semistability and stability coincide.

Let $\mathfrak{X}_{\mathrm{red}} := (\mu^{-1}(0) \cap \mathfrak{U})/G$ with its induced relative symplectic form and $\mathbb{S}$-action, and let $\psi \colon \mu^{-1}(0) \cap \mathfrak{U} \to \mathfrak{X}_{\mathrm{red}}$ be the natural projection. We'll further assume that the natural map $\mathbb{C}[\mu^{-1}(0)]^G \to \mathbb{C}[\mathfrak{X}_{\mathrm{red}}]$ is an isomorphism.

Let $\mathcal{D}_{\mathfrak{U}}$ and $\mathcal{Q}_{\mathfrak{U}}$ denote the restrictions of $\mathcal{D}$ and $\mathcal{Q}$ to $\mathfrak{U}$, and for any $\xi \in \chi(\mathfrak{g})$, let

$$\mathcal{R}_\xi := \mathcal{Q}_{\mathfrak{U}} \Big/ \mathcal{Q}_{\mathfrak{U}} \cdot \langle h\eta(x) - h\xi(x) \mid x \in \mathfrak{g} \rangle,$$

$$\mathcal{E}_\xi(0) := \mathcal{D}_{\mathfrak{U}}(0) \Big/ \mathcal{D}_{\mathfrak{U}}(-n) \cdot \langle \eta(x) - \xi(x) \mid x \in \mathfrak{g} \rangle,$$

$$\mathcal{E}_\xi := \mathcal{D}_{\mathfrak{U}} \Big/ \mathcal{D}_{\mathfrak{U}} \cdot \langle \eta(x) - \xi(x) \mid x \in \mathfrak{g} \rangle.$$

These are all sheaves on $\mathfrak{U}$ with support $\mu^{-1}(0) \cap \mathfrak{U}$, which we use to define sheaves of algebras on $\mathfrak{X}_{\mathrm{red}}$ as follows:

$$\mathcal{Q}_{\mathrm{red}} := \psi_* \mathcal{E}nd_{\mathcal{Q}_{\mathfrak{U}}}(\mathcal{R}_\xi)^{\mathrm{op}},$$

$$\mathcal{D}_{\mathrm{red}}(0) := \psi_* \mathcal{E}nd_{\mathcal{D}(0)_{\mathfrak{U}}}(\mathcal{E}_\xi(0))^{\mathrm{op}},$$

$$\mathcal{D}_{\mathrm{red}} := \psi_* \mathcal{E}nd_{\mathcal{D}_{\mathfrak{U}}}(\mathcal{E}_\xi)^{\mathrm{op}}.$$

Kashiwara and Rouquier [KR08, 2.8(i)] show that the first sheaf is an $\mathbb{S}$-equivariant quantization of $\mathfrak{X}_{\mathrm{red}}$ of weight n, and the second and third are related to the first in the usual way. Kashiwara and Rouquier work in the classical topology, but their argument works equally well in the Zariski topology.

Remark 3.12 Kashiwara and Rouquier also take the fixed points of G. Since we have assumed that G is connected, this is redundant; the pushforward is automatically invariant under $\mathfrak{g}$. Of course, a reader interested in quotients by disconnected groups can apply our results to the connected component of the identity, and then consider the residual action of the component group.

We observe that this geometric operation of symplectic reduction is closely related to an algebraic one. Let $Y_\xi := A\big/A \cdot \langle \eta(x) - \xi(x) \mid x \in \mathfrak{g}\rangle$, where as before we let $A = \Gamma_{\mathbb{S}}(\mathcal{D})$.

Proposition 3.13 *If* $A_{\mathrm{red}} = \Gamma_{\mathbb{S}}(\mathcal{D}_{\mathrm{red}})$*, then* $A_{\mathrm{red}} \cong \mathrm{End}_A(Y_\xi)$.

Proof: Restriction gives a natural map $A \to \Gamma_{\mathbb{S}}(\mathcal{D}_{\mathfrak{U}})$, which induces a map

$$A^G \to \Gamma_{\mathbb{S}}(\mathfrak{U}; \mathcal{E}nd_{\mathcal{D}_{\mathfrak{U}}}(\mathcal{E}_\xi)^{\mathrm{op}}) \cong A_{\mathrm{red}}.$$

This map kills any G-invariant element of the left ideal generated by $\eta(x) - \xi(x)$ for $x \in \mathfrak{g}$ and thus induces a map $Y_\xi^G \cong \mathrm{End}_A(Y_\xi) \to A_{\mathrm{red}}$. We wish to show that this map is an isomorphism.

By Nakayama, it's enough to check this after passing to associated graded. The associated graded of A^G is $\mathbb{C}[\mathfrak{X}]^G$ (since G is reductive), and the map $\mathbb{C}[\mathfrak{X}]^G \to \mathrm{gr}(A_{\mathrm{red}}) \subset \mathbb{C}[\mathfrak{X}_{\mathrm{red}}]$ is the obvious quotient map. The associated graded of Y_ξ^G is a quotient of $\mathbb{C}[\mathfrak{X}]^G/(\mu^*(\mathfrak{g})) \cong \mathbb{C}[\mu^{-1}(0)]^G$, so we have maps

$$\mathbb{C}[\mu^{-1}(0)]^G \twoheadrightarrow \mathrm{gr}(Y_\xi^G) \to \mathrm{gr}(A_{\mathrm{red}}) \hookrightarrow \mathbb{C}[\mathfrak{X}_{\mathrm{red}}].$$

The composition of these maps is a isomorphism. Since the first map is a surjection and the last is an injection, each of the intermediate steps is an isomorphism. $\square$

Next we describe the period of $\mathcal{Q}_{\mathrm{red}}$ in terms of the parameter ξ; this will prove to be an important technical tool that is needed for the proofs of Proposition 4.4 and Lemma 4.15. For simplicity, we assume that $\mathfrak{X}$ is symplectic over $\operatorname{Spec}\mathbb{C}$ (rather than over an arbitrary base) and $\mathbb{C}[\mathfrak{X}]^{\mathbb{S}\times G} = \mathbb{C}$, that $\mathcal{Q}$ is the canonical quantization of $\mathfrak{X}$, and that $\mathfrak{X}_{\mathrm{red}}$ satisfies our running assumptions on $\mathfrak{X}$.

The following general result about opposites and quantum Hamiltonian reduction will be used to prove Lemma 3.15, and may also be of independent interest.

Lemma 3.14 *Let A be an algebra with an action of a connected reductive affine algebraic group G with noncommutative moment map $\eta\colon U(\mathfrak{g}) \to A$. Then we have natural isomorphisms*

$$\mathrm{End}_A(A/A\eta(\mathfrak{g}))^{\mathrm{op}} \cong \mathrm{End}_{A^{\mathrm{op}}}(A/\eta(\mathfrak{g})A) \cong \mathrm{End}_{A^{\mathrm{op}}}(A^{\mathrm{op}}/A^{\mathrm{op}}\eta(\mathfrak{g})). \qquad (4)$$

That is, the left and right quantum Hamiltonian reductions are opposite to each other.

Proof: We can freely replace G with a finite cover, and thus assume that G is a product of simple groups. Since reducing by $G_1 \times G_2$ can be done in stages as reduction by G_1 and then by G_2, we can reduce to the case where G is simple.

Right (resp. left) multiplication define homomorphisms

$$\mathrm{End}_A(A/A\eta(\mathfrak{g}))^{\mathrm{op}} \cong (A/A\eta(\mathfrak{g}))^G \longleftarrow A^G \longrightarrow (A/\eta(\mathfrak{g})A)^G \cong \mathrm{End}_{A^{\mathrm{op}}}(A/\eta(\mathfrak{g})A).$$

Since G is reductive, the functor of invariants is exact and these maps are surjective, so we need only show their kernels agree. The kernel K_1 of the left map is $A^G \cap A\eta(\mathfrak{g})$ and the kernel K_2 of the right map is $A^G \cap \eta(\mathfrak{g})A$. If G is abelian then

$$A^G \cap \eta(\mathfrak{g})A = \eta(\mathfrak{g})A^G = A^G\eta(\mathfrak{g}) = A^G \cap A\eta(\mathfrak{g}),$$

so we can assume that G is non-abelian.

Thus, assume that $a = \sum_i y_i\eta(x_i)$ is an element of K_1, where x_i ranges over a basis of $\mathfrak{g}$. We can replace y_i with its projection to the isotypic component of A corresponding to the adjoint representation $\mathfrak{g} \cong \mathfrak{g}^*$ (since any other simple tensored with $\mathfrak{g}$ has no invariants). In this case, invariance shows that there is an equivariant map $\pi\colon \mathfrak{g} \to A$ sending $\pi(x^i) = y_i$ where x^i is the dual basis to x_i under the Killing form. Thus we have $a = \sum_i \eta(x_i)y_i + x_i \cdot y_i = \sum_i \eta(x_i)y_i + \pi([x_i, x^i])$ by the equivariance of π. Since $\sum_i [x_i, x^i]$ is invariant under the adjoint action, it is trivial, and we have that $a = \sum_i \eta(x_i)y_i \in K_2$. Applying a symmetric argument, we see that $K_1 = K_2$, so the first equality of (4) follows immediately. The second is just the equivalence of categories between right A-modules and left A^{op}-modules. $\square$

A quantized moment map $\eta\colon U(\mathfrak{g}) \to A$ is called **balanced** if, when $\xi = 0$, $\mathcal{Q}_{\mathrm{red}}$ is the canonical quantization of $\mathfrak{X}_{\mathrm{red}}$.

Lemma 3.15 *The canonical quantization of the variety $\mathfrak{X}$ admits a balanced quantized moment map.*

Proof: By Proposition 3.11, the set of quantized moment maps is a torsor for $\chi(\mathfrak{g})$. Since $\mathcal{Q}$ is the canonical quantization, we know that $\mathcal{Q} \cong \mathcal{Q}^{\mathrm{op}}$, and any choice of such an isomorphism (that is, any algebra anti-automorphism ϕ of $\mathcal{Q}$) sends a quantized moment map to minus a quantized moment map. Thus, $-\phi$ preserves the set of quantized moment maps, and is an anti-automorphism of $\chi(\mathfrak{g})$-torsors, so it fixes a unique point.

Recall that

$$\mathcal{Q}_{\mathrm{red}} = \psi_* \mathcal{E}nd_{\mathcal{Q}_{\mathfrak{U}}} \left(\mathcal{Q}_{\mathfrak{U}} \Big/ \mathcal{Q}_{\mathfrak{U}} \cdot \langle h\eta(x) - h\xi(x) \mid x \in \mathfrak{g} \rangle \right)^{\mathrm{op}}.$$

By Lemma 3.14, the opposite ring of $\mathcal{Q}_{\mathrm{red}}$ is obtained as the analogous reduction of the opposite ring of $\mathcal{Q}$:

$$\mathcal{Q}_{\mathrm{red}}^{\mathrm{op}} \cong \psi_* \mathcal{E}nd_{\mathcal{Q}_{\mathfrak{U}}} \left(\mathcal{Q}_{\mathfrak{U}}^{\mathrm{op}} \Big/ \mathcal{Q}_{\mathfrak{U}}^{\mathrm{op}} \cdot \langle -h\eta(x) + h\xi(x) \mid x \in \mathfrak{g} \rangle \right)^{\mathrm{op}}.$$

Twisting the action of $\mathcal{Q}$ by the action of ϕ, this sheaf is also isomorphic to

$$\psi_* \mathcal{E}nd_{\mathcal{Q}_{\mathfrak{U}}} \left(\mathcal{Q}_{\mathfrak{U}} \Big/ \mathcal{Q}_{\mathfrak{U}} \cdot \langle -h\phi(\eta(x)) + h\xi(x) \mid x \in \mathfrak{g} \rangle \right)^{\mathrm{op}}.$$

Thus, if we choose η to be the fixed point of $-\phi$ and take $\xi = 0$, the quantization $\mathcal{Q}_{\mathrm{red}}$ is isomorphic to its own opposite, and therefore to the canonical quantization. □

The following proposition is implicit in the principal results of [Los12], but does not seem to be explicitly stated in the generality that we need. Our proof is similar to the proof of [Los12, 5.3.1].

Proposition 3.16 *If $\mathfrak{X}$ is canonically quantized, η is a balanced quantized moment map, and $\xi \in \chi(\mathfrak{g})$ is arbitrary, then the period of $\mathcal{Q}_{\mathrm{red}}$ is equal to $[\omega_{\mathrm{red}}] + h\mathsf{K}(\xi)$, where $\mathsf{K}\colon \chi(\mathfrak{g}) \to H^2(\mathfrak{X}_{\mathrm{red}}; \mathbb{C})$ is the Kirwan map.*

Proof: Consider the inclusion $\chi(\mathfrak{g}) \cong (\mathfrak{g}^*)^G \subset \mathfrak{g}^*$, and let $\mathfrak{P} := \big(\mathfrak{U} \cap \mu^{-1}(\chi(\mathfrak{g}))\big)/G$, which is equipped with a natural map $\gamma\colon \mathfrak{P} \to \chi(\mathfrak{g})$. Since G acts freely on $\mathfrak{U}$, γ is a submersion and $\mathfrak{P}$ is a flat deformation of $\mathfrak{X}_{\mathrm{red}} = \gamma^{-1}(0)$, symplectic over the base $\chi(\mathfrak{g})$. The quantization

$$\hat{\mathcal{Q}}_{\mathrm{red}} \cong \gamma_* \mathcal{E}nd \Big(\mathcal{Q}_{\mathfrak{U}} \Big/ \mathcal{Q}_{\mathfrak{U}} \cdot \langle h\eta(x) \mid x \in [\mathfrak{g}, \mathfrak{g}] \rangle \Big)^G$$

of $\mathfrak{P}$ is self opposite, and thus canonical, so its period is equal to the class of the relative symplectic form $\omega_{\mathfrak{P}} \in \Omega^2(\mathfrak{P}/\chi(\mathfrak{g}))$. The quotient

$$\mathcal{Q}_{\mathrm{red}} = \hat{\mathcal{Q}}_{\mathrm{red}} / \hat{\mathcal{Q}}_{\mathrm{red}} \cdot \langle h\eta(x) - h\xi(x) \mid x \in \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \rangle,$$

which is supported on $\mathfrak{X}_{\mathrm{red}}$, can be thought of as the pullback of $\hat{\mathcal{Q}}_{\mathrm{red}}$ by the map $s\colon \Delta \to$

$\Delta \times \chi(\mathfrak{g})$ which is the identity on Δ and has the property that $s^*x = h \cdot \xi(x)$ for any element $x \in \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \cong \chi(\mathfrak{g})^*$. By Proposition 3.1, this quantization of $\mathfrak{X}_{\mathrm{red}}$ has period $s^*[\omega_{\mathfrak{P}}]$. The usual Duistermaat-Heckman theorem implies that $s^*[\omega_{\mathfrak{P}}] = [\omega_{\mathrm{red}}] + h\mathsf{K}(\xi)$. □

4 Modules over quantizations

Let $\mathcal{Q}$ be an $\mathbb{S}$-equivariant quantization of $\mathfrak{X}$, and consider the sheaves $\mathcal{D}$ and $\mathcal{D}(m)$ defined in the beginning of Section 3.3. An h-adically complete module over $\mathcal{Q}$ (respectively $\mathcal{D}(0)$) is called **coherent** if it is locally a quotient of a sheaf which is free of finite rank. By Nakayama's lemma, this is equivalent to the property that one obtains a coherent sheaf by setting h (respectively $h^{1/n}$) to zero.

Remark 4.1 Some other sources on modules over deformation quantizations contain an *a priori* stronger notion of "coherent" as in defined in [KS12, §1.1]. However, since $\mathfrak{X}$ (and thus $\mathcal{D}$) is Noetherian, [KS12, 1.2.5] shows that this notion coincides with the one we have given above. In general, we simplify many issues around finiteness by assuming that the modules we consider are coherent. Removing this condition would complicate matters substantially.

A $\mathbb{S}$-equivariant $\mathcal{D}$-module is a $\mathcal{D}$-module equipped with an $\mathbb{S}$-structure in the sense of Section 3.2, compatible with the $\mathbb{S}$-structure on $\mathcal{D}$. More precisely, it is a $\mathcal{D}$-module $\mathcal{N}$ along with an isomorphism $a_{\mathrm{tw}}^*\mathcal{N} \cong p^*\mathcal{N}$ satisfying the natural cocycle condition, such that the following diagram commutes.

$$\begin{tikzcd}
a_{\mathrm{tw}}^*\mathcal{D} \otimes a_{\mathrm{tw}}^*\mathcal{N} \arrow[r] \arrow[d, "\cong"'] & a_{\mathrm{tw}}^*\mathcal{N} \arrow[d, "\cong"] \\
p^*\mathcal{D} \otimes p^*\mathcal{N} \arrow[r] & p^*\mathcal{N}
\end{tikzcd}$$

An $\mathbb{S}$-equivariant $\mathcal{D}$-module $\mathcal{N}$ is called **good** if it admits a coherent $\mathbb{S}$-equivariant $\mathcal{D}(0)$-lattice $\mathcal{N}(0)$. Let $\mathcal{D}$-Mod be the category of arbitrary $\mathbb{S}$-equivariant modules over $\mathcal{D}$, and let $\mathcal{D}$-mod $\subset \mathcal{D}$-Mod be the full subcategory consisting of good modules. Note that the choice of lattice is not part of the data of an object of $\mathcal{D}$-mod. The reason for this is that we want an abelian category, which would fail if we worked with lattices: the quotient of a lattice by a sublattice is only a lattice after killing torsion.

Many of our important results require considering derived categories; unfortunately, there seems to be no single choice of finiteness condition on derived categories which will suit us once and for all. In order to define the cohomology of sheaves of $\mathcal{D}$-modules, it is most convenient to work in unbounded derived category $D(\mathcal{D}\text{-Mod})$ of arbitrary $\mathcal{D}$-modules (in order to use

Čech resolutions), but in most cases of interest to us, we can restrict to the bounded derived category $D^b(\mathcal{D}\text{-mod})$ of good $\mathcal{D}$-modules.

Remark 4.2 Note that if $\mathcal{C}$ is an abelian category and $\mathcal{C}_0$ an abelian subcategory closed under taking subobjects, we can consider both the derived category $D^b(\mathcal{C}_0)$ and the category $D^b_{\mathcal{C}_0}(\mathcal{C})$ of bounded complexes in $\mathcal{C}$ with cohomology in $\mathcal{C}_0$. There is an obvious functor $D^b(\mathcal{C}_0) \to D^b_{\mathcal{C}_0}(\mathcal{C})$ which is sometimes an equivalence and sometimes not. If $\mathcal{C}_0$ has enough projectives which remain projective in $\mathcal{C}$, then every complex in $D^b_{\mathcal{C}_0}(\mathcal{C})$ can be replaced by a quasi-isomorphic projective resolution in $\mathcal{C}_0$, which shows that this functor is an equivalence. In particular, this argument carries through when $\mathcal{C}$ is the category of all modules over some ring, and $\mathcal{C}_0$ is the subcategory of finitely generated modules.

If $\mathcal{C}$ is the category of quasi-coherent sheaves on a projective (over affine) scheme and $\mathcal{C}_0$ is the subcategory of coherent sheaves, then this functor is still an equivalence, even though coherent sheaves do not have enough projectives; this follows from considering the corresponding modules over the projective coordinate ring. Similarly, we will show that $\mathcal{D}$-mod admits an analogous description (Theorem 5.8), which implies that $D^b(\mathcal{D}\text{-mod})$ is equivalent to $D^b_{\mathcal{D}\text{-mod}}(\mathcal{D}\text{-Mod})$ (Corollary 5.11).

Remark 4.3 If $\mathfrak{X} = \mathfrak{M}$ is a conical symplectic resolution, there are heuristic reasons to treat $\mathcal{D}$-mod as an algebraic version of the Fukaya category of $\mathfrak{M}$ twisted by the B-field defined by $e^{2\pi i\lambda} \in H^2(\mathfrak{M}; \mathbb{C}^\times)$, where $h\lambda$ is the period of $\mathcal{D}$. The firmest justification at moment lies in the physical theory of A-branes, which the Fukaya category is an attempt to formalize. Kapustin and Witten [KW07] suggest that on a hyperkähler manifold, there are objects in an enlargement of the Fukaya category which correspond not just to Lagrangian submanifolds, but higher dimensional coisotropic submanifolds. In particular, there is an object in this category supported on all of $\mathfrak{M}$ called the **canonical coisotropic brane**. Following the prescription of Kapustin and Witten further shows that $\mathcal{D}$ is isomorphic to the sheaf of endomorphisms of this object. Nadler and Zaslow [NZ09] prove a related result in which $\mathfrak{M}$ is replaced by the cotangent bundle of an arbitrary real analytic manifold.

4.1 Cotangent bundles

Let us consider the special case of quantizations of $\mathfrak{X} = T^*X$ for some smooth projective variety X, where $\mathbb{S}$ acts by inverse scaling of the cotangent fibers[11]. Quantizations of cotangent bundles have been considered many times before in different contexts, but for the sake of completeness, we wish to show in detail how it fits in our schema. We will assume that

[11]This variety may not satisfy the property of being projective over an affine variety $\mathfrak{X}_0$, but we will not use that assumption in this section.

$H^1(X) = 0$ and $H^2(X) \cong H^{1,1}(X)$; in particular

$$H^i(\mathfrak{X}; \mathfrak{S}_{\mathfrak{X}})^{\mathbb{S}} \cong H^i(X; \mathfrak{S}_X) \cong H^{i,0}(X) = 0 \text{ for } i = 1, 2$$

and $H^2(X) \cong \mathrm{Pic}(X) \otimes \mathbb{C}$.

A **Picard Lie algebroid** $\mathcal{P}$ on X is an extension in the abelian category of Lie algebroids of the tangent sheaf $\mathcal{T}_X$, with its tautological Lie algebroid structure, by the structure sheaf $\mathfrak{S}_X$, with the trivial Lie algebroid structure. Such an extension in the category of coherent sheaves is classified by

$$\mathrm{Ext}^1(\mathcal{T}_X, \mathfrak{S}_X) \cong H^1(X; \mathcal{T}_X^*) \cong H^{1,1}(X; \mathbb{C}) \cong H^2(\mathfrak{X}; \mathbb{C}).$$

Since we have that $H^0(X; \wedge^2 \mathcal{T}_X^*) = 0$, there is a unique Picard Lie algebroid $\mathcal{P}_\lambda$ on X for each $\lambda \in H^2(\mathfrak{X}; \mathbb{C})$.

Let $\mathcal{U}_\lambda$ be the universal enveloping algebra of $\mathcal{P}_\lambda$ modulo the ideal that identifies the constant function $1 \in \mathfrak{S}_X$ with the unit of the algebra. If λ is the image of the Euler class of a line bundle $\mathcal{L}$ on X, then $\mathcal{U}_\lambda$ is isomorphic to the sheaf of differential operators on $\mathcal{L}$. More generally, $\mathcal{U}_\lambda$ is referred to as the **sheaf of λ-twisted differential operators** on X. A coherent sheaf of $\mathcal{U}_\lambda$-modules is called a **λ-twisted D-module** on X. The sheaf $\mathcal{U}_\lambda$ has an order filtration, and any coherent sheaf of $\mathcal{U}_\lambda$-modules admits a compatible filtration.

However, $\mathcal{U}_\lambda$ is a sheaf on X, and we wish to find one on T^*X. This requires the technique of **microlocalization** (see, for example, [Kas03, AVdBVO89] for more detailed discussion of this technique). The associated graded of $\mathcal{U}_\lambda$ with respect to the order filtration is isomorphic to $\mathrm{Sym}_{\mathfrak{S}_X}\mathcal{T}_X$; put differently, if

$$R_\lambda := \left\{ \sum u_i h^i \in \mathcal{U}_\lambda[h] \,\middle|\, u_i \text{ has order} \leq i \right\}$$

is the Rees algebra of the order filtration on λ, then $R_\lambda / hR_\lambda \cong \mathrm{Sym}_{\mathfrak{S}_X}\mathcal{T}_X$. Given an open subset $U \subset T^*X$, we obtain a multiplicative subset $S_U \subset \mathrm{Sym}_{\mathfrak{S}_X}\mathcal{T}_X(\pi(U))$ consisting of functions on $\pi^{-1}(\pi(U))$ which are invertible on U.

We can give a non-commutative version of this construction using an associated multiplicative system in $R_\lambda(\pi(U))$. Let

$$S'_U = \left\{ r \in R_\lambda(\pi(U)) \,\middle|\, \exists\, m \text{ such that } r \in h^m R_\lambda \text{ with } \overline{h^{-m} r} \in S_U \right\}.$$

This is a multiplicative system because $\mathrm{Sym}_{\mathfrak{S}_X}\mathcal{T}_X$ is a sheaf of domains. Furthermore, since $[r, R_\lambda] \subset hR_\lambda$, the operation of bracket with any algebra element is topologically nilpotent (the successive powers converge to 0 in the h-adic topology). Thus, in any quotient $R_\lambda / h^m R_\lambda$, the reduction of this set S'_U satisfies the Ore condition, and we can define the localization of R_λ by S'_U as the inverse limit $\mathcal{R}_\lambda(U) := \varprojlim (R_\lambda / h^m R_\lambda)_{S'_U}$. This defines an $\mathbb{S}$-equivariant sheaf of rings $\mathcal{R}_\lambda$ on $\operatorname{Spec}\big(\operatorname{Sym}_{\mathfrak{S}_X}\mathcal{T}_X\big) \cong T^*X = \mathfrak{X}$, which is free over $\mathbb{C}[[h]]$ and satisfies $\mathcal{R}/h\mathcal{R} \cong \mathfrak{S}_{T^*X}$, and is therefore a quantization of $\mathfrak{X}$.

Proposition 4.4 *The period of $\mathcal{R}_\lambda$ is $h(\lambda - \varpi/2)$, where $\varpi = c_1(T^*X) \in H^2(X;\mathbb{C}) \cong H^2(\mathfrak{X};\mathbb{C})$ is the canonical class.*

Proof: We begin by choosing line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_k$ on X and complex numbers $\zeta_1, \ldots, \zeta_k$ such that $\lambda = \sum_{i=1}^k \zeta_i c_1(\mathcal{L}_i)$. Let Y be the total space of $\oplus\mathcal{L}_i$ and let $T := (\mathbb{C}^\times)^k$ act on Y by scaling the fibers of the individual lines. Let $\mathbb{S}$ act on T^*Y via the inverse scaling action on the fibers, and let $\tilde{\mathcal{R}}$ be the $T \times \mathbb{S}$-equivariant quantization of T^*Y obtained by microlocalizing the sheaf of (untwisted) differential operators on Y. The action of T on $(T^*Y, \tilde{\mathcal{R}})$ admits a quantized moment map

$$\varphi\colon U(\mathfrak{t}) \to \Gamma_{\mathbb{S}}(\tilde{\mathcal{R}}) \cong \Gamma(Y, D_Y)$$

given on $\mathfrak{t}$ by the equation

$$\varphi(a_1, \ldots, a_k) = \sum_{i=1}^k a_i t_i \frac{\partial}{\partial t_i}, \tag{5}$$

where t_i is any coordinate on the fiber of $\mathcal{L}_i$ (the operator $t_i \frac{\partial}{\partial t_i}$ is independent of this choice). If we take $\zeta := (\zeta_1, \ldots, \zeta_k) \in \mathbb{C}^k \cong \chi(\mathfrak{t})$, then symplectic reduction of $(T^*Y, \tilde{\mathcal{R}})$ at the parameter ζ yields the pair $(\mathfrak{X}, \mathcal{R}_\lambda)$, as noted by Beilinson and Bernstein in [BB93, §2.5].

First, consider the special case where $k = 1$ and $\mathcal{L}_1 = \omega_X^{-1}$, the anti-canonical bundle of X. Then Y is Calabi-Yau and $\tilde{\mathcal{R}}$ is the canonical quantization, and so we can apply Proposition 3.16. In order to do this, we must find a quantized moment map with self-opposite reduction. By [BB93, §2.5], the reduction by φ at the parameter $\xi \in \mathbb{C} \cong \chi(\mathfrak{t})$ is isomorphic to the sheaf of differential operators on X twisted by $-\xi\varpi \in H^2(X;\mathbb{C})$, and this sheaf is self-opposite when $\xi = -1/2$. This implies that

$$\eta(a) := a\Big(t_1 \frac{\partial}{\partial t_1} + \frac{1}{2}\Big)$$

is a canonical quantized moment map. By Proposition 3.16, the reduction by η at the parameter ξ has period equal to $-h\xi\varpi$, and is isomorphic to the sheaf of differential operators twisted by $(-\xi + 1/2)\varpi$, confirming the result for multiples of the canonical class.

Now, assume that $\mathcal{L}_1 = \omega_X^{-1}$, which we can always arrange. If $\sigma\colon T^*Y \to T^*Y/G$ is the projection, then $\sigma_*\tilde{\mathcal{R}}^T$ is an $\mathbb{S}$-equivariant quantization of the relative Poisson scheme $T^*Y/G \to \mathfrak{t}^*$, and thus has period $[\omega_{T^*Y/G}] + h\epsilon$ for some $\epsilon \in H^2(X;\mathbb{C})$. If $s\colon \Delta \to \Delta \times \mathfrak{t}^*$ is the section corresponding to $\zeta_1 = -1/2$ and $\zeta_i = 0$ for $i > 1$, then we arrive at the conclusion that $s^*\sigma_*\tilde{\mathcal{R}}^T \cong \mathcal{R}_{-\varpi/2}$, which already know has period 0. Thus, we must have $s^*([\omega_{T^*Y/G}] + h\epsilon) = h(\varpi/2 + \epsilon) = 0$, so $\epsilon = -\varpi/2$.

For arbitrary ζ_i, we have a section $s_\zeta \colon \Delta \to \Delta \times \mathfrak{t}^*$, and

$$s_\zeta^*([\omega_{T^*Y/G}] - h\varpi/2) = h\Big(\sum_{i=1}^{n} \zeta_i c_1(\mathcal{L}_i) - \varpi/2\Big) = h(\lambda - \varpi/2).$$

Thus $\mathcal{R}_\lambda$ has the desired period. □

There is a natural $\mathbb{S}$-equivariant map $p^{-1}\mathcal{U}_\lambda \to \mathcal{R}_\lambda[h^{-1}]$, where $p\colon \mathfrak{X} \to X$ is the projection and $\mathbb{S}$ acts trivially on $p^{-1}\mathcal{U}_\lambda$. For any λ-twisted D-module $\mathcal{N}$ on X, the **microlocalization** of $\mathcal{N}$ is defined to be the $\mathcal{R}_\lambda[h^{-1}]$-module $\mathcal{R}_\lambda[h^{-1}] \otimes_{p^{-1}\mathcal{U}_\lambda} \mathcal{N}$. Proposition 4.5, which is well-known to the experts, may be regarded as a non-commutative version of the equivalence between coherent sheaves on $\mathfrak{X}$ and sheaves of coherent $\mathrm{Sym}_{\mathfrak{S}_X}\mathcal{T}_X$-modules on X.

Proposition 4.5 *Microlocalization defines an equivalence of categories from the category of finitely generated λ-twisted D-modules on X to $\mathcal{R}_\lambda[h^{-1}]$-mod.*

Proof: The adjoint equivalence is $\mathcal{J} \mapsto (p_*\mathcal{J})^{\mathbb{S}}$; we need only check this on the algebras themselves. It is clear that the microlocalization of $\mathcal{U}_\lambda$ is $\mathcal{R}_\lambda[h^{-1}]$. On the other hand, we have a map $\mathcal{U}_\lambda \to (p_*\mathcal{R}_\lambda[h^{-1}])^{\mathbb{S}}$ which is injective, and whose surjectivity is easily verified by passing to the associated graded. □

Remark 4.6 While the cotangent bundles of smooth projective varieties provide a large supply of conical symplectic varieties, these varieties very rarely are conical symplectic resolutions. In general they do not have enough global functions to be resolutions of their affinizations. For example, consider the case of a curve:

- If $X = \mathbb{P}^1$, T^*X is a resolution of a singular quadric.
- If X is elliptic, $T^*X \cong X \times \mathbb{A}^1$, so the affinization of T^*X is isomorphic to $\mathbb{A}^1$.
- If X has genus greater than 1, then T^*X is a line bundle of positive degree, and thus has no nonconstant global functions.

Example 4.7 One class of projective varieties whose cotangent bundles *are* conical symplectic resolutions are varieties of the form $X = G/P$, where G is a reductive algebraic group and $P \subset G$ is a parabolic subgroup. Philosophically, the reason is that X has a lot of vector fields (induced by the action of $\mathfrak{g}$), therefore its cotangent bundle has a lot of functions. It is conjectured (see for example [Kal09, 1.3]) that these are the *only* such projective varieties.

If P is a Borel subgroup, then T^*X is the Springer resolution of the nilpotent cone in $\mathfrak{g}$. More generally, the moment map $\mu\colon T^*X \to \mathfrak{g}^* \cong \mathfrak{g}$ is always generically finite, and its image is the closure $\bar{O}_P = G \cdot \mathfrak{p}^\perp$ of the Richardson orbit O_P associated with P. If $G = \mathrm{SL}(r;\mathbb{C})$, or if O_P is simply-connected, then μ is generically one to one, and T^*X is a symplectic resolution of $\bar{O}_P$ [Hes78, 1.3]. In other cases, it is still a symplectic resolution of its affinization, but this affinization may be a finite cover of $\bar{O}_P$.

4.2 Localization

We return to considering a general $\mathfrak{X}/S$ satisfying the assumptions of Section 3. We fix a quantization $\mathcal{D}$ of $\mathfrak{X}$, and we let $A := \Gamma_{\mathbb{S}}(\mathcal{D})$ be its section algebra. Let A-Mod be the category of arbitrary A-modules, and let A-mod be the full subcategory of finitely generated modules. As in the case of $\mathcal{D}$-modules, we will be interested in the unbounded derived category $D(A\text{-Mod})$ and the bounded derived category $D^b(A\text{-mod})$; by Remark 4.2, $D^b(A\text{-mod})$ is equivalent to the full subcategory of $D(A\text{-Mod})$ consisting of objects whose cohomology is both bounded and finitely generated.

We have a functor

$$\Gamma_{\mathbb{S}} \colon \mathcal{D}\text{-mod} \to A\text{-mod}$$

given by taking $\mathbb{S}$-invariant global sections. The left adjoint functor

$$\mathrm{Loc} \colon A\text{-mod} \to \mathcal{D}\text{-mod}$$

is defined by putting $\mathrm{Loc}(N) := \mathcal{D} \otimes_A N$, with the $\mathbb{S}$-action induced from the action on $\mathcal{D}$. To see that $\mathrm{Loc}(N)$ is indeed an object of $\mathcal{D}$-mod, let $Q \subset N$ be a finite generating set and define a filtration of N by putting $N(m) := A(m) \cdot Q$. We define the **Rees algebra** $R(A)$ to be the h-adic completion of

$$A(0)[[h^{1/n}]] + h^{1/n} A(1)[[h^{1/n}]] + h^{2/n} A(2)[[h^{1/n}]] + \ldots \subset A[[h^{1/n}]]$$

and the **Rees module** $R(N)$ to be the h-adic completion of

$$N(0)[[h^{1/n}]] + h^{1/n} N(1)[[h^{1/n}]] + h^{2/n} N(2)[[h^{1/n}]] + \ldots \subset N[[h^{1/n}]].$$

Note that $R(N)$ is a module over $R(A) \cong \Gamma(\mathcal{D}(0))$, and $\mathcal{D}(0) \otimes_{R(A)} R(N)$ is a coherent lattice in $\mathrm{Loc}(N)$.

Remark 4.8 If N is an object of A-mod, we have shown that $\mathrm{Loc}(N)$ always admits a coherent lattice, but the construction of that lattice depends on a choice of filtration of N. Conversely, any coherent lattice $\mathcal{N}(0)$ for an object $\mathcal{N}$ of $\mathcal{D}$-mod induces a filtration of $N := \Gamma_{\mathbb{S}}(\mathcal{N})$ by putting $N(m) := \Gamma_{\mathbb{S}}\big(h^{-m/n}\mathcal{N}(0)[h^{1/n}]\big)$.

If $\Gamma_{\mathbb{S}}$ and Loc are biadjoint equivalences of categories, we will say that **localization holds**

for $\mathcal{D}$ or that **localization holds at $\boldsymbol{\lambda}$**, where $[\omega_{\mathfrak{X}}] + h\lambda$ is the period of $\mathcal{D}$. Localization is known to hold for certain parameters in many special cases, including quantizations of the Hilbert scheme of points in the plane [KR08, 4.9], the cotangent bundle of G/P [BB81], resolutions of Slodowy slices [Gin09, 3.3.6] & [DK, 7.4], and hypertoric varieties [BK12, 5.8]. We conjecture that any conical symplectic resolution $\mathfrak{M}$ admits many quantizations for which localization holds.

Conjecture 4.9 *Let $\Lambda \subset H^2(\mathfrak{M};\mathbb{C})$ be the set of periods of quantizations for which localization holds. There exists*

- *a finite list of effective classes $x_1, \ldots, x_r \in H_2(\mathfrak{M};\mathbb{Z})$*
- *a finite list of rational numbers $a_i \in \mathbb{Q}$*

such that $\Lambda = H^2(\mathfrak{M};\mathbb{C}) \smallsetminus \bigcup_{i=1}^{r} D_i$, where

$$D_i := \left\{\lambda \in H^2(\mathfrak{M};\mathbb{C}) \;\middle|\; \langle x_i, \lambda\rangle - a_i \in \mathbb{Z}_{\leq 0}\right\}.$$

Remark 4.10 The classes $x_1, \ldots, x_r$ should exactly correspond to the effective curve classes in "generic non-affine deformations" of $\mathfrak{M}$ in the sense of [BMO, 1.15]. These classes play an important role in the formula for quantum cohomology of the Springer resolution [BMO, 1.1], and conjecturally of any conical symplectic resolution.

Though we cannot prove Conjecture 4.9, we will establish asymptotic results both in the derived (Theorem 4.17) and abelian (Corollary 5.17) settings.

4.3 Derived localization

In this section, we continue the assumptions of Section 4.2. We next wish to consider the derived functors $\mathbb{R}\Gamma_{\mathbb{S}}$ and $\mathbb{L}\mathrm{Loc}$ relating the triangulated categories $D(A\text{-Mod})$ and $D(\mathcal{D}\text{-Mod})$. Note that these derived functors are well-defined by [Spa88, Th. A]. First, let us establish certain homological properties of these functors.

Lemma 4.11 *For any good $\mathbb{S}$-equivariant module $\mathcal{N}$, the module $\mathbb{R}^k\Gamma(\mathfrak{X};\mathcal{N}(0))$ is finitely generated over $R(A)$ and the map*

$$\mathbb{R}^k\Gamma(\mathfrak{X};\mathcal{N}(0)) \to \varprojlim \mathbb{R}^k\Gamma(\mathfrak{X};\mathcal{N}(0)/\mathcal{N}(-nm))$$

is an isomorphism for all k.

Proof: Let

$$G^k(m) := \mathbb{R}^k\Gamma(\mathfrak{X};\mathcal{N}(-nm)) \qquad \text{and} \qquad G^k(m|p) := \mathbb{R}^k\Gamma(\mathfrak{X};\mathcal{N}(-nm)/\mathcal{N}(-np))$$

for $p \geq m$. We claim that the cohomology $G^k(0)$ is a finitely generated $\mathbb{S}$-equivariant $R(A)$-module. To see this, note that the cohomology long exact sequence of

$$0 \to \mathcal{N}(-n) \to \mathcal{N}(0) \to \mathcal{N}(0)/\mathcal{N}(-n) \to 0$$

gives an injective map $G^k(0)/hG^k(0) \hookrightarrow G^k(0|1) = \mathbb{R}^k\Gamma(\mathfrak{X}; \mathcal{N}(0)/\mathcal{N}(-n))$. The latter is the cohomology of a coherent sheaf, and thus finitely generated over $\mathbb{C}[\mathfrak{X}]$. Let P be a submodule of $G^k(0)$ generated by representatives of a finite generating set of $G^k(0)/hG^k(0)$, so we have $G^k(0) = P + hG^k(0)$. Then given any $x \in G^k(0)$, we can inductively find $p_i \in P$, $i = 0, 1, 2, \dots$, so that $x - \sum_{j=0}^{N} h^j p_j$ lies in $h^{N+1}G^k(0)$. Since $R(A)$ is complete in the h-adic topology, we can take the limit to obtain $p \in P$ such that $x - p$ lies in $\bigcap_{i=0}^{\infty} h^i G^k(m)$. But this intersection is zero, since $\bigcap_{i=0}^{\infty} h^i \mathcal{N}(-nm) = 0$, and so $G^k(0) = P$. Thus $G^k(0)$ is finitely generated as desired.

Thus, $G^k(0)$ is a quotient of a finite rank free module $R(A)^{\oplus n}$ by a submodule K. Consider the short exact sequence of projective systems

$$0 \to K/(K \cap h^m R(A)^{\oplus n}) \to (R(A)/h^m R(A))^{\oplus n} \to G^k(0)/h^m G^k(0) \to 0.$$

Since the kernel satisfies Mittag-Leffler, we obtain an isomorphism

$$G^k(0) \cong R(A)^{\oplus n}/K \cong \Big(\varprojlim (R(A)/h^m R(A))^{\oplus n}\Big)/\Big(\varprojlim K/(K \cap h^m R(A)^{\oplus n})\Big) \cong \varprojlim G^k(0)/h^m G^k(0).$$

Note that the long exact sequence associated to the short exact sequence of projective systems $h^m G^k(0) \to G^k(0) \to G^k(0)/h^m G^k(0)$ further shows that the first derived functor of $\varprojlim$ vanishes on the left hand system:

$$\varprojlim{}^1 h^m G^k(0) \cong \big(\varprojlim G^k(0)/h^m G^k(0)\big)/G^k(0) = 0.$$

All higher derived functors vanish, since this holds for any projective system over $\mathbb{Z}_{\geq 0}$ in the category of modules over a ring. Now, we consider the long exact sequence

$$\cdots \to G^{k-1}(0|m) \to G^k(m) \to G^k(0) \to G^k(0|m) \to G^{k+1}(m) \to \cdots. \tag{6}$$

This breaks into a series of short exact sequences

$$0 \to \mathrm{Tor}^1(\mathbb{C}[h]/(h^m), G^k(m)) \to G^k(m) \to G^k(0) \to G^k(0)/h^m G^k(0) \to 0.$$

The submodule of all h-torsion elements in $G^k(0)$ is finitely generated, so it is killed by h^M for some M. For $m > M$, the group $\mathrm{Tor}^1(\mathbb{C}[h]/(h^m), G^k(m))$ stabilizes, and the induced map in the projective system is multiplication by h. This projective system satisfies the property that the image of $\mathrm{Tor}^1(\mathbb{C}[h]/(h^{m+M}), G^k(M+m))$ in $\mathrm{Tor}^1(\mathbb{C}[h]/(h^m), G^k(m))$ is trivial, so the projective system has $\varprojlim \mathrm{Tor}^1(\mathbb{C}[h]/(h^m), G^k(m)) \cong \varprojlim^1 \mathrm{Tor}^1(\mathbb{C}[h]/(h^m), G^k(m)) = 0$ by Mittag-Leffler again. The short exact sequence

$$0 \to \mathrm{Tor}^1(\mathbb{C}[h]/(h^m), G^k(m)) \to G^k(m) \to h^m G^k(0) \to 0$$

shows that $\varprojlim G^k(m) = \varprojlim^1 G^k(m) = 0$ as well.

Since $G^k(0|m)$ is the extension of two projective systems with higher derived limits vanishing, the higher projective limits of $G^k(0|m)$ vanish as well. The long exact sequence (6) thus remains exact when we take the projective limit, since the higher derived functors of all its terms vanish. Therefore, we obtain the desired isomorphism $G^k(0) \cong \varprojlim G^k(0|m)$. □

Proposition 4.12 *The functor $\mathbb{R}\Gamma_{\mathbb{S}}$ induces a functor $D^b(\mathcal{D}\text{-mod}) \to D^b(A\text{-mod})$.*

Proof: Since any complex of A-modules with cohomology that is finitely generated and bounded is quasi-isomorphic to a bounded complex, we need only prove that $\mathbb{R}\Gamma_{\mathbb{S}}$ applied to any good $\mathcal{D}$-module $\mathcal{N}$ has finitely generated cohomology in finitely many degrees. By Lemma 4.11, the cohomology is finitely generated, and we need only check that $G^k(0|m)$ (using the notation of the lemma) is only non-zero in finitely many degrees. Since $\mathcal{N}(0)/\mathcal{N}(-nm)$ is just an iterated extension of $\mathcal{N}(0)/\mathcal{N}(-1)$ it suffices to show the same for $H^i(\mathfrak{X}; \mathcal{N}(0)/\mathcal{N}(-1))$. Since $\mathfrak{X}$ is projective over $\mathfrak{X}_0$, this group is finitely generated over $\mathbb{C}[\mathfrak{X}]$ and can only be non-zero if $0 \leq i \leq \dim \mathfrak{X}$. □

If the functors $\mathbb{R}\Gamma_{\mathbb{S}}$ and $\mathbb{L}\mathrm{Loc}$ induce biadjoint equivalences $D^b(\mathcal{D}\text{-mod}) \cong D^b(A\text{-mod})$, we say that **derived localization holds for $\boldsymbol{\mathcal{D}}$** or that **derived localization holds at $\boldsymbol{\lambda}$**, where $[\omega_{\mathfrak{X}}] + h\lambda$ is the period of $\mathcal{D}$. The following result of Kaledin [Kal08, §3.1] gives a sufficient condition for derived localization to hold. Let $\mathcal{D}^{\mathrm{op}}$ be the opposite ring of $\mathcal{D}$, and let A^{op} be its section algebra. Consider the sheaf of algebras $\mathcal{D}\hat{\boxtimes}_{\mathbb{C}((h))}\mathcal{D}^{\mathrm{op}}$ on $\mathfrak{X} \times \mathfrak{X}$, which has section algebra $A \otimes A^{\mathrm{op}}$. Let $\mathcal{D}_{\mathrm{diag}}$ be the $\mathcal{D}\hat{\boxtimes}_{\mathbb{C}((h))}\mathcal{D}^{\mathrm{op}}$-module obtained by pushing $\mathcal{D}$ forward along the diagonal inclusion from $\mathfrak{X}$ to $\mathfrak{X} \times \mathfrak{X}$, and let A_{diag} be the algebra A, regarded as a module over $A \otimes A^{\mathrm{op}}$.

Theorem 4.13 (Kaledin) *Suppose that the higher cohomology groups of $\mathcal{D}$ vanish.*[12] *Then derived localization holds if and only if the natural map $\mathbb{L}\mathrm{Loc}(A_{\mathrm{diag}}) \to \mathcal{D}_{\mathrm{diag}}$ is a quasi-isomorphism.*

[12]By Proposition 2.4, this condition is satisfied by any conical symplectic resolution.

Remark 4.14 Kaledin uses the bounded above derived categories $D^{-}(\mathcal{D}\text{-mod})$ and $D^{-}(A\text{-mod})$; however, this equivalent to the claim that the equivalence holds on bounded derived categories, by the argument of [Kal08, 1.2].

We will use Kaledin's result to prove Theorem A from the introduction. To do this, we first need to establish a technical result. As usual, we let $\mathfrak{M}$ be a conical symplectic resolution. Let $\mathscr{D}$ be any $\mathbb{S}$-equivariant quantization of its twistor deformation $\mathscr{M}_\eta$, and let t be the coordinate on $\mathbb{A}^1$. Let $\mathcal{N}$ be a $\mathscr{D}$-module supported on a Lagrangian subvariety of $\mathfrak{M} \subset \mathscr{M}_\eta$ for $\eta \in H^2(\mathfrak{M};\mathbb{Z})$ (in the sense that its pullback to the complement of this Lagrangian is zero).

Lemma 4.15 *There exists a nonzero polynomial $q(x) \in \mathbb{C}[x]$ such that $q(h^{-1}t) \in \mathscr{A}_\eta$ acts by zero on $\mathcal{N}$.*

Proof: Let $\mathscr{L}$ be the twistor line bundle on $\mathscr{M}_\eta$, i.e. the line bundle satisfying the statement of Proposition 2.10, and let $u\colon \mathsf{Tot}(\mathscr{L}^\times) \to \mathscr{M}_\eta$ be the projection. Then the total space $\mathsf{Tot}(\mathscr{L}^\times)$ is symplectic, and the fiberwise $\mathbb{C}^*$-action is Hamiltonian with moment map t, where t is the coordinate on $\mathbb{A}^1$, and the map u coinduces the Poisson structure on $\mathscr{M}_\eta$.

Since the quantization $\mathscr{D}$ is $\mathbb{S}$-equivariant, its period will be of the form $[\omega_{\mathscr{M}_\eta}]+h\lambda = t\eta+h\lambda$ for some $\lambda \in H^2(\mathscr{M}_\eta;\mathbb{C})$. Let $\mathscr{U}$ be the quantization of $\mathsf{Tot}(\mathscr{L}^\times)$ with period $u^*\lambda$. As noted by Bezrukavnikov and Kaledin [BK04, 6.2], the algebra $\mathscr{U}$ carries a $\mathbb{C}^*$-equivariant structure for the fiberwise action, commuting with the $\mathbb{S}$-equivariant structure. By Proposition 3.11, $\mathscr{U}$ has a quantized moment map for the $\mathbb{C}^*$-action; choose one, and let $\tau \in \Gamma_{\mathbb{S}}(\mathscr{U}[h^{-1/n}])$ be the image of the generator y of $\mathrm{Lie}(\mathbb{C}^*)$. By the definition of a non-commutative moment map, $y-\tau$ commutes with the action of any $\mathbb{C}^*$-invariant section of $\mathscr{U}$ on any $\mathbb{C}^*$-equivariant module over this algebra.

As noted in the proof of Proposition 3.16, the invariant pushforward $(u_*\mathscr{U}[h^{-1/n}])^{\mathbb{C}^*}$ is the quantization of $\mathscr{M}_\eta$ with period $h\lambda+t\eta$, and is therefore isomorphic to our given quantization $\mathscr{D}$. Thus, we have an equivalence between good $\mathbb{S}$-equivariant $\mathscr{D}$-modules and $\mathbb{S}\times\mathbb{C}^*$-equivariant $\mathscr{U}[h^{-1/n}]$-modules, induced by the adjoint functors u^* and $u_*^{\mathbb{C}^*}$.

Recall that we are given a $\mathscr{D}$-module $\mathcal{N}$ on $\mathscr{M}_\eta$ supported on a Lagrangian subvariety of $\mathfrak{M}$. Thus, $u^*\mathcal{N}$ is supported on the preimage of $\mathrm{Supp}(\mathcal{N})$ which is Lagrangian. By a finiteness theorem of Kashiwara and Schapira [KS12, 7.1.10], the self Ext-sheaf of $(u^*\mathcal{N})^{\mathrm{an}}$ is perverse, and in particular, its endomorphism algebra commuting with $\mathbb{S}$ is finite dimensional over $\mathbb{C}$. By Theorem 5.22, the same holds for $u^*\mathcal{N}$.[13] As with any element of any finite dimensional algebra over $\mathbb{C}$, the endomorphism $y-\tau$ has a minimal polynomial $q(x)$ such that $q(y-\tau)=0$. Since the structure map $\pi^{-1}\mathfrak{S}^1_{\mathbb{A}} \to \mathscr{D}$ is given by $t \mapsto h\tau$, we thus have that the action of $\tau = h^{-1}t$ on the reduction $\mathcal{N} = (u_*u^*\mathcal{N})^{\mathbb{C}^*}$ satisfies the same polynomial equation. □

[13]One can also use the theory of Euler classes from [KS12], which we will discuss later in Section 6.2, to show that such a module has finite length, imitating the usual proof for D-modules [HTT08, 3.1.2(ii)]. We thank the referee for this observation.

Remark 4.16 Lemma 4.15 almost certainly holds for general $\eta \in H^2(\mathfrak{M};\mathbb{C})$ rather than just classes in the image of integral cohomology; however the proof uses the line bundle $\mathcal{L}$ in a very strong way. Proving the general case will require understanding the theory of twistor deformations of line bundles over gerbes.

Theorem 4.17 *Fix a class $\eta \in H^2(\mathfrak{M};\mathbb{Z})$ such that $\mathscr{M}_\eta(\infty)$ is affine.*[14] *Derived localization holds at $\lambda + k\eta$ for all but finitely many $k \in \mathbb{C}$.*

Proof: Let $\mathcal{D}_k$ be the quantization with period $h(\lambda + k\eta)$. By Theorem 4.13, we need to show that the map $\mathbb{L}\mathrm{Loc}((A_k)_{\mathrm{diag}}) \to (\mathcal{D}_k)_{\mathrm{diag}}$ is an isomorphism for all but finitely many k; let $\mathcal{P}_k$ denote the cone of this map. Let $\mathscr{D}$ be the quantization of $\mathscr{M}_\eta$ with period $t\eta + h\lambda$, and let $\sigma_k \colon \Delta \to \mathbb{A}^1 \times \Delta$ be the map associated to the polynomial hk as in Section 3.1. Proposition 3.1 tells us that $\sigma_k^*\mathscr{D} \cong \mathcal{D}_k$, which implies that the morphism $\mathbb{L}\mathrm{Loc}((A_k)_{\mathrm{diag}}) \to (\mathcal{D}_k)_{\mathrm{diag}}$ on $\mathfrak{M} \times \mathfrak{M}$ is the pullback of the morphism $\phi \colon \mathbb{L}\mathrm{Loc}(\mathscr{A}_{\mathrm{diag}}) \to \mathscr{D}_{\mathrm{diag}}$ on $\mathscr{M}_\eta \times_{\mathbb{A}^1} \mathscr{M}_\eta$. It follows that $\mathcal{P}_k \cong \sigma_k^*\mathscr{P}$ where $\mathscr{P}$ is the cone of ϕ.

We now apply Lemma 4.15 to the symplectic resolution $\mathfrak{M} \times \mathfrak{M}$, with sheaf $\mathcal{N} = \mathscr{P}$ and cohomology class (η, η). The associated twistor deformation is $\mathscr{M}_\eta \times_{\mathbb{A}^1} \mathscr{M}_\eta$. The sheaf we will apply it to is $\mathcal{H}^j(\mathscr{P})$. This is supported on the preimage of $0 \in \mathbb{A}^1$, since all fibers over non-zero points of $\mathbb{A}^1$ are affine varieties, where obviously the map of interest is an isomorphism. If we localize $R(A_{\mathrm{diag}})$ to a sheaf on $\mathfrak{M}_0 \times \mathfrak{M}_0$, the result is supported on the diagonal. In fact, its classical limit is the structure sheaf of the diagonal $\Delta_{\mathfrak{M}_0}$. Thus, its localization is supported the preimage of the diagonal, that is, on the Steinberg variety $\mathfrak{M} \times_{\mathfrak{M}_0} \mathfrak{M} \subset \mathfrak{M} \times \mathfrak{M}$. Since $(\mathcal{D}_k)_{\mathrm{diag}}$ is also supported on diagonal $\Delta_{\mathfrak{M}}$, the sheaf $\mathscr{P}$ is also supported on the Steinberg. Since any symplectic resolution is semi-small, the Steinberg variety is isotropic. That means that either the Steinberg is Lagrangian or $\mathscr{P} = 0$, so the hypotheses of Lemma 4.15 are satisfied.

If k is not a root of the polynomial p provided by the Lemma, then $h^{-1}t - k$ acts invertibly on $\mathcal{H}^j(\mathscr{P})$, so the specialization of this sheaf at k is trivial, and so we have $\sigma_k^*\mathcal{H}^j(\mathscr{P}) = 0$. Thus, for any integer m, we can find a polynomial (the product of those for each individual homological degree) where $\mathcal{H}^j(\mathcal{P}_k)$ is trivial for $j \geq -m$.

By [Kal08, 3.3], $\mathcal{P}_k$ is trivial if and only if it it has trivial homology in degrees above $-\ell$ where ℓ is the global dimension of $\mathcal{D}_k \hat{\boxtimes}_{\mathbb{C}((h))} \mathcal{D}_k^{\mathrm{op}}$-mod (which is finite since the same is true for $\mathfrak{S}_{\mathfrak{M}\times\mathfrak{M}}$). By the argument above, this happens at all k other than the roots of a polynomial with complex coefficients, and thus for all but finitely many k. $\square$

[14] See Proposition 2.14 and the preceding paragraph for a discussion of $\mathscr{M}_\eta(\infty)$.

5 $\mathbb{Z}$-algebras

A **$\mathbb{Z}$-algebra** is an algebraic structure that mimics the homogeneous coordinate ring of a projective variety in a noncommutative setting. More precisely, it is an $\mathbb{N} \times \mathbb{N}$-graded vector space

$$Z = \bigoplus_{k \geq m \geq 0} {}_kZ_m$$

with a product that satisfies the condition ${}_kZ_\ell \cdot {}_\ell Z_m \subset {}_kZ_m$ for all $k \geq \ell \geq m$ and ${}_kZ_\ell \cdot {}_{\ell'}Z_m = 0$ if $\ell \neq \ell'$. While Z itself will usually not have a unit, each algebra ${}_kZ_k$ is required to be unital; we will also always assume that ${}_kZ_k$ is Noetherian, that ${}_kZ_m$ is finitely generated as a left ${}_kZ_k$-module and as a right ${}_mZ_m$-module, and that there exists a natural number r such that Z is generated as an algebra by those ${}_kZ_m$ with $k - m \leq r$. A left module over the $\mathbb{Z}$-algebra Z is an $\mathbb{N}$-graded vector space

$$N = \bigoplus_{m \geq 0} {}_mN$$

with an action of Z such that ${}_kZ_m \cdot {}_mN \subset {}_kN$ for all $k \geq m$ and ${}_kZ_m \cdot {}_{m'}N = 0$ if $m \neq m'$. It is called **bounded** if ${}_mN = 0$ for $m \gg 0$.

Remark 5.1 Some authors also discuss **torsion** modules, which are those isomorphic to a direct limit of bounded modules. We will only be interested in finitely generated Z-modules, and in this setting the conditions of being bounded and being torsion are equivalent.

We also assume that S is affine. Our goal is to define a $\mathbb{Z}$-algebra whose localized module category is equivalent to $\mathcal{D}$-mod.

5.1 Quantizations of line bundles

In this section, we continue the assumptions given above, though projectivity of $\mathfrak{X}$ and the affinity of S are not needed in this subsection. In addition to modules over quantizations, we will also need to consider bimodules over pairs of quantizations. Let $\mathcal{Q}$ and $\mathcal{Q}'$ be $\mathbb{S}$-equivariant quantizations. A $\mathcal{Q}' - \mathcal{Q}$ bimodule is a sheaf of modules over the sheaf $\mathcal{Q}' \otimes_{\mathbb{C}[[h]]} \mathcal{Q}^{\mathrm{op}}$ of algebras on $\mathfrak{X}$. Such a bimodule is called **coherent** if it is a quotient of a bimodule which is locally free of finite rank. The most important examples will be quantizations of line bundles.

Let $\mathcal{L}$ be an $\mathbb{S}$-equivariant line bundle on $\mathfrak{X}$ and let $\eta \in H^2_{DR}(\mathfrak{X}/S; \mathbb{C})$ be the image of the Euler class of $\mathcal{L}$. Fix an $\mathbb{S}$-equivariant quantization $\mathcal{Q}_0$ with period $[\omega_{\mathfrak{X}}] + h\lambda$, and for any integer k, let $\mathcal{Q}_k$ be the quantization with period $[\omega_{\mathfrak{X}}] + h(\lambda + k\eta)$.

Proposition 5.2 *For every pair of integers k and m, there exists a coherent $\mathbb{S}$-equivariant $\mathcal{Q}_k - \mathcal{Q}_m$ bimodule ${}_k\mathcal{T}_m$ with an isomorphism*[15] ${}_k\mathcal{T}_m/(h \cdot {}_k\mathcal{T}_m) \cong \mathcal{L}^{k-m}$. *This bimodule is unique*

[15] This is an isomorphism of $\mathfrak{S}_{\mathfrak{X}} - \mathfrak{S}_{\mathfrak{X}}$ bimodules, where the two actions of $\mathfrak{S}_{\mathfrak{X}}$ on $\mathcal{L}^{k-m}$ are the same.

up to canonical isomorphism, and tensor product with ${}_k\mathcal{T}_m$ defines an equivalence of categories from $\mathcal{Q}_m$-mod to $\mathcal{Q}_k$-mod.

Proof: By the usual sheaf theory, the locally free $\mathbb{S}$-equivariant modules of rank 1 over $\mathcal{Q}_k$ are in bijection with $H^1_{\mathbb{S}}(\mathfrak{X};\mathcal{Q}_k^\times)$. We have a surjective map of sheaves of groups $\mathcal{Q}_k^\times \longrightarrow \mathfrak{S}_{\mathfrak{X}}^\times$. The kernel of this map is $1+h\mathcal{Q}_k$. As a sheaf of groups, this possesses a filtration by the subgroups $1+h^n\mathcal{Q}_k$, with successive quotients isomorphic to the structure sheaf $\mathfrak{S}_{\mathfrak{X}}$ considered as a sheaf of abelian groups, since

$$(1+h^na)(1+h^nb) \equiv 1+h^n(a+b) \pmod{h^{n+1}}.$$

Since $\mathfrak{S}_{\mathfrak{X}}$ has vanishing higher cohomology, an argument as in [KR08, 2.12] shows the inverse limit $1+h\mathcal{Q}_k$ has vanishing higher cohomology as well. By the Hochschild-Serre spectral sequence, its higher equivariant cohomology also vanishes. In particular, we have an induced isomorphism

$$H^1_{\mathbb{S}}(\mathfrak{X};\mathcal{Q}_k^\times) \cong H^1_{\mathbb{S}}(\mathfrak{X};\mathfrak{S}_{\mathfrak{X}}^\times).$$

The line bundle $\mathcal{L}^{k-m}$ is classified by an element $[\mathcal{L}^{k-m}]$ of $H^1_{\mathbb{S}}(\mathfrak{X};\mathfrak{S}_{\mathfrak{X}}^\times)$, and we define ${}_k\mathcal{T}_m$ to be the locally free rank 1 left $\mathcal{Q}_k$-module given by the corresponding element $[{}_k\mathcal{T}_m]$ of $H^1_{\mathbb{S}}(\mathfrak{X};\mathcal{Q}_k^\times)$. The structure maps of ${}_k\mathcal{T}_m/(h\cdot {}_k\mathcal{T}_m)$ as a $\mathfrak{S}_{\mathfrak{X}}$-module are just the reduction mod h of the structure maps of ${}_k\mathcal{T}_m$, which tells us that ${}_k\mathcal{T}_m/(h\cdot {}_k\mathcal{T}_m) \cong \mathcal{L}^{k-m}$.

Now consider the sheaf of $\pi^{-1}\mathfrak{S}_S[[h]]$-algebras $\mathcal{Q}' = \operatorname{End}_{\mathcal{Q}_k}({}_k\mathcal{T}_m)^{\mathrm{op}}$. This sheaf is an $\mathbb{S}$-equivariant quantization of $\mathfrak{X}$, and it is obtained from $\mathcal{Q}_k$ by twisting the transition functions by the 1-cocycle representing ${}_k\mathcal{T}_m$. We want to show that this quantization is isomorphic to $\mathcal{Q}_m$. In order to show this, it suffices to calculate the period of $\mathcal{Q}'$ and see that it agrees with that of $\mathcal{Q}_m$.

If we can show this in the case where S is a point, then it will imply that these periods agree after pullback to every single point in S. Any two sections of $H^2_{DR}(\mathfrak{X}/S;\mathbb{C})$ that agree after pullback to every point in S are the same. Thus we can assume that $S=\operatorname{Spec}\mathbb{C}$. Since $\mathcal{Q}'$ is $\mathbb{S}$-equivariant, its period must be of the form $[\omega_{\mathfrak{X}}]+h\lambda'$ by Proposition 3.5. By definition, the period is the obstruction to lifting the torsor corresponding to $\mathcal{Q}'$ to G in the notation of Bezrukavnikov and Kaledin [BK04, (3.2)]. The class λ' is determined by the reduction $\mathcal{Q}'/h^3\mathcal{Q}'$, since the obstruction to lifting this to G_3 is $[\omega_{\mathfrak{X}}]+h\lambda' \in H^2_{DR}(\mathfrak{X}/S;\mathbb{C})\otimes\mathbb{C}[h]/(h^3)$.

As shown in the proof of [BK04, 1.8], the set of quantizations of a given symplectic structure up to second order is a torsor over $H^1_{DR}(\mathfrak{X};\mathcal{H})$ where $\mathcal{H}$ is, in the language of [BK04], the localization $\mathsf{Loc}(\mathcal{M}_s,\mathsf{H})$ of the module H of Hamiltonian vector fields on the formal disk for the Harish-Chandra torsor.

It is helpful to think about the classical rather than Zariski topology in order to understand this action. As we discuss in Section 5.4, associated to $\mathcal{Q}$, there is a quantization of the structure sheaf of the complex manifold $\mathfrak{M}^{\mathrm{an}}$, which we denote $\mathcal{Q}^{\mathrm{an}}$, and analytic versions of all the sheaves we have considered. Since the higher pushforwards $R^n\pi_*\mathfrak{S}_{\mathfrak{X}}$ or $R^n\pi_*\mathfrak{S}^{\mathrm{an}}_{\mathfrak{X}^{\mathrm{an}}}$ vanish, we have an isomorphisms of groups

$$H^1_{DR}(\mathfrak{X};\mathcal{H}) \cong H^2_{DR}(\mathfrak{X};\mathbb{C}) \qquad H^1_{DR}(\mathfrak{X};\mathcal{H}^{\mathrm{an}}) \cong H^2_{DR}(\mathfrak{X}^{\mathrm{an}};\mathbb{C})$$

via the boundary map δ for the short exact sequence of sheaves

$$0 \to \mathfrak{S}_{\mathfrak{X}} \longrightarrow J_\infty \mathfrak{S}_{\mathfrak{X}} \xrightarrow{H} \mathcal{H} \to 0,$$

(or its counterpart in the classical topology). By a classical result of Grothendieck, algebraic and analytic de Rham cohomology of the structure sheaf agree, so the same holds for $H^1_{DR}(\mathfrak{X};\mathcal{H}) \cong H^1_{DR}(\mathfrak{X}^{\mathrm{an}};\mathcal{H}^{\mathrm{an}})$.

The classical topology has the advantage that the de Rham cohomology of $\mathfrak{S}^{\mathrm{an}}_{\mathfrak{X}}$ and $\mathcal{H}^{\mathrm{an}}$ agree with the usual sheaf cohomology of their flat sections, which are locally constant functions and Hamiltonian vector fields $\mathscr{H}^{\mathrm{an}}$ respectively; thus we can think of an element of $H^1_{DR}(\mathfrak{X}^{\mathrm{an}};\mathcal{H}^{\mathrm{an}}) \cong H^1(\mathscr{H}^{\mathrm{an}})$ as a 1-cocycle in Hamiltonian vector fields. In the torsor action, a 1-cocycle acts on a first order quantization $\mathcal{Q}^{\mathrm{an}}/h^3\mathcal{Q}^{\mathrm{an}}$ by twisting it via the action of $\mathcal{H}^{\mathrm{an}}$ on $\mathcal{Q}^{\mathrm{an}}/h^3\mathcal{Q}^{\mathrm{an}}$ by $X \cdot a = a + h^2 X(\bar{a})$ where $X(\bar{a})$ denotes the usual action of a vector field on the function $\bar{a}$, which is the image of a in $\mathcal{Q}^{\mathrm{an}}/h\mathcal{Q}^{\mathrm{an}} \cong \mathfrak{S}^{\mathrm{an}}_{\mathfrak{X}}$. Note that this does not change the underlying Poisson bracket. The period mod h^2 changes by the image under the boundary map $h\delta$. Note that the period map is normalized so that the nth order describes the $(n+1)$st order of the quantization; for example, the 0th order part, the symplectic form, describes the 1st order part of the quantization.

Now, we have a map of abelian groups $\beta\colon (\mathcal{Q}^{\mathrm{an}}_k)^\times \to \mathcal{H}^{\mathrm{an}}$ uniquely determined by $a^{-1}qa = q + h^2\beta(a)(\bar{q})$, which thus matches actions on $\mathcal{Q}^{\mathrm{an}}_k/h^3\mathcal{Q}^{\mathrm{an}}_k$. Thus, when we twist by a 1-cocycle in $(\mathcal{Q}^{\mathrm{an}}_k)^\times$, this is the same as twisting by its image under β. That is, the period mod h^2 of $\mathrm{End}_{\mathcal{Q}^{\mathrm{an}}_k}\big({}_k\mathcal{T}_m{}^{\mathrm{an}}\big)^{\mathrm{op}}$ is $[\omega_{\mathfrak{X}}] + h(\lambda + k\eta + \delta \circ \beta_*([\mathcal{L}^{k-m}]))$ where β_* is the induced map $H^1(\mathcal{Q}^\times_k) \to H^1((\mathcal{Q}^{\mathrm{an}}_k)^\times) \to H^1(\mathscr{H}^{\mathrm{an}}) \cong H^1_{DR}(\mathcal{H})$ induced by the map $\beta\colon (\mathcal{Q}^{\mathrm{an}}_k)^\times \to \mathscr{H}^{\mathrm{an}}$.

Now, we calculate that

$$aqa^{-1} = q - a^{-1}[a,q] \equiv q - h^2\{\log \bar{a}, q\} \pmod{h^3}$$

so $\beta(a) = -H(\log \bar{a}) = -H(\bar{a})/a$. Thus, we wish to understand the map induced on the composition $\delta \circ \beta$ in first cohomology. Consider the diagram of sheaves in the analytic topology (we leave off superscripts to avoid clutter) with short exact rows, along with the relevant piece of the associated long exact sequences:

$$\begin{tikzcd}
 & & \mathcal{Q}_k^\times \arrow[d] \\
\mathbb{Z}_{\mathfrak{X}} \arrow[r] \arrow[d, "-1"'] & \mathfrak{S}_{\mathfrak{X}} \arrow[r, "\exp"] \arrow[d, "-1"'] & \mathfrak{S}_{\mathfrak{X}}^\times \arrow[d, "-H \circ \log"] \\
\pi^{-1}\mathfrak{S}_S \arrow[r] & \mathfrak{S}_{\mathfrak{X}} \arrow[r, "H"'] & \mathcal{H}
\end{tikzcd}
\qquad
\begin{tikzcd}
H^1(\mathfrak{X}; \mathcal{Q}_k^\times) \arrow[d] & \\
H^1(\mathfrak{X}; \mathfrak{S}_{\mathfrak{X}}^\times) \arrow[r, "c_1"] \arrow[d, "-H \circ \log"'] & H^2(\mathfrak{X}; \mathbb{Z}) \arrow[d, "-1"] \\
H^1(\mathfrak{X}; \mathcal{H}) \arrow[r, "\delta"'] & H^2(\mathfrak{X}; \pi^{-1}\mathfrak{S}_S)
\end{tikzcd}$$

This shows that

$$\delta \circ \beta([\mathcal{L}^{k-m}]) = -c_1(\mathcal{L}^{k-m}) = (m-k)\eta,$$

so by our previous calculation $\mathrm{End}_{\mathcal{Q}_k}\big({}_k\mathcal{T}_m\big)^{\mathrm{op}}$ and $\mathcal{Q}_m$ have identical periods and thus are isomorphic as $\pi^{-1}\mathfrak{S}_S[[h]]$-algebras.

Also, we wish to show that on ${}_k\mathcal{T}_m/(h \cdot {}_k\mathcal{T}_m)$, the quotient $\mathcal{Q}_m/h\mathcal{Q}_m \cong \mathfrak{S}_{\mathfrak{X}}$ acts by the usual module structure on $\mathcal{L}^{k-m}$. This is a local question, so we may assume that the line bundle $\mathcal{L}$ is trivial, in which case, $\mathcal{Q}_k \cong {}_k\mathcal{T}_m \cong \mathcal{Q}_m$ with the left and right actions just being left and right multiplication, which both coincide with the usual $\mathfrak{S}_{\mathfrak{X}}$-action after killing h.

Of course, ${}_k\mathcal{T}_m \otimes_{\mathcal{Q}_m} {}_m\mathcal{T}_k$ is a quantization of $\mathcal{L}^{k-m} \otimes_{\mathfrak{S}_{\mathfrak{M}}} \mathcal{L}^{m-k} \cong \mathfrak{S}_{\mathfrak{M}}$, so by uniqueness, ${}_k\mathcal{T}_m \otimes_{\mathcal{Q}_m} {}_m\mathcal{T}_k \cong \mathcal{Q}_k$, and tensor product is indeed an equivalence. □

Remark 5.3 In the next proposition and later in Section 6 we will want to vary the periods of the quantizations in more than one-dimensional families, so we will use an alternate notation and label the quantizations and bimodules by elements of $H^2_{DR}(\mathfrak{X}/S; \mathbb{C})$ instead of integers. In other words, the quantization $\mathcal{Q}_k$ will be written $\mathcal{Q}_{\lambda+k\eta}$ and the bimodule denoted ${}_1\mathcal{T}_0$ in the notation of Proposition 5.2 will be written ${}_{\lambda+\eta}\mathcal{T}_\lambda$.

We conclude this section by studying quantizations of line bundles in the context of Hamiltonian reduction. Let $(\mathfrak{X}, \mathcal{Q})$ be a quantization with a Hamiltonian action of a complex algebraic group G. For any $\xi \in \chi(G)$, let $\mathcal{L}_\xi$ be the line bundle on $\mathfrak{X}_{\mathrm{red}}$ descending from the trivial bundle on $\mathfrak{X}$ with G-structure given by ξ. Fix a quantized moment map η for the action of G on and a pair of elements $\xi, \xi' \in \chi(\mathfrak{g})$, and let $\mathcal{Q}_{\mathrm{red}} = \mathcal{Q}_{\mathsf{K}(\xi)}$ and $\mathcal{Q}'_{\mathrm{red}} = \mathcal{Q}_{\mathsf{K}(\xi')}$ be the corresponding reductions. Consider the $\mathcal{Q}'_{\mathrm{red}} - \mathcal{Q}_{\mathrm{red}}$ bimodule

$${}_{\xi'}\mathcal{S}_\xi := \psi_*(\mathcal{H}om_{\mathcal{Q}_{\mathfrak{U}}}(\mathcal{R}_{\xi'}, \mathcal{R}_\xi)).$$

Proposition 5.4 *If $\xi' - \xi$ does not integrate to a character of G, then ${}_{\xi'}\mathcal{S}_{\xi}$ is trivial. If it does, then it is isomorphic to the quantization ${}_{\mathsf{K}(\xi')}\mathcal{T}_{\mathsf{K}(\xi)}$ of $\mathcal{L}_{\xi'-\xi}$.*

Proof: First, note that the sheaf $\mathcal{R}_{\xi}$ inherits a left $\mathfrak{g}$-module structure via the action of left multiplication by $\eta(x) - \xi(x)$; furthermore $\mathcal{R}_{\xi}/h\mathcal{R}_{\xi} \cong \mathfrak{S}_{\mathfrak{U}\cap\mu^{-1}(\chi(\mathfrak{g}))}$, with the induced $\mathfrak{g}$-action coinciding with the natural one on $\mathfrak{S}_{\mathfrak{U}\cap\mu^{-1}(\chi(\mathfrak{g}))}$. In particular, it integrates to the group G.

The sheaf $\mathcal{E}nd(\mathcal{R}_{\xi})^{\mathrm{op}}$ is naturally isomorphic to the $\mathfrak{g}$-invariant subsheaf of $\mathcal{R}_{\xi}$ via the map that takes an endomorphism over any open set to the image of $\bar{1} \in \mathcal{R}_{\xi}(\mathfrak{U})$. Similarly, a map $\mathcal{R}_{\xi'} \to \mathcal{R}_{\xi}$ must take $\bar{1} \in \mathcal{R}_{\xi'}(\mathfrak{U})$ to a section r killed by $\eta(x) - \xi'(x)$, that is, one on which the $\mathfrak{g}$-action is of the form

$$x \cdot r = (\eta(x) - \xi(x))r = (\xi'(x) - \xi(x))r.$$

Since this action must integrate to an action of the group G, there can be no such maps if $\xi' - \xi$ does not integrate. If it does, then the pushforward ${}_{\xi'}\mathcal{S}_{\xi}$ is a quantization of the line bundle $\mathcal{L}_{\xi'-\xi}$ and thus isomorphic to ${}_{\mathsf{K}(\xi')}\mathcal{T}_{\mathsf{K}(\xi)}$. □

5.2 The quantum homogeneous coordinate ring of $\mathfrak{X}$

Fix an $\mathbb{S}$-equivariant quantization $\mathcal{Q}$ of $\mathfrak{X}$ with period $[\omega_{\mathfrak{X}}] + h\lambda \in H^2_{DR}(\mathfrak{X}/S;\mathbb{C})[[h]]$ and an $\mathbb{S}$-equivariant line bundle $\mathcal{L}$ on $\mathfrak{X}$ that is very ample relative to the affinization of $\mathfrak{X}$. To these data we will associate a $\mathbb{Z}$-algebra $Z = Z(\mathfrak{X}, \mathcal{Q}, \mathcal{L})$. Let $\eta \in H^2_{DR}(\mathfrak{X}/S;\mathbb{C})$ be the Euler class of $\mathcal{L}$, let $\mathcal{Q}_k$ be the quantization with period $[\omega_{\mathfrak{X}}] + h(\lambda + k\eta)$, let $\mathcal{D} := \mathcal{Q}[h^{-1/n}]$ and $\mathcal{D}_k := \mathcal{Q}_k[h^{-1/n}]$, and let ${}_k\mathcal{T}_m$ be the $\mathcal{Q}_k - \mathcal{Q}_m$ bimodule that quantizes the line bundle $\mathcal{L}^{k-m}$.

Definition 5.5 Let ${}_k\mathcal{T}'_m := {}_k\mathcal{T}_m[h^{-1/n}]$ be the $\mathcal{D}_k - \mathcal{D}_m$ bimodule associated to the $\mathcal{Q}_k - \mathcal{Q}_m$ bimodule ${}_k\mathcal{T}_m$.

Definition 5.6 Let ${}_kZ_m := \Gamma_{\mathbb{S}}({}_k\mathcal{T}'_m)$ with products induced by the canonical isomorphisms ${}_k\mathcal{T}'_\ell \otimes_{\mathcal{D}_\ell} {}_\ell\mathcal{T}'_m \cong {}_k\mathcal{T}'_m$. We call Z **the quantum homogeneous coordinate ring of $\mathfrak{X}$.**

We filter the sheaf ${}_k\mathcal{T}'_m$ by setting ${}_k\mathcal{T}'_m(0) = {}_k\mathcal{T}_m[h^{1/n}]$ and ${}_k\mathcal{T}'_m(\ell) = h^{\ell/n}{}_k\mathcal{T}'_m(0)$, and give ${}_kZ_m$ the induced filtration; it is compatible with the multiplication, so it makes Z into a filtered $\mathbb{Z}$-algebra.

Note that the associated graded of ${}_kZ_m$ is isomorphic to $\Gamma(\mathfrak{X};\mathcal{L}^{k-m})$, and for any Z-module N with a compatible filtration, the associated graded of N is a module over the $\mathbb{Z}$-algebra

$$\bigoplus_{k\geq m\geq 0} \Gamma(\mathfrak{X};\mathcal{L}^{k-m}).$$

We will use without comment the obvious equivalence between modules over this $\mathbb{Z}$-algebra and graded modules over the section ring $R(\mathcal{L}) := \bigoplus_{k\geq 0} \Gamma(\mathfrak{X}; \mathcal{L}^k)$. A filtration of N is called **good** if its associated graded is a finitely generated module over $R(\mathcal{L})$. Then N has a good filtration if and only if it is finitely generated over Z.

Let Z-mod be the category of finitely generated modules over Z, and let $Z\text{-mod}_{\mathrm{bd}}$ be the full subcategory of Z-mod consisting of bounded modules. We define the functors

$$\Gamma_{\mathbb{S}}^{\mathbb{Z}} \colon \mathcal{D}\text{-mod} \to Z\text{-mod} \qquad \text{and} \qquad \mathrm{Loc}^{\mathbb{Z}} \colon Z\text{-mod} \to \mathcal{D}\text{-mod}$$

by putting

$$\Gamma_{\mathbb{S}}^{\mathbb{Z}}(\mathcal{N}) := \bigoplus_{k\geq 0} \Gamma_{\mathbb{S}}\big({}_k\mathcal{T}_0' \otimes_{\mathcal{D}} \mathcal{N}\big) \qquad \text{and} \qquad \mathrm{Loc}^{\mathbb{Z}}(N) := \left(\bigoplus_{k\geq 0} {}_0\mathcal{T}_k'\right) \otimes_Z N.$$

Lemma 5.7 *If N is finitely generated over Z, then $\mathrm{Loc}^{\mathbb{Z}}(N)$ is finitely generated over $\mathcal{D}$.*

Proof: There is some integer K such that $\bigoplus_{k=0}^K N_k$ generates N. Thus $\mathrm{Loc}^{\mathbb{Z}}(N)$ is a quotient of $\bigoplus_{k=0}^K {}_0\mathcal{T}_k' \otimes_{{}_kZ_k} N_k$. Since the latter module is clearly finitely generated, the former is as well. □

A coherent lattice $\mathcal{N}(0)$ in $\mathcal{N}$ induces a filtration on $\Gamma_{\mathbb{S}}^{\mathbb{Z}}(\mathcal{N})$, which is good because we have an injection

$$\operatorname{gr} \Gamma_{\mathbb{S}}^{\mathbb{Z}}(\mathcal{N}) \hookrightarrow \bigoplus_{m\geq 0} \Gamma(\mathfrak{X};\ \overline{\mathcal{N}} \otimes \mathcal{L}^{\otimes m}),$$

where we put $\overline{\mathcal{N}} := \mathcal{N}(0)/\mathcal{N}(-1)$. The cokernel of this map is bounded, since if $m \gg 0$, then $H^1(\mathfrak{X}; \overline{\mathcal{N}} \otimes \mathcal{L}^{\otimes m}) = 0$, and consequently, $\operatorname{gr} \Gamma_{\mathbb{S}}\big({}_m\mathcal{T}_0' \otimes_{\mathcal{D}} \mathcal{N}\big) \cong \Gamma(\mathfrak{X}; \overline{\mathcal{N}} \otimes \mathcal{L}^{\otimes m})$. This shows, in particular, that

$$\overline{\mathrm{Loc}}(\operatorname{gr} \Gamma_{\mathbb{S}}^{\mathbb{Z}}(\mathcal{N})) \cong \overline{\mathcal{N}}, \tag{7}$$

where $\overline{\mathrm{Loc}}$ is the usual functor sending a graded module over $R(\mathcal{L})$ to a coherent sheaf on $\mathfrak{X}$ by the localization theorem for sheaves on a projective (over affine) variety. Conversely, a good filtration on a Z-module N induces a lattice in $\mathrm{Loc}^{\mathbb{Z}}(N)$, which is coherent because we have

$$\overline{\mathrm{Loc}}\big(\operatorname{gr} N\big) \cong \overline{\mathrm{Loc}^{\mathbb{Z}}(N)}. \tag{8}$$

The functor $\mathrm{Loc}^{\mathbb{Z}}$ is left-adjoint to $\Gamma_{\mathbb{S}}^{\mathbb{Z}}$; let $\iota_N \colon N \to \Gamma_{\mathbb{S}}^{\mathbb{Z}}\big(\mathrm{Loc}^{\mathbb{Z}}(N)\big)$ and $\epsilon_{\mathcal{N}} \colon \mathrm{Loc}^{\mathbb{Z}}\big(\Gamma_{\mathbb{S}}^{\mathbb{Z}}(\mathcal{N})\big) \to \mathcal{N}$ be the unit and co-unit of the adjunction. The following theorem justifies our name for Z.

Theorem 5.8 *The co-unit $\epsilon_{\mathcal{N}}$ is always an isomorphism and the unit ι_N is an isomorphism in sufficiently high degree. Furthermore, $\mathrm{Loc}^{\mathbb{Z}}$ kills all bounded modules, thus $\Gamma_{\mathbb{S}}^{\mathbb{Z}}$ and $\mathrm{Loc}^{\mathbb{Z}}$ are biadjoint equivalences between $\mathcal{D}$-mod and the quotient of Z-mod by $Z\text{-mod}_{\mathrm{bd}}$.*

Remark 5.9 We note that this theorem is quite close in flavor to several others in the theory of $\mathbb{Z}$-algebras, such as [SvdB01, 11.1.1], but these typically assume finiteness hypotheses that are too strong for our situation.

Remark 5.10 If we dropped the assumption that S is affine, we would expect to be able to prove a Theorem similar to Theorem 5.8 in which the $\mathbb{Z}$-algebra is replaced by a sheaf of $\mathbb{Z}$-algebras over S.

Proof of Theorem 5.8: Combining Equations (7) and (8), we have that the induced map

$$\overline{\epsilon_{\mathcal{N}}}\colon \overline{\mathrm{Loc}^{\mathbb{Z}}(\Gamma^{\mathbb{Z}}_{\mathbb{S}}(\mathcal{N}))} \to \overline{\mathrm{Loc}}(\mathrm{gr}\,\Gamma^{\mathbb{Z}}_{\mathbb{S}}(\mathcal{N})) \cong \overline{\mathcal{N}}$$

is an isomorphism. By Nakayama's lemma, $\epsilon_{\mathcal{N}}$ is an isomorphism as well. Similarly, the map

$$\mathrm{gr}(\iota_N)\colon \overline{\mathcal{N}} \to \overline{\mathrm{Loc}^{\mathbb{Z}}(\Gamma^{\mathbb{Z}}_{\mathbb{S}}(\mathcal{N}))}$$

is an isomorphism in high degree, thus the same is true for ι_N. If N is bounded, then $\overline{\mathrm{Loc}^{\mathbb{Z}}(N)} \cong \overline{\mathrm{Loc}}\big(\mathrm{gr}\,N\big)$ is the zero sheaf, thus $\mathrm{Loc}^{\mathbb{Z}}(N) = 0$, as well. □

Corollary 5.11 *The functor $D^b(\mathcal{D}\text{-mod}) \to D^b_{\mathcal{D}\text{-mod}}(\mathcal{D}\text{-Mod})$ is fully faithful.*

Proof: Let $\mathcal{N}$ be a good $\mathcal{D}$-module, and let $N := \Gamma^{\mathbb{Z}}_{\mathbb{S}}(\mathcal{N})$. Since N is finitely generated, there is some m such that the evaluation map ${}_pZ_m \otimes_{\mathbb{C}} {}_mN \to {}_pN$ is surjective for all $p \geq m$. Localizing, this shows we have a surjective map ${}_0\mathcal{T}_m \otimes_{\mathbb{C}} {}_mN \to \mathcal{N}$. Taking a classical limit (possibly after increasing m), we obtain a surjection $\mathcal{L}^{-m} \otimes_{\mathbb{C}} {}_mN \to \mathcal{N}/h\mathcal{N}$; thus we have described the quantization of the familiar construction of such a map in algebraic geometry. Applying this inductively, we can resolve $\mathcal{N}$ as a complex of locally free sheaves over $\mathcal{D}$, each step given by sums of ${}_0\mathcal{T}_{m_i}$ with $m_0 < m_1 < m_2 < \cdots$.

By taking m_0 sufficiently large, we can assure that for any fixed good $\mathcal{M}$, we have $H^i(\mathfrak{M}; \mathcal{L}^{m_j} \otimes_{\mathfrak{S}_{\mathfrak{M}}} \mathcal{M}/h\mathcal{M}) = 0$ for all $i > 0, j \geq 0$. Thus, we also have $\mathrm{Ext}^i_{\mathcal{D}}({}_0\mathcal{T}_{m_j}, \mathcal{M}) = 0$ for all $i > 0, j \geq 0$. It follows that we can use this resolution to compute $\mathrm{Ext}(\mathcal{N}, \mathcal{M})$ in either $D^b(\mathcal{D}\text{-mod})$ or $D^b_{\mathcal{D}\text{-mod}}(\mathcal{D}\text{-Mod})$ and we see that the results are canonically isomorphic. □

5.3 $\mathbb{Z}$-algebras and abelian localization

First, we discuss some basic results that hold whenever $\mathfrak{X}/S$ satisfies our running assumptions for this section. We call a bimodule between two rings **Morita** if it induces a Morita equivalence between the two rings. We call a $\mathbb{Z}$-algebra Z **Morita** if for all $k \geq m \geq 0$ the

${}_kZ_k - {}_mZ_m$-bimodule ${}_kZ_m$ is Morita and the natural map

$$ {}_kZ_{k-1} \otimes {}_{k-1}Z_{k-2} \otimes \cdots \otimes {}_{m+1}Z_m \to {}_kZ_m \tag{9} $$

is an isomorphism. In the terminology of [GS05, §5.4], this means that Z is isomorphic to the Morita $\mathbb{Z}$-algebra attached to the bimodules ${}_{m+1}Z_m$.

Definition 5.12 For any natural number p, let $Z[p]$ be the $\mathbb{Z}$-algebra defined by putting ${}_kZ[p]_m := {}_{k+p}Z_{m+p}$. For any Z-module N, we define a $Z[p]$-module $N[p]$ by $N[p]_k = N_{p+k}$.

It is clear that $Z[p]$ is isomorphic to the $\mathbb{Z}$-algebra $Z(\mathfrak{X}, \mathcal{Q}_p, \mathcal{L})$.

Proposition 5.13 *The $\mathbb{Z}$-algebra Z constructed in Section 5 is Morita if and only if, for all $k \geq 0$, localization holds for $\mathcal{D}_k$.*

Proof: Consider the functor $\gamma(M) = \bigoplus_k {}_kZ_0 \otimes_A M$ from finitely generated modules over $A = {}_0Z_0$ to Z-mod $/Z$-mod$_{\mathrm{bd}}$. Let β denote the adjoint to this functor; one description of β is that $\beta(\{{}_jN\}) = {}_0Z_j \otimes_{A_j} {}_jN$ for $j \gg 0$. There is a natural transformation $\beta(\Gamma_{\mathbb{S}}^{\mathbb{Z}}(\mathcal{M})) \to \Gamma_{\mathbb{S}}(\mathcal{M})$, induced by the natural transformation $\mathrm{Loc}(M) \to \mathrm{Loc}^{\mathbb{Z}}(\{{}_kZ_0 \otimes_A M\})$. The latter natural transformation has inverse given by the multiplication map of sections ${}_0\mathcal{T}'_k \otimes_{A_k} {}_kZ_0 \to \mathcal{D}_0$, tensored with M over A. Thus the former natural transformation is an isomorphism as well.

In particular, if we assume that Z is Morita then Gordon and Stafford [GS05, §5.5] show that γ and β are equivalences. Thus combining this result with Theorem 5.8, we see that $\Gamma_{\mathbb{S}} = \beta \circ \Gamma_{\mathbb{S}}^{\mathbb{Z}}$ is the composition of two equivalences, and thus an equivalence itself and localization holds for $\mathcal{D}$. Furthermore, if Z is Morita, then $Z[k]$ is Morita for all $k \geq 0$, so localization holds for $\mathcal{D}_k$ for all $k \geq 0$.

Conversely, suppose that localization holds for $\mathcal{D}_k$ for all $k \geq 0$. We have a natural isomorphism of functors

$$ {}_{k+1}Z_k \otimes - \cong \Gamma_{\mathbb{S}}({}_{k+1}\mathcal{T}'_k \otimes \mathrm{Loc}(-)) $$

from A_k-mod to A_{k+1}-mod. Since the right hand side is an equivalence, so is the left hand side; this proves that the bimodule ${}_{k+1}Z_k$ is Morita for all $k \geq 0$. Similarly, this implies that

$$ {}_{k+1}Z_k \otimes {}_kZ_m \cong \Gamma_{\mathbb{S}}\Big({}_{k+1}\mathcal{T}'_k \otimes \mathrm{Loc}({}_kZ_m)\Big) \cong \Gamma_{\mathbb{S}}({}_{k+1}\mathcal{T}'_m) \cong {}_{k+1}Z_m. $$

By induction, this implies that the map (9) is an isomorphism. Thus, Z is Morita. $\square$

For the remainder of the subsection, we consider the case of a conical symplectic resolution $\mathfrak{M}$. As in Proposition 2.10, let $\pi\colon \mathscr{M}_\eta \to \mathbb{A}^1$ be the twistor family of $\mathfrak{M}$ with $\mathscr{N}_\eta$ the affinization of $\mathscr{M}_\eta$, and let $\mathscr{L}$ be the line bundle on $\mathscr{M}_\eta$ extending $\mathcal{L}$. Let $\mathscr{Q}_k$ be the $\mathbb{S}$-equivariant quantization of $\mathscr{M}_\eta$ with period $[\omega_{\mathscr{M}_\eta}] + h(\lambda + k\eta)$.

Lemma 4.15 has an algebraic counterpart. Assume N is a $\mathscr{A}$-module such that:

1. $N \cong H^i(\mathscr{M}_\eta; \mathcal{N})^{\mathbb{S}}$ for any sheaf $\mathcal{N}$ satisfying the hypotheses of Lemma 4.15.
2. The preimage $\pi^{-1}(S)$ of the support S of the coherent sheaf $\operatorname{gr} N$ on $\mathscr{N}_\eta$ is contained in a Lagrangian subvariety of $\mathfrak{M}$.

Lemma 5.14 *There exists a nonzero polynomial $q(x) \in \mathbb{C}[x]$ such that $q(h^{-1}t) \in \mathscr{A}$ acts by zero on N.*

Proof: If $N \cong H^i(\mathscr{M}_\eta; \mathcal{N})^{\mathbb{S}}$, then the minimal polynomial q of $\mathcal{N}$ provides the desired polynomial. If hypothesis (2) holds, then $\operatorname{Loc}(N)$ is supported on $\pi^{-1}(S)$, so Lemma 4.15 applies to $\operatorname{Loc}(N)$. Since the map $N \hookrightarrow \Gamma_{\mathbb{S}}(\operatorname{Loc}(N))$ is injective, the polynomial q such that $q(h^{-1}t)$ kills $\operatorname{Loc}(N)$ applies equally to N. □

One particularly important application is to the product $\mathfrak{M} \times \mathfrak{M}$, and its twistor deformation $\mathscr{M}_\eta \times_{\mathbb{A}^1} \mathscr{M}_\eta$. The completed outer tensor product $\mathscr{Q}_k \hat{\boxtimes}_{\mathfrak{S}[\mathbb{A}^1]} \mathscr{Q}_\ell^{\mathrm{op}}$ is a quantization of this product, with $\mathbb{S}$-invariant section algebra $\mathscr{A}_k \otimes_{\mathbb{C}[h^{-1}t]} \mathscr{A}_\ell^{\mathrm{op}}$. Modules over this section algebra are just $\mathscr{A}_k$ - $\mathscr{A}_\ell$-bimodules with the left and right actions of $h^{-1}t$ coinciding. An important example of such a bimodule is ${}_k\mathscr{Z}_\ell := \Gamma_{\mathbb{S}}({}_k\mathscr{T}_\ell[h^{-1/n}])$ or a tensor product of such bimodules. These have the further special property that $\operatorname{gr}({}_k\mathscr{Z}_\ell)$ is supported on the diagonal in $\mathscr{N}_\eta \times_{\mathbb{A}^1} \mathscr{N}_\eta$; the same is thus true of any tensor product of these modules.

The preimage of the diagonal under $\pi \times \pi$ is just $\mathscr{M}_\eta \times_{\mathscr{N}_\eta} \mathscr{M}_\eta$, so its intersection with the preimage of any $a \in \mathbb{A}^1$ is Lagrangian (by the semi-small property). Thus, we have that:

Lemma 5.15 *Let B be a filtered $\mathscr{A}_k$ - $\mathscr{A}_m$-bimodule which is a subquotient of a tensor product of filtered bimodules of the form ${}_{k'}\mathscr{Z}_{m'}$, and whose support lies in $\mathfrak{M} \times \mathfrak{M}$ (i.e. whose classical limit $\operatorname{gr} B$ is killed by t). Then there exists a nonzero polynomial $q_B(x) \in \mathbb{C}[x]$ such that $q_B(h^{-1}t)$ acts by zero on B.*

Proposition 5.16 *There is a positive integer p such that $Z[p]$ is Morita.*

Proof: The statement that $Z[p]$ is Morita can be broken down into 3 smaller statements:

(a) There exists p such that the bimodule ${}_kZ_{k-1}$ is Morita for all $k \geq p$.

(b) There exists p such that the map (9) is surjective for all $k > m \geq p$.

(c) There exists p such that the map (9) is injective for all $k > m \geq p$.

We first prove (a). The bimodule ${}_kZ_{k-1}$ is Morita if and only if the maps

$$ {}_kZ_{k-1} \otimes_{A_{k-1}} {}_{k-1}Z_k \to A_k \tag{10} $$

and

$$ {}_{k-1}Z_k \otimes_{A_k} {}_kZ_{k-1} \to A_{k-1} \tag{11} $$

are both isomorphisms. Let ${}_0\mathscr{T}_{-1}$ be the $\mathscr{Q}_0 - \mathscr{Q}_{-1}$ bimodule quantizing $\mathscr{L}$, and let ${}_{-1}\mathscr{T}_0$ be the $\mathscr{Q}_{-1} - \mathscr{Q}_0$ bimodule quantizing $\mathscr{L}^{-1}$. Using notation similar to that of Proposition 3.1, we have

$$ {}_k\mathcal{T}_{k-1} \cong \sigma_k^*\big({}_0\mathscr{T}_{-1}\big) := {}_0\mathscr{T}_{-1}/(t-kh)\cdot {}_0\mathscr{T}_{-1}|\mathfrak{m} $$

$$ {}_{k-1}\mathcal{T}_k \cong \sigma_k^*\big({}_{-1}\mathscr{T}_0\big) := {}_{-1}\mathscr{T}_0/(t-hk)\cdot {}_{-1}\mathscr{T}_0|\mathfrak{m}, $$

which induces maps

$$ \sigma_k^*\big({}_0\mathscr{Z}_{-1}\big) := {}_0\mathscr{Z}_{-1}/(t-kh)\cdot {}_0\mathscr{Z}_{-1} \to {}_kZ_{k-1} \tag{12} $$

$$ \sigma_k^*\big({}_{-1}\mathscr{Z}_0\big) := {}_{-1}\mathscr{Z}_0/(t-kh)\cdot {}_{-1}\mathscr{Z}_0 \to {}_{k-1}Z_k. \tag{13} $$

Consider the short exact sequence

$$ 0 \longrightarrow {}_0\mathscr{T}_{-1} \xrightarrow{t-kh} {}_0\mathscr{T}_{-1} \longrightarrow {}_k\mathcal{T}_{k-1} \longrightarrow 0. $$

Adjoining $h^{-1/n}$ and taking sections, we obtain a long exact sequence

$$ 0 \longrightarrow {}_0\mathscr{Z}_{-1} \xrightarrow{t-kh} {}_0\mathscr{Z}_{-1} \longrightarrow {}_kZ_{k-1} \longrightarrow H^1(\mathscr{M}_\eta; {}_k\mathscr{T}_{k-1}[h^{-1/n}]) \longrightarrow \cdots . $$

This tells us that the map (12) is injective, with cokernel equal to the submodule of $H^1(\mathscr{M}_\eta; {}_k\mathscr{T}_{k-1}[h^{-1/n}])$ annihilated by $t-kh$. Note that the associated graded of the bimodule $H^1(\mathscr{M}_\eta; {}_k\mathscr{T}_{k-1}[h^{-1/n}])$ is supported over $0 \in \mathbb{A}^1$, since all other fibers of π are affine (Proposition 2.14). By Lemma 5.14, there exists a nonzero polynomial $f(x)$ such that $f(h^{-1}t)$ acts by zero on $H^1(\mathscr{M}_\eta; {}_k\mathscr{T}_{k-1}[h^{-1/n}])$. If $t-kh$ fails to act injectively on $H^1(\mathscr{M}_\eta; {}_k\mathscr{T}_{k-1}[h^{-1/n}])$, then so does $h^{-1}t-k$, which implies that k is a root of $f(x)$. Since there are only finitely many roots, there exists a p such that $t-kh$ acts injectively for all $k \geq p$, and therefore the map (12) is an isomorphism. The same argument with k and $k-1$ reversed applies to the map (13).

Now, consider the tensor product map

$$ {}_0\mathscr{Z}_{-1} \otimes_{\mathscr{A}_{-1}} {}_{-1}\mathscr{Z}_0 \to \mathscr{A}_0. \tag{14} $$

Let K be the kernel and E be the cokernel of this map, which are bimodules over $\mathscr{A}_0$. Over non-zero elements of $\mathbb{A}^1$, the fibers are affine, so this map is an isomorphism. Thus $\operatorname{gr} K$ and $\operatorname{gr} E$ are killed by t, and Lemma 5.15 applies. Thus, there are minimal polynomials for $h^{-1}t$ acting on these modules given by q_K and q_E.

The usual spectral sequence for tensor product shows that the cokernel of the map (10) is

$\sigma_k^* E = E\big/(h - tk) \cdot E$, and the kernel of this map is an extension of $\mathbb{R}^1\sigma_k^*(E) \cong \mathrm{Tor}_k^1(\mathbb{C}, E)$ and $\sigma_k^*(K)$. Possibly increasing the p introduced earlier, we can assume that for $k \geq p$, the element $h^{-1}t - k$ acts invertibly on E and K. Thus, we have

$$\sigma_k^*(E) = \sigma_k^*(K) = \mathbb{R}^1\sigma_k^*(E) = 0.$$

This shows that (10) is an isomorphism. A completely symmetric argument shows that after increasing p again, we may also conclude that the map (11) is an isomorphism, and so (a) is established.

We next prove (b). Fix an integer r such that $R(\mathcal{L})$ is generated in degrees less than or equal to r; it follows that Z is generated by ${}_kZ_m$ for $k - m \leq r$. For k and m such that $k - m \leq r$, we can proceed exactly as in the proof of (a) to find a p such that the map (9) is a surjection whenever $m \geq p$. For the rest of the cases, we can induct on the quantity $k - m - r$. Our inductive hypothesis tells us that the image of the map (9) contains the image of the multiplication map ${}_kZ_q \otimes_{A_q} {}_qZ_m$ for all $k > q > m$. Thus, the associated graded of the image of (9) contains all elements of $R(\mathcal{L})$ of degree $k - m$ which can be written as a sum of products of lower degree elements. Since elements of degree $r \leq k - m$ generate $R(\mathcal{L})$, this implies that the map (9) is indeed surjective, and (b) is proved.

Finally, we use (a) to prove (c). Choose p such that the map ${}_{j+1}Z_j \otimes {}_jZ_{j+1} \to {}_{j+1}Z_{j+1}$ is an isomorphism for all $j \geq p$. Now let $k > m \geq p$ be given, and consider the maps

$$\begin{array}{c} {}_kZ_{k-1} \otimes {}_{k-1}Z_{k-2} \otimes \cdots \otimes {}_{m+1}Z_m \otimes {}_mZ_{m+1} \otimes \cdots \otimes {}_{k-2}Z_{k-1} \otimes {}_{k-1}Z_k \\ \downarrow \\ {}_kZ_m \otimes {}_mZ_{m+1} \otimes \cdots \otimes {}_{k-2}Z_{k-1} \otimes {}_{k-1}Z_k \\ \downarrow \\ {}_kZ_k = A_k. \end{array}$$

(15), (16)

By our choice of p, the composition of the maps (15) and (16) is an isomorphism. It follows that (15) is injective. Since the map (15) is the tensor product of the map (9) with the Morita bimodule ${}_mZ_{m+1} \otimes \cdots \otimes {}_{k-2}Z_{k-1} \otimes {}_{k-1}Z_k$, the map (9) must also be injective. □

Propositions 5.13 and 5.16 immediately yield the following corollary.

Corollary 5.17 *There is an integer p such that localization holds for $\mathcal{D}_k$ for all $k \geq p$.*

Remark 5.18 Corollary 5.17 is precisely the first statement of Corollary B.1 from the introduction for very ample line bundles. If η is only ample, then there exists a positive integer

r such that $r\eta$ is very ample, and we obtain Corollary B.1 by applying Corollary 5.17 with $\lambda' = \lambda + j\eta$ and $\eta' = r\eta$ for $j = 0, 1, \ldots, r-1$.

It is still desirable to have a non-asymptotic result; that is, a necessary and sufficient condition for localization to hold for $\mathcal{D}$ itself in terms of $\mathbb{Z}$-algebras. Let $Z^{(p)}$ be the $\mathbb{Z}$-algebra defined by ${}_kZ_m^{(p)} \cong {}_{kp}Z_{mp}$ with the obvious product structure. It is clear that $Z^{(p)}$ is isomorphic to the $\mathbb{Z}$-algebra $Z(\mathfrak{M}, \mathcal{Q}, \mathcal{L}^p)$.

Lemma 5.19 *For all p, the restriction functor $Z\text{-mod}\,/Z\text{-mod}_{\mathrm{bd}} \to Z^{(p)}\text{-mod}\,/Z^{(p)}\text{-mod}_{\mathrm{bd}}$ is an equivalence of categories.*

Proof: By Theorem 5.8, both the source and the target are equivalent to $\mathcal{D}$-mod, and it is easy to check that these equivalences are compatible with the restriction functor. □

Proposition 5.20 *Localization holds for $\mathcal{D}$ if and only if $Z^{(p)}$ is Morita for some p.*

Proof: If $Z^{(p)}$ is Morita, then the functor $\Gamma_{\mathbb{S}} \colon \mathcal{D}\text{-mod} \to A\text{-mod}$ factors as

$$\mathcal{D}\text{-mod} \to Z\text{-mod}\,/Z\text{-mod}_{\mathrm{bd}} \to Z^{(p)}\text{-mod}\,/Z^{(p)}\text{-mod}_{\mathrm{bd}} \to A\text{-mod},$$

where the first functor is the equivalence of Theorem 5.8, the second is the equivalence of Lemma 5.19, and the last is the equivalence of [GS05, §5.5]. Thus localization holds for $\mathcal{D}$.

Conversely, assume that localization holds for $\mathcal{D}$. By Theorem 5.16, there is an integer p such that $Z[p]$ is Morita, which easily implies that $Z^{(p)}[1]$ is Morita. We need to extend this to show that $Z^{(p)}$ is Morita, which involves showing that the bimodule ${}_pZ_0$ is Morita and the multiplication map ${}_{2p}Z_p \otimes {}_pZ_0 \to {}_{2p}Z_0$ is an isomorphism. The fact that ${}_pZ_0$ is Morita follows from the natural isomorphism of functors

$${}_pZ_0 \otimes - \;\;\cong\;\; \Gamma_{\mathbb{S}}\big({}_p\mathcal{T}_0' \otimes \mathrm{Loc}(-)\big)$$

along with the fact that localization holds for both $\mathcal{D}$ and $\mathcal{D}_p$. Similarly, the fact that the multiplication map is an isomorphism follows from the natural isomorphism of functors

$${}_{2p}Z_p \otimes - \;\;\cong\;\; \Gamma_{\mathbb{S}}\big({}_{2p}\mathcal{T}_p' \otimes \mathrm{Loc}(-)\big)$$

applied to the module ${}_pZ_0$. □

Remark 5.21 The "if" direction of Proposition 5.20 is very close in content to [KR08, 2.10] (though they do not use the language of $\mathbb{Z}$-algebras) and our proof draws heavily on theirs. We note, however, that Proposition 5.16 and Corollary 5.17 have no analogues in [KR08].

5.4 Comparison of the analytic and algebraic categories

We keep our running assumptions from the start of Section 5, and assume for simplicity that S is smooth and that $\mathbb{C}[\mathfrak{X}]^{\mathbb{S}} = \mathbb{C}$. Up until this point we have worked exclusively in the algebraic category, quantizing the sheaf of regular functions in the Zariski topology. On the other hand, some other important papers have considered quantizations of the functions on an analytic variety, for example [KR08, KS12]. We will need to apply some results from these papers below, so we must prove a comparison theorem relating quantizations and their module categories for the nondegenerate Poisson scheme $\mathfrak{X}$ and its analytification $\mathfrak{X}^{\mathrm{an}}$.

First, we note that every quantization $\mathcal{Q}$ in the Zariski topology introduces a corresponding quantization $\mathcal{Q}^{\mathrm{an}}$ of the structure sheaf in the analytic category. To see this, we can consider the jet bundle $J_\infty\mathcal{Q}$, which is a pro-vector bundle on $\mathfrak{X}$ with flat connection whose sheaf of flat sections is $\mathcal{Q}$, as explained in [BK04, 1.4]. The corresponding sheaf of analytic sections $(J_\infty\mathcal{Q})^{\mathrm{an}}$ again has a flat connection, and we let $\mathcal{Q}^{\mathrm{an}}$ be its sheaf of flat sections. We have a map $\alpha^{-1}\mathcal{Q} \to \mathcal{Q}^{\mathrm{an}}$ where $\alpha\colon \mathfrak{X}^{\mathrm{an}} \to \mathfrak{X}$ is the identity on points. If $\mathcal{Q}$ is $\mathbb{S}$-equivariant, so is $\mathcal{Q}^{\mathrm{an}}$.

As in the Zariski topology, we let $\mathcal{D}^{\mathrm{an}} := \mathcal{Q}^{\mathrm{an}}[h^{-1/n}]$. Similarly, for any $\mathcal{D}$-module M, we let $M^{\mathrm{an}} := \alpha^{-1}M \otimes_{\alpha^{-1}\mathcal{D}} \mathcal{D}^{\mathrm{an}}$. As in [KR08], we call an $\mathbb{S}$-equivariant $\mathcal{D}^{\mathrm{an}}$-module **good** if it admits a coherent $\mathbb{S}$-equivariant $\mathcal{Q}^{\mathrm{an}}|_U$-lattice on every relatively compact open subset of $\mathfrak{X}$. If M is a good $\mathcal{D}$-module, M^{an} is a good $\mathcal{D}^{\mathrm{an}}$-module.

Theorem 5.22 *The functor* $(-)^{\mathrm{an}}\colon \mathcal{D}\text{-mod} \to \mathcal{D}^{\mathrm{an}}\text{-mod}$ *is an equivalence of categories.*

Proof: In essence, the proof is simply to observe that a version Theorem 5.8 holds in the analytic topology. More precisely, we define the quantum homogeneous coordinate ring Z^{an} exactly as we defined Z. There is a canonical map from Z to Z^{an}, and we claim that it is an isomorphism.

In bidegree $(0,0)$, this map is the map from $\Gamma_{\mathbb{S}}(\mathcal{D})$ to $\Gamma_{\mathbb{S}}(\mathcal{D}^{\mathrm{an}})$. To see that this is an isomorphism, it is enough to show that the associated graded map $\Gamma(\mathfrak{S}_{\mathfrak{M}}) \to \Gamma(\mathfrak{S}_{\mathfrak{M}}^{\mathrm{an}})^{\mathrm{fin}}$ is an isomorphism, where $(-)^{\mathrm{fin}}$ denotes the subalgebra of $\mathbb{S}$-locally finite vectors. Since all $\mathbb{S}$-weights on $\mathbb{C}[\mathfrak{M}]$ are positive, any $\mathbb{S}$-weight vector in $\Gamma(\mathfrak{S}_{\mathfrak{M}}^{\mathrm{an}})$ can be interpreted as a section of a line bundle on the projectivization of $\mathfrak{X}_0$ for the $\mathbb{S}$ action; by the classic GAGA theorem of Serre [Ser56], this is in fact algebraic, and thus arises from an algebraic function on $\mathfrak{X}_0$. The argument in arbitrary bidegree follows from a similar analysis of sections of line bundles.

Now that we know that Z and Z^{an} are isomorphic, we have a functor from $\mathcal{D}^{\mathrm{an}}$-mod to $\mathcal{D}$-mod given by the composition

$$\mathcal{D}^{\mathrm{an}}\text{-mod} \xrightarrow{(\Gamma_{\mathbb{S}}^{\mathbb{Z}})^{\mathrm{an}}} Z^{\mathrm{an}}\text{-mod} \cong Z\text{-mod} \xrightarrow{\mathrm{Loc}^{\mathbb{Z}}} \mathcal{D}\text{-mod}\,.$$

This functor splits $(-)^{\mathrm{an}}$ and is exact (since the cohomology of a sufficiently high twist with $\mathcal{L}$

vanishes), so to check that it gives an equivalence, we need only check that it kills no module $\mathcal{K}$. Thus, we need only show that for any good $\mathbb{S}$-equivariant $\mathcal{D}$-module, we must have that $\Gamma_{\mathbb{S}}^{\mathbb{Z}}$ is not 0. Since $\mathcal{L}$ is ample, $\mathcal{K}/h\mathcal{K} \otimes \mathcal{L}^k$ has non-zero sections for $k \gg 0$ unless $\mathcal{K}/h\mathcal{K} = 0$; then Nakayama's lemma tells us that ${}_k\mathcal{T}_0'^{\mathrm{an}} \otimes_{\mathcal{D}_0} \mathcal{K}$ has non-zero sections as well unless $\mathcal{K} = 0$. This completes the proof. □

Remark 5.23 Hou-Yi Chen [Che10] proves a version of Theorem 5.22 in the more general context of DQ-algebroids, but subject to the hypothesis that $\mathfrak{X}$ is projective over a point (which is never the case for a conical symplectic resolution of positive dimension). Chen uses a more direct reduction to Serre's classic GAGA theorem than we do; it is possible that his techniques could be adapted to our setting, as well.

Remark 5.24 It might worry the reader that we used some analytic techniques in the proof of Proposition 5.2, used that result in the proof of Theorem 5.8, and then used that in the proof of Theorem 5.22; at first glance, this looks as though it may be circular. In fact, in the proof of Proposition 5.2, we use only the comparison theorem between algebraic and analytic de Rham cohomology; nothing in the vein of GAGA.

Similarly, it might worry the reader that we use Theorem 5.22 in the proof of Lemma 4.15 earlier in the paper, but Lemma 4.15 is only used in the proof of the localization results, Theorem 4.17 and Proposition 5.16, which are not used in this section.

5.5 Twisted modules and the Kirwan functor

In this section, we return to the assumptions of Section 3.4, while keeping those introduced at the start of 5. That is we additionally assume that we have a Hamiltonian action of a connected reductive algebraic group G on $(\mathfrak{X}, \mathcal{Q})$ such that $\mathbb{C}[\mathfrak{X}]^{G\times\mathbb{S}}$ with quantized moment map $\eta\colon U(\mathfrak{g}) \to A$, which induces a flat commutative moment map $\mathfrak{X} \to \mathfrak{g}^*$. We fix a G-equivariant ample line bundle $\mathcal{L}$ on $\mathfrak{X}$ and we let $\mathfrak{U}$ be its semistable locus. We assume that the G action on $\mathfrak{U}$ is free and, and if $\mathfrak{X}_{\mathrm{red}}$ is the reduced space (with $\mathcal{L}_{\mathrm{red}}$ its induced ample line bundle), that we have an induced isomorphism $\mathbb{C}[\mu^{-1}(0)]^G = \mathbb{C}[\mathfrak{X}_{\mathrm{red}}]$, and more generally an isomorphism $\Gamma(\mu^{-1}(0); \mathcal{L}^k)^G = \Gamma(\mathfrak{X}_{\mathrm{red}}, \mathcal{L}^k)$.

Fix an element $\xi \in \chi(\mathfrak{g})$. We'll let $\mathcal{D}_{\mathrm{red}}$ be the quantization of $\mathfrak{X}_{\mathrm{red}}$ defined by reduction by $\eta - \xi$ (as defined in Section 3.4). We call a G-equivariant object $\mathcal{N}$ of $\mathcal{D}$-mod (respectively $\mathcal{D}_{\mathfrak{U}}$-mod) **$\xi$-twisted** if, for all $x \in \mathfrak{g}$, the action of x on $\mathcal{N}$ induced by the G-structure coincides with left multiplication by the element $\eta(x) - \xi(x) \in A$. Let $\mathcal{D}\text{-mod}_\xi$ (respectively $\mathcal{D}_{\mathfrak{U}}\text{-mod}_\xi$) denote the full subcategory of ξ-twisted objects of $\mathcal{D}$-mod (respectively $\mathcal{D}_{\mathfrak{U}}$-mod). Kashiwara and Rouquier [KR08, 2.8(ii)] prove that $\mathcal{D}_{\mathfrak{U}}\text{-mod}_\xi$ is equivalent to $\mathcal{D}_{\mathrm{red}}$-mod via the functor that takes $\mathcal{N}$ to $\psi_*\mathcal{H}om(\mathcal{E}_\xi, \mathcal{N})$, where $\mathcal{E}_\xi$ is the sheaf defined in Section 3.4.

Define the functor $\kappa\colon \mathcal{D}\text{-mod} \to \mathcal{D}_{\mathrm{red}}\text{-mod}$ by putting

$$\kappa(\mathcal{N}) := \psi_*\mathcal{H}om(\mathcal{E}_\xi, \mathcal{N}_{\mathfrak{U}})$$

for all $\mathcal{N}$ in $\mathcal{D}$-mod. We call κ the **Kirwan functor** in analogy with the Kirwan map in (equivariant) cohomology. Our main result in this section will be Theorem 5.31, which says that the Kirwan functor is essentially surjective. To prove this theorem, we introduce all of the analogous constructions in the context of $\mathbb{Z}$-algebras.

In Section 5, we defined a $\mathbb{Z}$-algebra $Z = Z(\mathfrak{X}, \mathcal{D}, \mathcal{L})$ and functor $\Gamma^{\mathbb{Z}}_{\mathbb{S}}\colon \mathcal{D}\text{-mod} \to Z\text{-mod}$. We may also define the $\mathbb{Z}$-algebra $Z_{\mathrm{red}} = Z(\mathfrak{X}_{\mathrm{red}}, \mathcal{D}_{\mathrm{red}}, \mathcal{L}_{\mathrm{red}})$, with its own sections functor $\Gamma^{\mathbb{Z}}_{\mathbb{S},\mathrm{red}}\colon \mathcal{D}_{\mathrm{red}}\text{-mod} \to Z_{\mathrm{red}}\text{-mod}$.

By assumption, we have a ring homomorphism

$$\eta\colon U(\mathfrak{g}) \longrightarrow A = \Gamma_{\mathbb{S}}(\mathcal{D}) \cong {}_0Z_0.$$

Moreover, for all $m \geq 0$, there is a unique homomorphism

$$\eta_m\colon U(\mathfrak{g}) \longrightarrow \Gamma_{\mathbb{S}}(\mathcal{D}_m) \cong {}_mZ_m$$

such that $\eta_0 = \eta$ and for all $x \in \mathfrak{g}$, the action of x on $\mathcal{L}$ induced by the G-structure coincides with that induced by the adjoint action, via η_{m+1} and η_m, on the $\mathcal{D}_{m+1} - \mathcal{D}_m$ bimodule ${}_{m+1}\mathcal{T}_m$ that quantizes $\mathcal{L}$. By Proposition 3.13, we can describe A_{red} as an algebraic reduction of A, and similarly, we have a map ${}_jZ_k \to \Gamma(\mathfrak{U}; {}_j\mathcal{T}_k)$, which induces a map

$${}_iY_j := {}_iZ_j \Big/ {}_iZ_j \cdot \langle \eta_j(x) - \xi(x) \mid x \in \mathfrak{g}\rangle \to \Gamma(\mathfrak{U}; {}_j\mathcal{T}_k \otimes \mathcal{E}_\xi).$$

Lemma 5.25 *The induced map ${}_iY_j{}^G \to {}_i(Z_{\mathrm{red}})_j$ is an isomorphism.*

Proof: The proof is essentially the same as Proposition 3.13. The associated graded map $\Gamma(\mu^{-1}(0); \mathcal{L}^{i-j})^G \to \Gamma(\mathfrak{X}_{\mathrm{red}}, \mathcal{L}^{i-j}_{\mathrm{red}})$ is an isomorphism by assumption, so this implies the same for the map under consideration. □

We say that a G-equivariant Z-module $N = \bigoplus_m {}_mN$ is **ξ-twisted** if, for all $x \in \mathfrak{g}$, the action of x on ${}_mN$ induced by the G-structure coincides with left multiplication by the element $\eta_m(x) - \xi(x) \in {}_mZ_m$. We denote the category of such modules $Z\text{-mod}_\xi$.

Lemma 5.25 tells us that Y is a naturally a $Z - Z_{\mathrm{red}}$ bimodule. We define the **$\mathbb{Z}$-Kirwan functor**

$$\kappa^{\mathbb{Z}} := \mathrm{Hom}_Z(Y, -)\colon Z\text{-mod} \to Z_{\mathrm{red}}\text{-mod}$$

along with its left adjoint

$$\kappa_!^{\mathbb{Z}} := Y \otimes_{Z_{\mathrm{red}}} - : Z_{\mathrm{red}}\text{-mod} \to Z\text{-mod}\,.$$

Remark 5.26 Every Z-module N has a largest submodule N_ξ on which $\eta_j(x) - \xi(x)$ acts locally finitely and on which the $\mathfrak{g}$-action integrates to a G-action. The G-action makes N_ξ a ξ-twisted equivariant module in a canonical way. Because of the ξ-twisted condition, the G-invariant part of N_ξ is already a module over Z_{red} in the obvious way, and we have a canonical isomorphism $\kappa^{\mathbb{Z}}(N) \cong N_\xi^G$.

Proposition 5.27 *The functors $\kappa^{\mathbb{Z}}$ and $\kappa_!^{\mathbb{Z}}$ both preserve boundedness and thus induce functors*

$$\kappa^{\mathbb{Z}}\colon Z\text{-mod}\,/Z\text{-mod}_{\mathrm{bd}} \to Z_{\mathrm{red}}\text{-mod}\,/\,Z_{\mathrm{red}}\text{-mod}_{\mathrm{bd}}$$

and

$$\kappa_!^{\mathbb{Z}}\colon Z_{\mathrm{red}}\text{-mod}\,/\,Z_{\mathrm{red}}\text{-mod}_{\mathrm{bd}} \to Z\text{-mod}\,/Z\text{-mod}_{\mathrm{bd}}\,.$$

Proof: The functor $\kappa^{\mathbb{Z}}$ obviously sends bounded modules to bounded modules. To see that $\kappa_!^{\mathbb{Z}}$ preserves boundedness, find integers N and M such that all of the higher cohomology groups of $\mathcal{L}^N$ and $\mathcal{L}_{\mathrm{red}}^M$ vanish. Then for any non-negative integers i, j, k with $i \geq j + N \geq k + N + M$ the associated graded of the multiplication map ${}_iY_j \otimes {}_j(Z_{\mathrm{red}})_k \to {}_iY_k$. is

$$\Gamma(\mu^{-1}(0); \mathcal{L}^{i-j})^G \otimes \Gamma(\mathfrak{X}_{\mathrm{red}}; \mathcal{L}_{\mathrm{red}}^{j-k}) \to \Gamma(\mu^{-1}(0); \mathcal{L}^{i-k})^G.$$

Since $\oplus_n \Gamma(\mu^{-1}(0); \mathcal{L}^n)^G$ is a finitely generated module over $\oplus_n \Gamma(\mathfrak{X}_{\mathrm{red}}; \mathcal{L}_{\mathrm{red}}^n)$, there is some N' such that $\oplus_{N \leq n \leq N'} \Gamma(\mu^{-1}(0); \mathcal{L}^n)^G$ generates $\oplus_{N \leq n} \Gamma(\mu^{-1}(0); \mathcal{L}^n)^G$. That is, if we fix i and k such that $i - k \geq N$, then $\Gamma(\mathfrak{X}; \mathcal{L}^{i-k})^G$ is spanned by the images of the maps

$$\Gamma(\mu^{-1}(0); \mathcal{L}^{i-j})^G \otimes \Gamma(\mathfrak{X}_{\mathrm{red}}; \mathcal{L}_{\mathrm{red}}^{j-k}) \to \Gamma(\mu^{-1}(0); \mathcal{L}^{i-k})^G \text{ for all } j \text{ such that } N \leq i - j \leq N'.$$

We may as well assume that $N' \geq M + N$. Thus, since a map whose associated graded is surjective is itself surjective, we see that the map

$$\bigoplus_{j \geq i - N'} {}_iY_j \otimes {}_j(Z_{\mathrm{red}})_k \to {}_iY_k$$

is surjective for all $k \leq i - N'$. If M is a Z_{red}-module, it follows that if $i \geq N'$ then $\kappa_!^{\mathbb{Z}}(M)_i$ is spanned by the images of ${}_iY_j \otimes M_j$ for $j \geq i - N'$. Then if $M_p = 0$ for $p \geq P$, we have $\kappa_!^{\mathbb{Z}}(M)_i = 0$ whenever $i > P + N'$.

This shows that both $\kappa^{\mathbb{Z}}$ and $\kappa_!^{\mathbb{Z}}$ preserve bounded modules and thus induce functors on the quotient categories. □

Proposition 5.28 *The following diagram commutes.*

$$\begin{tikzcd}
\mathcal{D}\text{-mod} \arrow[r, "\Gamma^{\mathbb{Z}}_{\mathbb{S}}"] \arrow[d, "\kappa"'] & Z\text{-mod}\,/\,Z\text{-mod}_{\mathrm{bd}} \arrow[d, "\kappa^{\mathbb{Z}}"] \\
\mathcal{D}_{\mathrm{red}}\text{-mod} \arrow[r, "\Gamma^{\mathbb{Z}}_{\mathbb{S},\mathrm{red}}"] & Z_{\mathrm{red}}\text{-mod}\,/\,Z_{\mathrm{red}}\text{-mod}_{\mathrm{bd}}
\end{tikzcd}$$

(Note that the horizontal arrows are equivalences by Theorem 5.8.)

Proof: Fix an object $\mathcal{N}$ of $\mathcal{D}$-mod. First, note that we can assume that $\mathcal{N} = \mathcal{N}_\xi$, that is, that $\mathcal{N}$ has an G-equivariant structure agreeing with that induced by $\eta - \xi$. This is because passing to the largest submodule that has such a structure doesn't change κ or $\kappa^{\mathbb{Z}}$.

With this assumption, we have a restriction map

$$\kappa^{\mathbb{Z}}\left(\Gamma^{\mathbb{Z}}_{\mathbb{S}}(\mathcal{N})\right) = \left(\Gamma^{\mathbb{Z}}_{\mathbb{S}}(\mathfrak{X};\mathcal{N})\right)^G \longrightarrow \left(\Gamma^{\mathbb{Z}}_{\mathbb{S}}(\mathfrak{U};\mathcal{N})\right)^G \cong \Gamma^{\mathbb{Z}}_{\mathbb{S},\mathrm{red}}(\kappa(\mathcal{N}))$$

where $\Gamma^{\mathbb{Z}}_{\mathbb{S}}(\mathfrak{X};-) = \Gamma^{\mathbb{Z}}_{\mathbb{S}}$, and $\Gamma^{\mathbb{Z}}_{\mathbb{S}}(\mathfrak{U};-)$ denotes the same functor defined using the set $\mathfrak{U}$ of stable points. As in the proof of Theorem 5.8, let $\bar{\mathcal{N}} := \mathcal{N}(0)/\mathcal{N}(-1)$.

For each $m \in \mathbb{Z}$, the restriction from $\mathfrak{X}$ to $\mathfrak{U}$ gives the following long exact sequence in local cohomology.

$$H^0_{\mathfrak{X}\setminus\mathfrak{U}}\left(\bar{\mathcal{N}} \otimes \mathcal{L}^m\right)^G \longrightarrow \Gamma\left(\mathfrak{X};\bar{\mathcal{N}} \otimes \mathcal{L}^m\right)^G \longrightarrow \Gamma\left(\mathfrak{U};\bar{\mathcal{N}} \otimes \mathcal{L}^m\right)^G \longrightarrow H^1_{\mathfrak{X}\setminus\mathfrak{U}}\left(\bar{\mathcal{N}} \otimes \mathcal{L}^m\right)^G \longrightarrow \cdots$$

The space

$$\bigoplus_{m\geq 0} H^0_{\mathfrak{X}\setminus\mathfrak{U}}\left(\bar{\mathcal{N}} \otimes \mathcal{L}^m\right) \tag{17}$$

of sections of twists of $\bar{\mathcal{N}}$ which are supported on $\mathfrak{X} \setminus \mathfrak{U}$ is finitely generated over the ring

$$\bigoplus_{m\geq 0} \Gamma\left(\mathfrak{X} \setminus \mathfrak{U}; \mathcal{L}^m\right) \tag{18}$$

of sections of powers of the restriction of $\mathcal{L}$ to $\mathfrak{X} \setminus \mathfrak{U}$. Since G is reductive, the invariant part of (17) is finitely generated over the invariant part of (18). The invariant part of (18) is a single copy of $\mathbb{C}$, since any invariant section of $\mathcal{L}^m$ for $m > 0$ vanishes on all unstable points. Thus $H^0_{\mathfrak{X}\setminus\mathfrak{U}}\left(\bar{\mathcal{N}} \otimes \mathcal{L}^m\right)^G$ vanishes for $m \gg 0$.

The module $H^1_{\mathfrak{X}\setminus\mathfrak{U}}\left(\bar{\mathcal{N}} \otimes \mathcal{L}^m\right)$ is *not* in general finitely generated as a module over the invariant section ring. On the other hand, the module $\bigoplus_{m\geq 0} \Gamma\left(\mathfrak{U};\bar{\mathcal{N}} \otimes \mathcal{L}^m\right)^G$ is the sections of the twists of a coherent sheaf on the quotient $\mathfrak{U}/G$, which is projective over an affine variety, and thus finitely generated over the invariant section ring $\bigoplus_{m\geq 0}\Gamma\left(\mathfrak{U};\mathcal{L}^m\right)^G$. In particular, its image in $\bigoplus_{m\geq 0}H^1_{\mathfrak{X}\setminus\mathfrak{U}}\left(\bar{\mathcal{N}}\otimes\mathcal{L}^m\right)^G$ under the boundary map is finitely generated over the same ring.

Since any positive degree invariant section of $\mathcal{L}$ vanishes on $\mathfrak{X}\setminus\mathfrak{U}$, its action on local cohomology is locally nilpotent; this implies that there is some integer k such that all invariants of degree $\geq k$ act trivially on the image of $\bigoplus_{m\geq 0}\Gamma\left(\mathfrak{U};\bar{\mathcal{N}}\otimes\mathcal{L}^m\right)^G$ under the boundary map. This in turn implies that the image is trivial for m sufficiently large. Note that we used the fact that the image is finitely generated in both of these steps.

It follows that the restriction map

$$\left(\Gamma(\mathfrak{X};\bar{\mathcal{N}}\otimes\mathcal{L}^m)\right)^G\longrightarrow\left(\Gamma(\mathfrak{U};\bar{\mathcal{N}}\otimes\mathcal{L}^m)\right)^G$$

is an isomorphism for $m\gg 0$. We next observe that

$$\left(\Gamma(\mathfrak{X};\bar{\mathcal{N}}\otimes\mathcal{L}^m)\right)^G\cong\ \mathrm{gr}\left(\Gamma^{\mathbb{Z}}_{\mathbb{S}}(\mathfrak{X};{}_m\mathcal{T}'_0\otimes_{\mathcal{D}}\mathcal{N})\right)^G\cong\ \mathrm{gr}\left(\Gamma^{\mathbb{Z}}_{\mathbb{S}}(\mathfrak{X};\mathcal{N})[m]\right)^G$$

and similarly

$$\left(\Gamma(\mathfrak{U};\bar{\mathcal{N}}\otimes\mathcal{L}^m)\right)^G\cong\ \mathrm{gr}\left(\Gamma^{\mathbb{Z}}_{\mathbb{S}}(\mathfrak{U};\mathcal{N})[m]\right)^G,$$

where $[m]$ denotes a shift as in Definition 5.12. Since maps that induce isomorphism on associated graded are isomorphisms, we may conclude that the restriction map

$$\left(\Gamma^{\mathbb{Z}}_{\mathbb{S}}(\mathfrak{X};\mathcal{N})[m]\right)^G\longrightarrow\left(\Gamma^{\mathbb{Z}}_{\mathbb{S}}(\mathfrak{U};\mathcal{N})[m]\right)^G$$

is an isomorphism for $m\gg 0$. This is equivalent to the statement that the kernel and cokernel of the map

$$\left(\Gamma^{\mathbb{Z}}_{\mathbb{S}}(\mathfrak{X};\mathcal{N})\right)^G\longrightarrow\left(\Gamma^{\mathbb{Z}}_{\mathbb{S}}(\mathfrak{U};\mathcal{N})\right)^G$$

are bounded, as desired. □

Lemma 5.29 *The Kirwan functor κ has a left adjoint $\kappa_!$ such that $\kappa\circ\kappa_!$ is isomorphic to the identity functor on $\mathcal{D}_{\mathrm{red}}$-mod.*

Proof: By Theorem 5.28, we may work instead with the $\mathbb{Z}$-Kirwan functor $\kappa^{\mathbb{Z}}$ and its left adjoint $\kappa^{\mathbb{Z}}_!$. Let ${}_iY'_j\subset{}_iY_j$ be the sum of all non-trivial G-isotypic components. Since G is reductive, ${}_iY_j$ is isomorphic to ${}_iY'_j\oplus{}_iY^G_j$. There is a natural map from ${}_iY^G_j$ to ${}_i(Z_{\mathrm{red}})_j$ whose associated graded is the map $\Gamma(\mu^{-1}(0);\mathcal{L}^{i-j})^G\to\Gamma(\mathfrak{X}_{\mathrm{red}};\mathcal{L}^{i-j}_{\mathrm{red}})$. This map is an isomorphism when $i-j$ is sufficiently large, which implies that the same is true of the map ${}_iY^G_j$ to ${}_i(Z_{\mathrm{red}})_j$.

Thus, modulo bounded modules, we have $Y \cong Y' \oplus Z_{\mathrm{red}}$ as a right module over Z_{red}. Then for any Z_{red}-module N, we have

$$\kappa^{\mathbb{Z}} \circ \kappa_!^{\mathbb{Z}}(N) = \kappa^{\mathbb{Z}}(Y \otimes_{Z_{\mathrm{red}}} N) \cong (Y \otimes_{Z_{\mathrm{red}}} N)^G \cong Z_{\mathrm{red}} \otimes_{Z_{\mathrm{red}}} N \cong N,$$

modulo bounded modules. □

Remark 5.30 One can use similar principles to construct a right adjoint as well as a left to κ. One considers the $Z_{\mathrm{red}} - Z$ bimodule

$${}_iW_j := {}_iZ_j \Big/ \langle \eta_j(x) - \xi(x) \mid x \in \mathfrak{g} \rangle \cdot {}_iZ_j.$$

The obvious guess for the right adjoint based on general nonsense is $\mathrm{Hom}_{Z_{\mathrm{red}}}(W, -)$; however, we need to exercise care here since W is not finitely generated as a left module. On the other hand, it is (as a left module) the direct sum $W = \bigoplus_{\chi \in \hat{G}} W^\chi$ of its isotypic components W^χ according to the natural G action, and each isotypic component is finitely generated even after taking the associated graded by a classical theorem of Hilbert. We should emphasize that here $\hat{G}$ is the set of all finite dimensional representations, not just 1-dimensional ones.

A replacement for $\mathrm{Hom}_{Z_{\mathrm{red}}}(W, -) = \prod_{\chi \in \hat{G}} \mathrm{Hom}_{Z_{\mathrm{red}}}(W^\chi, -)$ with better finiteness properties is the direct sum $\kappa_*(-) = \bigoplus_{\chi \in \hat{G}} \mathrm{Hom}_{Z_{\mathrm{red}}}(W^\chi, -)$ which we can consider as the subspace of $\mathrm{Hom}_{Z_{\mathrm{red}}}(W, -)$ which kills all but finitely many isotypic components. This is closed under the action of Z acting on the right since the G-action on Z is locally finite.

It is still not obvious that κ_* takes finitely generated modules to finitely generated modules. When X is the cotangent bundle to a smooth affine G-variety, this is proved in a recent preprint by McGerty and Nevins [MN, 6.1(3)].

The following theorem, which is an immediate consequence of Lemma 5.29, may be regarded as a categorical, quantum version of Kirwan surjectivity.

Theorem 5.31 *The Kirwan functor κ is essentially surjective.*

Proof: For any object of $\mathcal{D}_{\mathrm{red}}$-mod, we can apply the left adjoint from Lemma 5.29 to obtain a witness to essential surjectivity. □

Remark 5.32 McGerty and Nevins [MN14] always work with symplectic quotients of affine schemes, and the category of quantizations that they consider is by definition the essential image of the Kirwan functor. Thus Theorem 5.31 establishes that their module category is the same as ours.

6 Convolution and twisting

Throughout Section 6, we'll only consider conical symplectic resolutions $\mathfrak{M}$. Let $\nu\colon \mathfrak{M} \to \mathfrak{M}_0$ be the resolution map with Steinberg variety $\mathfrak{Z} := \mathfrak{M} \times_{\mathfrak{M}_0} \mathfrak{M}$. Consider the three different projections $p_{ij}\colon \mathfrak{M} \times \mathfrak{M} \times \mathfrak{M} \to \mathfrak{M} \times \mathfrak{M}$ as well as the two projections $p_i\colon \mathfrak{M} \times \mathfrak{M} \to \mathfrak{M}$. The cohomology $H^{2\dim \mathfrak{M}}_{\mathfrak{Z}}(\mathfrak{M} \times \mathfrak{M}; \mathbb{C})$ with supports in $\mathfrak{Z}$ has a convolution product given by the formula

$$\alpha \star \beta := (p_{13})_*(p_{12}^*\alpha \cdot p_{23}^*\beta),$$

making it into a semisimple algebra [CG97, 8.9.8]. For any closed subvariety $\mathfrak{L} \subset \mathfrak{M}$ with the property that $\mathfrak{L} = \nu^{-1}(\nu(\mathfrak{L}))$, there is a degree-preserving action of this algebra on the cohomology $H^*_{\mathfrak{L}}(\mathfrak{M}; \mathbb{C})$ given by the formula

$$\alpha \star \gamma := (p_2)_*(\alpha \cdot p_1^*\gamma).$$

Example 6.1 When $\mathfrak{M}$ is the cotangent bundle of the flag variety, $H^{2\dim \mathfrak{M}}_{\mathfrak{Z}}(\mathfrak{M} \times \mathfrak{M}; \mathbb{C})$ is isomorphic to the group ring of the Weyl group [CG97, 3.4.1], and $H^*(\mathfrak{M}; \mathbb{C})$ is isomorphic to the regular representation.

In this section, we explain how to categorify this action. In Section 6.1, we define the category of Harish-Chandra bimodules over a pair of quantizations. There is both an algebraic and a geometric version of this definition, and they are related by the localization and invariant section functors. In Section 6.2, we show that a (geometric) Harish-Chandra bimodule has a characteristic cycle in $H^{2\dim \mathfrak{M}}_{\mathfrak{Z}}(\mathfrak{M} \times \mathfrak{M}; \mathbb{C})$, and tensor products of bimodules categorify convolution product of cycles. Furthermore, an object $\mathcal{N}$ of $D^b(\mathcal{D}\text{-mod})$ has a characteristic cycle in $H^{\dim \mathfrak{M}}_{\mathfrak{L}}(\mathfrak{M}; \mathbb{C})$ for any $\mathfrak{L} \subset \mathfrak{M}$ containing $\operatorname{Supp} \mathcal{N}$, and we show that the tensor product action of bimodules on modules categorifies the convolution action. In Section 6.3 we define a particularly nice collection of (algebraic) Harish-Chandra bimodules, which we use in Section 6.4 to study a certain collection of auto-equivalences of $D^b(A\text{-mod})$ related to twisting functors on BGG category $\mathcal{O}$.

6.1 Harish-Chandra bimodules

Recall that, for any $\lambda \in H^2(\mathfrak{M}; \mathbb{C})$, we let $A_\lambda := \Gamma_{\mathbb{S}}(\mathcal{D}_\lambda)$ be the section ring of the quantization of $\mathfrak{M}$ with period λ. Recall from Proposition 3.6 that we can write this ring for each λ as a quotient of the sections $\mathscr{A}$ of a canonical quantization of the universal deformation $\mathscr{M}$. Let H be a finitely generated $A_{\lambda'}$-A_λ bimodule. Recall that $\operatorname{gr} A_\lambda \cong \operatorname{gr} A_{\lambda'} \cong \mathbb{C}[\mathfrak{M}]$, thus for any filtration $H(0) \subset H(1) \subset \ldots \subset H$ which is compatible with the filtrations on A_λ and $A_{\lambda'}$, the $\mathbb{C}[\mathfrak{M}] \otimes \mathbb{C}[\mathfrak{M}]$-module $\operatorname{gr} H$ may be interpreted as an $\mathbb{S}$-equivariant sheaf on $\mathfrak{M}_0 \times \mathfrak{M}_0 \cong \operatorname{Spec}\big(\mathbb{C}[\mathfrak{M}] \otimes \mathbb{C}[\mathfrak{M}]\big)$.

When n is greater than 1, we will be interested in a **thickened associated graded** $\mathrm{gr}_n H := R(H)/hR(H)$. This is a module over $R(A_\lambda \otimes A_{\lambda'}^{\mathrm{op}})/hR(A_\lambda \otimes A_{\lambda'}^{\mathrm{op}}) \cong \mathbb{C}[\mathfrak{M}_0] \otimes \mathbb{C}[\mathfrak{M}_0] \otimes \mathbb{C}[h^{1/n}]/(h)$, and thus over $\mathbb{C}[\mathfrak{M}_0] \otimes \mathbb{C}[\mathfrak{M}_0]$. The module $\mathrm{gr}_n H$ is an n-fold self-extension of $\mathrm{gr}\, H$, but this can be a non-split extension, so $\mathrm{gr}_n H$ contains more information.

Definition 6.2 We say that H is **Harish-Chandra** if it is finitely generated and it admits a filtration such that $\mathrm{gr}_n H$ is scheme-theoretically supported on the diagonal. Equivalently, we require that if $a_\lambda \in A_\lambda(k)$ and $a_{\lambda'} \in A_{\lambda'}(k)$ are specializations of the same element $a \in \mathscr{A}$, then for all $h \in H(m)$, we have $a_\lambda \cdot h - h \cdot a_{\lambda'} \in H(k + m - n)$.

Let ${}_{\lambda'}\mathbf{HC}^{\mathrm{a}}_\lambda$ be the category of Harish-Chandra bimodules, and let $D^b_{\mathbf{HC}}(A_{\lambda'}\text{-mod-}A_\lambda)$ be the full subcategory of $D^b(A_{\lambda'}\text{-mod-}A_\lambda)$ consisting of objects H whose cohomology $\mathbb{H}^i(H)$ is Harish-Chandra.

Proposition 6.3 *If $A_{\lambda'}$ has finite global dimension, $H_1 \in D^b_{\mathbf{HC}}(A_{\lambda'}\text{-mod-}A_\lambda)$, and $H_2 \in D^b_{\mathbf{HC}}(A_{\lambda''}\text{-mod-}A_{\lambda'})$, then $H_2 \overset{L}{\otimes} H_1 \in D^b_{\mathbf{HC}}(A_{\lambda''}\text{-mod-}A_\lambda)$.*

Proof: Consider the Rees modules $R(H_1), R(H_2)$ associated to some good filtration. These modules have locally free resolutions over $R(A_{\lambda'} \otimes A_\lambda^{\mathrm{op}})$ and $R(A_{\lambda''} \otimes A_{\lambda'}^{\mathrm{op}})$ such that, if f is congruent to f' modulo h, then $f' \otimes 1 - 1 \otimes f$ acts trivially on the cohomology of $R(H_i)$ modulo h. By a standard result of homological algebra, there exists a homotopy p_i on the resolution of $R(H_i)$ such that $f' \otimes 1 - 1 \otimes f + \partial p_i + p_i \partial$ acts trivially modulo h on the complex itself. Then $p_2 \otimes 1 + 1 \otimes p_1$ is a homotopy that plays the same role for $f'' \otimes 1 - 1 \otimes f$ acting on the tensor product of these complexes. This shows that $f'' \otimes 1 - 1 \otimes f$ acts by 0 modulo h on the cohomology of $R(H_2) \overset{L}{\otimes} R(H_1)$, so $H_2 \overset{L}{\otimes} H_1 \in D^b_{\mathbf{HC}}(A_{\lambda''}\text{-mod-}A_\lambda)$. □

We can view $A_{\lambda'} \otimes A_\lambda^{\mathrm{op}}$ as the ring of $\mathbb{S}$-invariant sections of a sheaf $\mathcal{D}_{\lambda'} \hat{\boxtimes} \mathcal{D}_\lambda^{\mathrm{op}}$ on $\mathfrak{M} \times \mathfrak{M}$; we must complete the naive tensor product in the h-adic topology in order to satisfy the hypotheses of a quantization. As a quantization, $\mathcal{D}_{\lambda'} \hat{\boxtimes} \mathcal{D}_\lambda^{\mathrm{op}}$ has period $(\lambda', -\lambda)$. By a $\mathcal{D}_{\lambda'} - \mathcal{D}_\lambda$ bimodule, we mean an $\mathbb{S}$-equivariant sheaf of $\mathcal{D}_{\lambda'} \hat{\boxtimes} \mathcal{D}_\lambda^{\mathrm{op}}$-modules on $\mathfrak{M} \times \mathfrak{M}$. We let $D^b(\mathcal{D}_{\lambda'}\text{-mod-}\mathcal{D}_\lambda)$ be the bounded derived category of good $\mathcal{D}_{\lambda'} \hat{\boxtimes} \mathcal{D}_\lambda^{\mathrm{op}}$-modules.

Definition 6.4 For any $\lambda, \lambda' \in H^2(\mathfrak{M}; \mathbb{C})$, let ${}_{\lambda'}\mathbf{HC}^{\mathrm{g}}_\lambda$ be the category of good $\mathcal{D}_{\lambda'} - \mathcal{D}_\lambda$ bimodules $\mathcal{H}$ with "thick classical limits" that are scheme-theoretically supported on the Steinberg $\mathfrak{Z} \subset \mathfrak{M} \times \mathfrak{M}$. More precisely, if $\mathscr{Q}$ is the canonical quantization of $\mathscr{M}$, we require $\mathcal{H}$ to admit a lattice $\mathcal{H}(0)$ such that for all sections $\tilde{f}$ of $\mathscr{Q}$, $\mathcal{H}(0)$ is invariant under $h^{-1}(\tilde{f}_{\lambda'} \otimes 1 - 1 \otimes \tilde{f}_\lambda)$, where $\tilde{f}_{\lambda'}$ and $\tilde{f}_\lambda$ are the specializations of $\tilde{f}$ at λ' and λ, respectively. As in the algebraic setting, we define $D^b_{\mathbf{HC}}(\mathcal{D}_{\lambda'}\text{-mod-}\mathcal{D}_\lambda)$ to be the full subcategory of $D^b(\mathcal{D}_{\lambda'}\text{-mod-}\mathcal{D}_\lambda)$ consisting of objects $\mathcal{H}$ whose cohomology $\mathbb{H}(\mathcal{H})$ lies in ${}_{\lambda'}\mathbf{HC}^{\mathrm{g}}_\lambda$.

Considering these bimodules as modules over the quantization $\mathcal{D}_{\lambda',-\lambda}$ of $\mathfrak{M}\times\mathfrak{M}$, we can apply the (derived) localization and sections functors as in previous sections.

Theorem 6.5 *For every λ, λ', we have $\mathbb{R}\Gamma_{\mathbb{S}}(D^b_{\mathbf{HC}}(\mathcal{D}_{\lambda'}\text{-mod-}\mathcal{D}_\lambda)) \subset D^b_{\mathbf{HC}}(A_{\lambda'}\text{-mod-}A_\lambda)$. If A_λ and $A_{\lambda'}$ have finite global dimension*[16], *then* $\mathbb{L}\mathrm{Loc}(D^b_{\mathbf{HC}}(A_{\lambda'}\text{-mod-}A_\lambda)) \subset D^b_{\mathbf{HC}}(\mathcal{D}_{\lambda'}\text{-mod-}\mathcal{D}_\lambda)$.

Proof: Let $\mathcal{H}$ be an object in ${}_{\lambda'}\mathbf{HC}^{\mathrm{g}}_\lambda$, and let $\mathcal{H}(0)\subset\mathcal{H}$ be a lattice satisfying the required condition. For every m, we have a long exact sequence showing that $H^p(\mathfrak{M};\mathcal{H}(0)/\mathcal{H}(-mn))$ is a extension of a submodule of $H^p(\mathfrak{M};\mathcal{H}(0)/\mathcal{H}(-(m-1)n))$ and quotient of

$$H^p(\mathfrak{M};\mathcal{H}(-(m-1)n)/\mathcal{H}(-mn)) \cong H^p(\mathfrak{M};\mathcal{H}(0)/\mathcal{H}(-n)).$$

Thus, $H^p(\mathfrak{M};\mathcal{H}(0)/\mathcal{H}(-mn))$ has an m step filtration compatible with $\mathcal{H}(0)\supset\mathcal{H}(-n)\supset\cdots$ such that elements of $A_{\lambda'}\otimes A_\lambda^{\mathrm{op}}$ of the form $\tilde{f}_{\lambda'}\otimes 1 - 1\otimes\tilde{f}_\lambda$ act trivially on the associated graded. Since $H^p(\mathfrak{M};\mathcal{H}(0)) = \varprojlim H^p(\mathfrak{M};\mathcal{H}(0)/\mathcal{H}(-mn))$ by Lemma 4.11, we have an induced filtration on this group such that $\tilde{f}_{\lambda'}\otimes 1 - 1\otimes\tilde{f}_\lambda$ acts trivially modulo h. This shows that the cohomology of $\mathbb{R}\Gamma^{\mathbb{S}}(\mathcal{H})$ is Harish-Chandra as well.

Now let H be an object of ${}_{\lambda'}\mathbf{HC}^{\mathrm{a}}_\lambda$ and put $\mathcal{H} := \mathbb{L}\mathrm{Loc}(H)$. A filtration of H induces a lattice in $\mathbb{H}^p(\mathcal{H})$. For any $\tilde{f}\in\Gamma(\mathfrak{M};\mathscr{D}(0))$, we have that

$$(\tilde{f}_{\lambda'}\otimes 1 - 1\otimes\tilde{f}_\lambda)\cdot R(H)\subset h\cdot R(H);$$

thus, on any projective resolution, the map induced by $(\tilde{f}_{\lambda'}\otimes 1 - 1\otimes\tilde{f}_\lambda)$ is null-homotopic mod h; this implies that our lattice in $\mathbb{H}^p(\mathcal{H})$ has the required property. □

Corollary 6.6 *If derived localization holds at λ' and $-\lambda$, then* $\mathbb{L}\mathrm{Loc}$ *and* $\mathbb{R}\Gamma_{\mathbb{S}}$ *are inverse equivalences between $D^b_{\mathbf{HC}}(A_{\lambda'}\text{-mod-}A_\lambda)$ and $D^b_{\mathbf{HC}}(\mathcal{D}_{\lambda'}\text{-mod-}\mathcal{D}_\lambda)$. If localization holds at λ' and $-\lambda$, then* Loc *and* $\Gamma_{\mathbb{S}}$ *are inverse equivalences between ${}_{\lambda'}\mathbf{HC}^{\mathrm{a}}_\lambda$ and ${}_{\lambda'}\mathbf{HC}^{\mathrm{g}}_\lambda$.*

Consider the convolution product defined by the formula

$$\mathcal{H}_1\star\mathcal{H}_2 := (p_{13})_*(p_{12}^{-1}\mathcal{H}_1 \overset{L}{\otimes}_{p_2^{-1}\mathcal{D}_{\lambda'}} p_{23}^{-1}\mathcal{H}_2), \tag{19}$$

where p_{ij} is one of the three projections from $\mathfrak{M}\times\mathfrak{M}\times\mathfrak{M}$ to $\mathfrak{M}\times\mathfrak{M}$.

Proposition 6.7 *If $\mathcal{M}\in D^b(\mathcal{D}_{\lambda''}\hat{\boxtimes}\mathcal{D}^{\mathrm{op}}_{\lambda'}\text{-mod})$ and $\mathcal{N}\in D^b(\mathcal{D}_{\lambda'}\hat{\boxtimes}\mathcal{D}^{\mathrm{op}}_{\lambda}\text{-mod})$, then we have $\mathcal{M}\star\mathcal{N}\in D^b(\mathcal{D}_{\lambda''}\hat{\boxtimes}\mathcal{D}^{\mathrm{op}}_{\lambda}\text{-mod})$. If furthermore $\mathcal{M}\in D^b_{\mathbf{HC}}(\mathcal{D}_{\lambda''}\text{-mod-}\mathcal{D}_{\lambda'})$, and $\mathcal{N}\in D^b_{\mathbf{HC}}(\mathcal{D}_{\lambda'}\text{-mod-}\mathcal{D}_\lambda)$, then $\mathcal{M}\star\mathcal{N}\in D^b_{\mathbf{HC}}(\mathcal{D}_{\lambda''}\text{-mod-}\mathcal{D}_\lambda)$.*

[16]The finite global dimension hypothesis is truly necessary. If A_λ does not have finite global dimension, the derived localization $\mathbb{L}\mathrm{Loc}(A_\lambda)$ as a bimodule may not be bounded and thus not in $D^b_{\mathbf{HC}}(\mathcal{D}_\lambda\text{-mod-}\mathcal{D}_\lambda)$.

Proof: The modules $\mathcal{M}(0)$ and $\mathcal{N}(0)$ have finite resolutions

$$\cdots \to M_1^{(1)} \hat{\boxtimes} M_1^{(2)} \to M_0^{(1)} \hat{\boxtimes} M_0^{(2)} \to \mathcal{M}(0) \qquad \text{and} \qquad \cdots \to N_1^{(1)} \hat{\boxtimes} N_1^{(2)} \to N_0^{(1)} \hat{\boxtimes} N_0^{(2)} \to \mathcal{N}(0)$$

with $M_j^{(1)}$ (resp. $M_j^{(2)}, N_j^{(1)}, N_j^{(2)}$) locally free over $\mathcal{Q}_{\lambda''}$ (resp. $\mathcal{Q}_{\lambda'}^{\mathrm{op}}, \mathcal{Q}_{\lambda'}, \mathcal{Q}_{\lambda}^{\mathrm{op}}$), since the same is true of coherent sheaves over $\mathfrak{S}_{\mathfrak{M}\times\mathfrak{M}}$. Thus, we can apply convolution to these modules by taking the naive tensor product over $p_2^{-1}\mathcal{D}_{\lambda'}$:

$$\mathcal{M} \star \mathcal{N}(0) := M_\bullet^{(1)} \,\hat{\boxtimes}\, \mathbb{H}^\bullet\big(\mathfrak{M}; M_\bullet^{(2)} \otimes_{\mathcal{D}_{\lambda'}} N_\bullet^{(1)}\big) \,\hat{\boxtimes}\, N_\bullet^{(2)},$$

where the middle term is considered as a complex of vector spaces, which is of finite length since $\mathfrak{M}$ is finite dimensional. This shows that $\mathcal{M} \star \mathcal{N}$ is a bounded length complex.

The argument that $\mathcal{M} \star \mathcal{N}$ lies in ${}_{\lambda''}\mathbf{HC}^{\mathrm{g}}{}_{\lambda}$ if $\mathcal{M}, \mathcal{N}$ are Harish-Chandra is exactly as in Proposition 6.3. The action of $f_\lambda \otimes 1 - 1 \otimes f_{\lambda'}$ on any resolution of $\mathcal{M}(0)$ is homotopic to 0 modulo h for a global function f, as is the action of $f_{\lambda'} \otimes 1 - 1 \otimes f_{\lambda''}$ on any resolution of $\mathcal{N}(0)$. Thus, tensoring these homotopies gives one for $f_\lambda \otimes 1 - 1 \otimes f_{\lambda''}$ on $\mathcal{M} \star \mathcal{N}(0)$. This function thus kills the cohomology of the classical limit $\mathcal{M} \star \mathcal{N}(0)/h \cdot \mathcal{M} \star \mathcal{N}(0)$. $\square$

Proposition 6.8 *Suppose that derived localization holds for $\lambda, \lambda', -\lambda'$, and $-\lambda''$. The derived sections functor $\mathbb{R}\Gamma_{\mathbb{S}}$ intertwines the convolution of bimodules with derived tensor product. That is, given Harish-Chandra bimodules $\mathcal{H}_1 \in D^b_{\mathbf{HC}}(\mathcal{D}_{\lambda'}\text{-mod-}\mathcal{D}_\lambda)$ and $\mathcal{H}_2 \in D^b_{\mathbf{HC}}(\mathcal{D}_{\lambda''}\text{-mod-}\mathcal{D}_{\lambda'})$, we have an isomorphism*

$$\mathbb{R}\Gamma_{\mathbb{S}}(\mathcal{H}_1 \star \mathcal{H}_2) \cong \mathbb{R}\Gamma_{\mathbb{S}}(\mathcal{H}_1) \overset{L}{\otimes} \mathbb{R}\Gamma_{\mathbb{S}}(\mathcal{H}_2).$$

In particular, if $\lambda = \lambda' = \lambda''$ and derived localization holds for $\pm\lambda$, then the derived localization and sections functors are inverse equivalences of tensor categories.

Proof: The complex of modules $\mathbb{R}\Gamma_{\mathbb{S}}(\mathcal{H}_1)$ has a free resolution over

$$A_{\lambda''} \otimes (A_{\lambda'})^{\mathrm{op}} = \Gamma_{\mathbb{S}}(\mathcal{D}_{\lambda''} \hat{\boxtimes} \mathcal{D}_{\lambda'}^{\mathrm{op}})$$

of the form

$$\cdots \to A_{\lambda''} \otimes U_1 \otimes A_{\lambda'} \to A_{\lambda''} \otimes U_0 \otimes A_{\lambda'} \to \cdots, \tag{20}$$

and similarly $\mathbb{R}\Gamma_{\mathbb{S}}(\mathcal{H}_2)$ has a free resolution over $A_{\lambda'} \otimes A_\lambda^{\mathrm{op}} = \Gamma_{\mathbb{S}}(\mathcal{D}_{\lambda'} \hat{\boxtimes} \mathcal{D}_\lambda^{\mathrm{op}})$

$$\cdots \to A_{\lambda'} \otimes V_1 \otimes A_\lambda \to A_{\lambda'} \otimes V_0 \otimes A_\lambda \to \cdots. \tag{21}$$

Since derived localization holds, the sheaves $\mathcal{H}_1$ and $\mathcal{H}_2$ have resolutions

$$\cdots \to \mathcal{D}_{\lambda''} \otimes U_1 \otimes \mathcal{D}_{\lambda'} \to \mathcal{D}_{\lambda''} \otimes U_0 \otimes \mathcal{D}_{\lambda'} \to \cdots$$

$$\cdots \to \mathcal{D}_{\lambda'} \otimes V_1 \otimes \mathcal{D}_{\lambda} \to \mathcal{D}_{\lambda'} \otimes V_0 \otimes \mathcal{D}_{\lambda} \to \cdots .$$

Thus, the convolution $\mathcal{H}_1 \star \mathcal{H}_2$ is given by the complex

$$\cdots \to \bigoplus_{i+j=k+1} \mathcal{D}_{\lambda''} \otimes U_i \otimes A_{\lambda'} \otimes V_j \otimes \mathcal{D}_{\lambda} \to \bigoplus_{i+j=k} \mathcal{D}_{\lambda''} \otimes U_i \otimes A_{\lambda'} \otimes V_j \otimes \mathcal{D}_{\lambda} \to \cdots . \tag{22}$$

The sections of (22) is the complex

$$\cdots \to \bigoplus_{i+j=k+1} A_{\lambda''} \otimes U_i \otimes A_{\lambda'} \otimes V_j \otimes A_{\lambda} \to \bigoplus_{i+j=k} A_{\lambda''} \otimes U_i \otimes A_{\lambda'} \otimes V_j \otimes A_{\lambda} \to \cdots . \tag{23}$$

This is also the tensor product of the complexes (20) and (21), so this shows that the convolutions and tensor products agree. $\square$

Following Căldăraru and Willerton [CW10], we define a 2-category $\mathsf{Qua}^{\mathrm{g}}$ where

- objects are elements of $H^2(\mathfrak{M}; \mathbb{C})$,
- 1-morphisms from λ to λ' are objects of $D^b_{\mathbf{HC}}(\mathcal{D}_{\lambda'}\text{-mod-}\mathcal{D}_{\lambda})$ with composition given by $\star$, and
- 2-morphisms are the usual morphisms in $D^b_{\mathbf{HC}}(\mathcal{D}_{\lambda'}\text{-mod-}\mathcal{D}_{\lambda})$.

Similarly, we can define a 2-category $\mathsf{Qua}^{\mathrm{a}}$ whose objects are those λ for which A_λ has finite global dimension (we should consider only these because of Proposition 6.3) and whose 1-morphisms are objects of $D^b_{\mathbf{HC}}(A_{\lambda'}\text{-mod-}A_{\lambda})$, with composition given by derived tensor product.

Let Cat denote the 2-category of all categories, and consider the functors

$$F^g \colon \mathsf{Qua}^{\mathrm{g}} \to \mathsf{Cat} \qquad \text{and} \qquad F^a \colon \mathsf{Qua}^{\mathrm{a}} \to \mathsf{Cat}$$

taking λ to $D^b(\mathcal{D}_\lambda\text{-mod})$ and $D^b(A_\lambda\text{-mod})$, respectively. On 1-morphisms, F^g takes an object $\mathcal{H}$ to the functor given by convolution with $\mathcal{H}$, defined exactly as in Equation (19). Similarly, F^a takes an object H to the functor given by tensor product with H.

Let $\mathfrak{L}_0 \subset \mathfrak{M}_0$ be an $\mathbb{S}$-equivariant closed subscheme, and let $\mathfrak{L} \subset \mathfrak{M}$ be its scheme-theoretic preimage. We would like to use $\mathfrak{L}_0$ and $\mathfrak{L}$ to define subcategories of A-mod and $\mathcal{D}$-mod in a way that is analogous to the definitions of algebraic and geometric Harish-Chandra bimodules

(Definitions 6.2 and 6.4). In fact, those definitions will specialize to these when $\mathfrak{M}$ is replaced by $\mathfrak{M} \times \mathfrak{M}$ and $\mathfrak{L}_0$ is the diagonal subscheme of $\mathfrak{M}_0 \times \mathfrak{M}_0$.

Definition 6.9 Let $C_\lambda^{\mathfrak{L}_0}$ be the full subcategory of A_λ-mod consisting of modules N admitting a filtration with thickened associated graded $\mathrm{gr}_n N$ scheme-theoretically supported on $\mathfrak{L}_0$. Equivalently, we require that if the symbol of $a_\lambda \in A_\lambda(k)$ vanishes on $\mathfrak{L}_0$, then $a_\lambda \cdot N(m) \subset N(k+m-n)$. Let $D^b_{\mathfrak{L}_0}(A_\lambda\text{-mod})$ be the full subcategory of $D^b(A_\lambda\text{-mod})$ consisting of objects with cohomology in $C_\lambda^{\mathfrak{L}_0}$.

Definition 6.10 Let $\mathcal{C}_\lambda^{\mathfrak{L}}$ be the full subcategory of $\mathcal{D}_\lambda$-mod consisting of modules $\mathcal{N}$ that have thick classical limits that are scheme-theoretically supported on $\mathfrak{L}$. More precisely, we require a lattice $\mathcal{N}(0)$ such that for any section $\tilde{f}$ of $\mathcal{Q}$ whose reduction modulo h lies in the ideal sheaf of $\mathfrak{L}$, $\mathcal{N}(0)$ is preserved by the action of $h^{-1}\tilde{f}$. Let $D^b_{\mathfrak{L}}(\mathcal{D}_\lambda\text{-mod})$ be the full subcategory of $D^b(\mathcal{D}_\lambda\text{-mod})$ consisting of objects with cohomology in $\mathcal{C}_\lambda^{\mathfrak{L}}$.

Proposition 6.7, along with an easy extension of the proof of Proposition 6.3, show that we have functors

$$F^g_{\mathfrak{L}}\colon \mathsf{Qua}^{\mathrm{g}} \to \mathsf{Cat} \qquad \text{and} \qquad F^a_{\mathfrak{L}}\colon \mathsf{Qua}^{\mathrm{a}} \to \mathsf{Cat}$$

taking λ to $D^b_{\mathfrak{L}}(\mathcal{D}_\lambda\text{-mod})$ and $D^b_{\mathfrak{L}_0}(A_\lambda\text{-mod})$, respectively.

Example 6.11 Suppose that $\mathfrak{L}_0 \subset \mathfrak{M}$ is the unique $\mathbb{S}$-fixed point; then $\mathfrak{L} = \nu^{-1}(0)$ is the core of $\mathfrak{M}$ (Remark 2.6), possibly with a non-reduced scheme structure. If the weight n of the symplectic form is equal to 1, then $\mathfrak{L}$ is Lagrangian, and $C_\lambda^{\mathfrak{L}_0}$ is the category of finite-dimensional A_λ-modules. When n is greater than 1, the core may be too small, in which case $C_\lambda^{\mathfrak{L}_0}$ will be zero. For example, if $\mathfrak{M}$ is the Hilbert scheme of points on $\mathbb{C}^2$ and $\mathbb{S}$ acts by scaling $\mathbb{C}^2$ (with $n = 2$), then the core is the punctual Hilbert scheme, which has dimension one less than half the dimension of $\mathfrak{M}$.

Example 6.12 Suppose that $\mathfrak{M}$ is equipped with a Hamiltonian action of $\mathbb{T} := \mathbb{C}^\times$ that commutes with the action of $\mathbb{S}$ and has finite fixed point set $\mathfrak{M}^{\mathbb{T}}$, and consider the Lagrangian subvariety

$$\mathfrak{L}_0 := \left\{ p \in \mathfrak{M}_0 \;\middle|\; \lim_{t \to 0} t \cdot p \text{ exists} \right\}.$$

In this case, $C_\lambda^{\mathfrak{L}_0}$ is the category of finitely generated A_λ-modules that are locally finite for the action of A_λ^+, where A_λ^+ is the subring of A_λ consisting of elements with non-negative $\mathbb{T}$-weight. This is an analogue of a block of BGG category $\mathcal{O}$, and will be the primary object of study in our forthcoming paper [BLPW16] with Licata.

To explain the connection with BGG category $\mathcal{O}$, take $\mathfrak{M} = T^*(G/B)$ and let $\rho \in H^2(\mathfrak{M}; \mathbb{C})$ be half of the Euler class of the canonical bundle. Then the ring $A_{\lambda+\rho}$ is a central quotient of $U(\mathfrak{g})$, and $C_{\lambda+\rho}^{\mathfrak{L}}$ is the category of finitely generated, $U(\mathfrak{b})$-locally finite $U(\mathfrak{g})$-modules with same central character as the Verma module V_λ with highest weight λ, where $H^2(\mathfrak{M};\mathbb{C})$ is identified with the dual Cartan $\mathfrak{h}^*$ via the Chern class map. When λ is a regular integral weight, this category is equivalent in a non-obvious way to the block $\mathcal{O}_\lambda$ of BGG category $\mathcal{O}$ by [Soe86, Th. 1].

6.2 Characteristic cycles

Let $\mathcal{D}$ be a quantization of $\mathfrak{M}$, and let $\mathcal{N} \in D^b(\mathcal{D}\text{-mod})$ be an object of the bounded derived category. We have isomorphisms

$$\mathcal{H}om^\bullet_{\mathcal{D}}(\mathcal{N},\mathcal{N}) \cong \mathcal{H}om^\bullet_{\mathcal{D}}(\mathcal{N},\mathcal{D}) \overset{L}{\otimes}_{\mathcal{D}} \mathcal{N} \cong \mathcal{D}_\Delta \overset{L}{\otimes}_{\mathcal{D}\hat{\boxtimes}\mathcal{D}^{\mathrm{op}}} \big(\mathcal{N}\hat{\boxtimes}\mathcal{H}om^\bullet_{\mathcal{D}}(\mathcal{N},\mathcal{D})\big),$$

and evaluation defines a canonical map to the Hochschild homology

$$\mathcal{HH}(\mathcal{D}) := \mathcal{D}_\Delta \overset{L}{\otimes}_{\mathcal{D}\hat{\boxtimes}\mathcal{D}^{\mathrm{op}}} \mathcal{D}_\Delta.$$

All this is completely general, and holds in both the Zariski and the classical topology. In the classical topology, we also have an isomorphism $\mathcal{HH}(\mathcal{D}^{\mathrm{an}}) \cong \mathbb{C}_{\mathfrak{M}_\Delta}[\dim \mathfrak{M}]((h))$ by [KS12, 6.3.1]. (This is a local calculation, so it suffices to check for the Weyl algebra, where it follows from a Koszul resolution.)

We define the **characteristic cycle**

$$\mathrm{CC}(\mathcal{N}) \in H^0(\mathcal{HH}(\mathcal{D}^{\mathrm{an}})) \cong H^{\dim \mathfrak{M}}\big(\mathfrak{M};\mathbb{C}((h))\big)$$

to be the image of id $\in H^0(\mathcal{H}om^\bullet_{\mathcal{D}}(\mathcal{N}^{\mathrm{an}},\mathcal{N}^{\mathrm{an}}))$ along this map. More generally, if $\mathcal{N}$ is supported on a subvariety $j\colon \mathfrak{L} \hookrightarrow \mathfrak{M}$, then we may consider the identity map of $\mathcal{N}^{\mathrm{an}}$ to be a section of $j^!\mathcal{H}om^\bullet_{\mathcal{D}}(\mathcal{N}^{\mathrm{an}},\mathcal{N}^{\mathrm{an}})$. Applying our map then gives us a class in

$$\mathrm{CC}(\mathcal{N}) \in H^0(j^!\mathcal{HH}(\mathcal{D}^{\mathrm{an}})) \cong H^{\dim \mathfrak{M}}_{\mathfrak{L}}\big(\mathfrak{M};\mathbb{C}((h))\big).$$

Our abuse of notation is justified by the fact that this class is functorial for inclusions of subvarieties. If we replace the conical symplectic resolution $\mathfrak{M}$ with the product $\mathfrak{M}\times\mathfrak{M}$, then this construction associates to a Harish-Chandra bimodule $\mathcal{H} \in D^b_{\mathbf{HC}}(\mathcal{D}_{\lambda'}\text{-mod-}\mathcal{D}_\lambda)$ a class $\mathrm{CC}(\mathcal{H}) \in H^{2\dim \mathfrak{M}}_{\mathfrak{Z}}\big(\mathfrak{M}\times\mathfrak{M};\mathbb{C}((h))\big)$.

Kashiwara and Schapira [KS12, 7.3.5] show that the characteristic cycle of a holonomic $\mathcal{D}$-module (that is, one with Lagrangian support) may be computed in terms of its classical limit.

Proposition 6.13 (Kashiwara and Schapira) *If $\mathcal{N} \in \mathcal{D}\text{-mod}$ is supported on a Lagrangian* *subvariety $\mathfrak{L}$ with components $\{\mathfrak{L}_i\}$, then for any $\mathcal{D}(0)$-lattice $\mathcal{N}(0) \subset \mathcal{N}$,*

$$\mathrm{CC}(\mathcal{N}) = \sum_i \mathrm{rk}_{\mathfrak{L}_i}\big(\mathcal{N}(0)/\mathcal{N}(-1)\big) \cdot [\mathfrak{L}_i] \in H^{\dim \mathfrak{M}}_{\mathfrak{L}}(\mathfrak{M};\mathbb{C}) \subset H^{\dim \mathfrak{M}}_{\mathfrak{L}}\big(\mathfrak{M};\mathbb{C}((h))\big),$$

where $\mathrm{rk}_{\mathfrak{L}_i}$ denotes the rank at the generic point of $\mathfrak{L}_i$.

We can also take characteristic cycles in families for modules over quantizations of twistor deformations. For $\eta \in H^2(\mathfrak{M};\mathbb{C})$, let $\mathscr{M}_\eta \to \mathbb{A}^1$ be the twistor deformation defined in Section 2.1 with quantization $\mathscr{D}$ extending $\mathcal{D}$. Let $\mathscr{N}$ be a good $\mathscr{D}$-module, and consider the image of the identity via the natural morphisms

$$\begin{aligned}\mathcal{H}om^\bullet_{\mathscr{D}}(\mathscr{N},\mathscr{N}) \cong \mathcal{H}om^\bullet_{\mathscr{D}}(\mathscr{N},\mathscr{D}) \overset{L}{\otimes}_{\mathscr{D}} \mathscr{N} \cong \mathscr{D}_\Delta \overset{L}{\otimes}_{\mathscr{D}\hat{\boxtimes}_{\mathbb{A}^1}\mathscr{D}^{\mathrm{op}}} \big(\mathscr{N}\hat{\boxtimes}_{\mathbb{A}^1}\mathcal{H}om^\bullet_{\mathscr{D}}(\mathscr{N},\mathscr{D})\big) \\ \to \mathscr{D}^{\mathrm{an}}_\Delta \otimes_{\mathscr{D}^{\mathrm{an}}\hat{\boxtimes}_{\mathbb{A}^1}\mathscr{D}^{\mathrm{an,op}}} \mathscr{D}^{\mathrm{an}}_\Delta \cong \pi^{-1}\mathfrak{S}_{\mathbb{A}^1}[\dim \mathfrak{M}]((h)). \quad (24)\end{aligned}$$

This defines a class in relative cohomology $\mathrm{CC}(\mathscr{N}) \in H^{\dim \mathfrak{M}}_{\mathscr{L}}(\mathscr{M}_\eta/\mathbb{A}^1;\mathbb{C}((h)))$ for any Lagrangian $\mathscr{L} \supset \mathrm{Supp}(\mathscr{N})$. If we let $\mathfrak{L} = \mathfrak{M} \cap \mathscr{L}$, then we have a natural restriction map

$$H^{\dim \mathfrak{M}}_{\mathscr{L}}(\mathscr{M}_\eta/\mathbb{A}^1;\mathbb{C}((h))) \to H^{\dim \mathfrak{M}}_{\mathfrak{L}}\big(\mathfrak{M};\mathbb{C}((h))\big)$$

given by dividing by the coordinate t on $\mathbb{A}^1$. We also have a natural functor of restriction from $\mathscr{D}$-mod $\to \mathcal{D}$-mod given by $\mathscr{N}|_{\mathfrak{M}} = \mathscr{N} \overset{L}{\otimes}_{\mathbb{C}[t]} \mathbb{C}$. The following lemma says that these operations are compatible.

Lemma 6.14 *If $\mathscr{N}$ is a good $\mathscr{D}$-module, then* $\mathrm{CC}(\mathscr{N}|_{\mathfrak{M}}) = \mathrm{CC}(\mathscr{N})|_{\mathfrak{M}}$.

Proof: Consider the complex (24) of $\pi^{-1}\mathfrak{S}_{\mathbb{A}^1}$ modules, and take the derived tensor product with $\mathbb{C}$ over $\mathbb{C}[t]$. We claim that we obtain corresponding sequence for $\mathscr{N}|_{\mathfrak{M}}$. That is, we obtain

$$\begin{aligned}\mathcal{H}om^\bullet_{\mathscr{D}}(\mathscr{N}|_{\mathfrak{M}},\mathscr{N}|_{\mathfrak{M}}) \cong \mathcal{H}om^\bullet_{\mathscr{D}}(\mathscr{N}|_{\mathfrak{M}},\mathcal{D}) \overset{L}{\otimes}_{\mathcal{D}} \mathscr{N}|_{\mathfrak{M}} \cong \mathcal{D}_\Delta \overset{L}{\otimes}_{\mathcal{D}\hat{\boxtimes}\mathcal{D}^{\mathrm{op}}} \big(\mathscr{N}|_{\mathfrak{M}}\hat{\boxtimes}\mathcal{H}om^\bullet_{\mathcal{D}}(\mathscr{N}|_{\mathfrak{M}},\mathcal{D})\big) \\ \to \mathcal{D}^{\mathrm{an}}_\Delta \otimes_{\mathcal{D}^{\mathrm{an}}\hat{\boxtimes}\mathcal{D}^{\mathrm{an,op}}} \mathcal{D}^{\mathrm{an}}_\Delta \cong \mathbb{C}_{\mathfrak{M}}[\dim \mathfrak{M}]((h)). \quad (25)\end{aligned}$$

It suffices to prove this for $\mathscr{N}$ locally free. In this case, $\mathcal{H}om^\bullet(\mathscr{N},\mathscr{D})$ is concentrated in degree 0 and is itself locally free, so the statement is clear.

Thus $\mathrm{CC}(\mathscr{N})|_{\mathfrak{M}}$ can be obtained as the image of the identity under the map (25). By definition $\mathrm{CC}(\mathscr{N}|_{\mathfrak{M}})$ is the image of the identity under (25), so we are done. □

Consider the category $K(\mathfrak{Z})$ with objects $H^2(\mathfrak{M};\mathbb{C})$ and morphisms $H^{2\dim \mathfrak{M}}_{\mathfrak{Z}}\big(\mathfrak{M}\times\mathfrak{M};\mathbb{C}\big)$ between any two objects, with composition given by the convolution structure defined at the beginning of this section. We also have a category $K(\mathbf{HC}^{\mathrm{g}})$ with objects $H^2(\mathfrak{M};\mathbb{C})$ and morphisms $K({}_{\lambda'}\mathbf{HC}^{\mathrm{g}}_{\lambda})$ from λ to λ', with composition given by convolution; this is simply the decategorification of the 2-category defined in the previous section.

Proposition 6.15 *The characteristic cycle map defines a functor $K(\mathbf{HC}^{\mathrm{g}}) \to K(\mathfrak{Z})$.*

Proof: The fact that the characteristic cycle of a morphism in $K(\mathbf{HC}^{\mathrm{g}})$ is an element of $H^{2\dim\mathfrak{M}}_{\mathfrak{Z}}(\mathfrak{M}\times\mathfrak{M};\mathbb{C})$ rather than $H^{2\dim\mathfrak{M}}_{\mathfrak{Z}}\big(\mathfrak{M}\times\mathfrak{M};\mathbb{C}((h))\big)$ follows from Proposition 6.13. Since the map $\mathfrak{Z}\times_{\mathfrak{M}}\mathfrak{Z}\to\mathfrak{Z}$ is proper, the rest of the proposition follows from [KS12, 6.5.4] and the fact that the functor $(-)^{\mathrm{an}}$ is monoidal and preserves Hom-spaces. $\square$

Now fix a subvariety $\mathfrak{L}_0 \subset \mathfrak{M}_0$, and let $\mathfrak{L}\subset\mathfrak{M}$ be its scheme-theoretic preimage as in Section 6.1. We assume for convenience that $\mathfrak{L}$ is Lagrangian. Consider the functor

$$G_{\mathfrak{L}} \colon K(\mathbf{HC}^{\mathrm{g}}) \to \mathsf{Ab}$$

taking:

- The class λ to $K(\mathcal{C}^{\mathfrak{L}}_{\lambda})$, the Grothendieck group of objects in $\mathcal{C}^{\mathfrak{L}}_{\lambda}$ with finitely generated cohomology concentrated in finitely many degrees. Note that by its definition, $\mathcal{C}^{\mathfrak{L}}_{\lambda}$ may not be a Serre subcategory, in which case we consider the subgroup of the Grothendieck group of all holonomic $\mathcal{D}$-modules generated by the objects in $\mathcal{C}^{\mathfrak{L}}_{\lambda}$.
- The class $[\mathcal{H}]\in K({}_{\lambda'}\mathbf{HC}^{\mathrm{g}}_{\lambda})$ to the convolution operator

  $$[\mathcal{H}]\star - : K(\mathcal{C}^{\mathfrak{L}}_{\lambda}) \to K(\mathcal{C}^{\mathfrak{L}}_{\lambda'})$$

  defined by the formula

  $$[\mathcal{N}]\star\alpha := (p_{13})_*(p_{12}^*[\mathcal{N}]\cdot p_{23}^*\alpha).$$

We also have a functor

$$H_{\mathfrak{L}} \colon K(\mathbf{HC}^{\mathrm{g}}) \to \mathsf{Ab}$$

taking every object λ to $H^{\dim\mathfrak{M}}_{\mathfrak{L}}(\mathfrak{M};\mathbb{C})$, where the map on morphisms is defined by the convolution action of $H^{2\dim\mathfrak{M}}_{\mathfrak{Z}}(\mathfrak{M}\times\mathfrak{M};\mathbb{C})$ on $H^{\dim\mathfrak{M}}_{\mathfrak{L}}(\mathfrak{M};\mathbb{C})$.

Proposition 6.16 *The characteristic cycle map*

$$\mathrm{CC}\colon K(\mathcal{C}^{\mathfrak{L}}_{\lambda}) \to H^{\dim\mathfrak{M}}_{\mathfrak{L}}(\mathfrak{M};\mathbb{C})$$

defines a natural transformation from $G_{\mathfrak{L}}$ to $H_{\mathfrak{L}}$. That is, for all $\mathcal{H}\in D^b_{\mathbf{HC}}(\mathcal{D}_{\lambda'}\text{-mod-}\mathcal{D}_{\lambda})$ and $\mathcal{N}\in D^b(\mathcal{C}^{\mathfrak{L}}_{\lambda})$,

$$\mathrm{CC}(\mathcal{H})\star\mathrm{CC}(\mathcal{N}) = \mathrm{CC}(\mathcal{H}\star\mathcal{N}).$$

Proof: Since the map $\mathfrak{Z} \times_{\mathfrak{M}} \mathfrak{Z} \to \mathfrak{L}$ is proper, this follows immediately from [KS12, 6.5.4]. $\square$

Thus, these bimodules provide a natural categorification of the convolution algebra of a symplectic singularity, and at least certain of its natural convolution modules. Of course, the characteristic cycle maps need not be isomorphisms, but in many contexts, they are.

Example 6.17 In the case where $\mathfrak{M} = T^*(G/B)$, the category ${}_\lambda\mathbf{HC}^{\mathrm{g}}_\lambda$ is equivalent to the category of regular twisted D-modules on $G/B \times G/B$ for the twist $(\lambda + \rho, -\lambda + \rho)$ which are smooth on diagonal G-orbits; as long as $\lambda + \rho$ is integral, this is the same as the category of perverse sheaves smooth along the same stratification. The fact that these categorify the symmetric group (and thus, implicitly, that CC is an isomorphism in this case) goes back at least as far as [Spr82]. This perspective is Koszul dual to the usual categorification of the Weyl group by projective functors [BG99, 5.16].

Example 6.18 In the case where $\mathfrak{M}$ is a hypertoric variety, the map from $K({}_\lambda\mathbf{HC}^{\mathrm{g}}_\lambda)$ to $H^{2\dim\mathfrak{M}}_{\mathfrak{Z}}(\mathfrak{M} \times \mathfrak{M}; \mathbb{C})$ is surjective by [BLPW12, 7.11], which allows us to conclude that every irreducible representation of the convolution algebra remains irreducible over $K(\mathbf{HC}^{\mathrm{g}})$. The dimensions of these representations are computed in [PW07] to be h-numbers of various matroids.

Example 6.19 In the case of Nakajima quiver varieties, it is more natural to consider all quiver varieties associated to a highest weight μ jointly, and thus define a 2-subcategory $\mathsf{Qua}(\mu)$ of modules over the exterior products of quantizations of quiver varieties associated to λ and possibly different dimension vectors.

However, even with different dimension vectors, we still have a notion of "diagonal" in the product of two quiver varieties with the same highest weight. The affinization of a quiver variety is the moduli space of semi-simple representations of the pre-projective algebra of a given dimension, and we say a pair of such representations lies in the **stable diagonal** if they become isomorphic after the addition of trivial representations. We can define a 2-categories $\mathbf{HC}^{\mathrm{g}}(\mu)$ by replacing the diagonal and its vanishing ideal with that of the stable diagonal. The third author [Web, Theorem A] relates this construction to works by Cautis and Lauda [CL] and Nakajima [Nak98].

Proposition 6.20 (Webster) *There is a 2-functor from the version of the 2-quantum group $\mathcal{U}$ defined by Cautis and Lauda to $\mathbf{HC}^{\mathrm{g}}(\mu)$ with the property that the induced map of K-groups is exactly the geometric construction of $U(\mathfrak{g})$ defined by Nakajima.*

6.3 Twisting bimodules

For the rest of this paper, we will assume that the Picard group of $\mathfrak{M}$ is torsion-free, so that a line bundle is determined by its Euler class in $H^2(\mathfrak{M};\mathbb{C})$. This assumption is not strictly necessary, but it greatly simplifies the notation (see Remark 6.22).

Consider the universal Poisson deformation $\mathscr{M}$ of $\mathfrak{M}$. Let $\mathscr{L}$ be a line bundle on $\mathscr{M}$, let $\mathcal{L}$ be its restriction to $\mathfrak{M}$, and let $\gamma \in H^2(\mathfrak{M};\mathbb{Z}) \cong H^2(\mathscr{M};\mathbb{Z})$ be the Euler class of $\mathscr{L}$ or $\mathcal{L}$. Let ${}_\gamma\mathscr{T}_0$ be the quantization of $\mathscr{L}$ constructed in Proposition 5.2, and let ${}_\gamma\mathscr{T}_0' := {}_\gamma\mathscr{T}_0[h^{-1/n}]$. This is a right $\mathscr{D}$-module and a left module over $\mathscr{D}_\gamma$, the quantization with period $I + h\gamma$.[17] Then $\Gamma_{\mathbb{S}}(\mathscr{M};{}_\gamma\mathscr{T}_0')$ is a family over $H^2(\mathfrak{M};\mathbb{C})$ via the right action of $\mathscr{A} = \Gamma_{\mathbb{S}}(\mathscr{M};\mathscr{D})$.

Recall the map $c\colon \mathbb{C}[H^2(\mathfrak{M};\mathbb{C})] \to \Gamma(\mathscr{M};\mathscr{D})$ from Section 3.3, and the fact that $h^{-1}c(x) \in \mathscr{A}$ for all $x \in H^2(\mathfrak{M};\mathbb{C})^*$. Also recall that, by Proposition 3.6, the specialization of $\mathscr{A}$ at $h^{-1}c(x) = \lambda(x)$ for all $x \in H^2(\mathfrak{M};\mathbb{C})^*$ is isomorphic to A_λ.

Definition 6.21 Let ${}_{\lambda+\gamma}T_\lambda$ denote the $A_{\lambda+\gamma} - A_\lambda$ bimodule that we obtain by specializing $\Gamma_{\mathbb{S}}(\mathscr{M};{}_\gamma\mathscr{T}_0')$ at $h^{-1}c(x) = \lambda(x)$ for all $x \in H^2(\mathfrak{M};\mathbb{C})^*$.

Remark 6.22 The purpose of the assumption at the beginning of this section was to ensure that the bimodule ${}_{\lambda+\gamma}T_\lambda$ is actually determined by λ and γ; without the assumption, the bimodule would depend on an additional choice of a line bundle with Euler class γ.

Proposition 6.23 *The bimodule ${}_{\lambda+\gamma}T_\lambda$ is Harish-Chandra.*

Proof: By definition, ${}_{\lambda+\gamma}T_\lambda$ is a specialization of $\Gamma_{\mathbb{S}}(\mathscr{M};{}_\gamma\mathscr{T}_0')$. It carries a natural filtration, where ${}_{\lambda+\gamma}T_\lambda(m)$ is the same specialization of $\Gamma_{\mathbb{S}}(\mathscr{M};h^{-m/n}{}_\gamma\mathscr{T}_0[h^{1/n}])$. We claim that the associated graded module with respect to this filtration is scheme-theoretically supported on the diagonal.

To see this, consider a function $f \in \mathbb{C}[\mathfrak{M}]$ of $\mathbb{S}$-weight ℓ. We can choose a lift $\tilde{f} \in \Gamma_{\mathbb{S}}(\mathscr{D}(\ell))$ so that its image in $\operatorname{gr}\Gamma_{\mathbb{S}}(\mathscr{D}) \cong \mathbb{C}[\mathscr{M}]$ restricts to f on $\mathfrak{M}$. Let $\mathscr{D}_\gamma$ be the quantization of $\mathscr{M}$ with period γ; since $\operatorname{gr}\Gamma_{\mathbb{S}}(\mathscr{D}_\gamma) \cong \operatorname{gr}\Gamma_{\mathbb{S}}(\mathscr{D})$, we can choose a lift $\tilde{f}_\gamma \in \Gamma_{\mathbb{S}}(\mathscr{D}_\gamma(\ell))$ of f similarly. To show that $f \otimes 1 - 1 \otimes f$ annihilates $\operatorname{gr}_n({}_{\lambda+\gamma}T_\lambda)$, it is sufficient to show that $\tilde{f}_\gamma \otimes 1 - 1 \otimes \tilde{f}$ takes $\Gamma_{\mathbb{S}}(\mathscr{M};h^{-m/n}{}_\gamma\mathscr{T}_0[h^{1/n}])$ to $\Gamma_{\mathbb{S}}(\mathscr{M};h^{1-\ell+m/n}{}_\gamma\mathscr{T}_0[h^{1/n}])$. This follows from the fact that ${}_\gamma\mathscr{T}_0$ is the quantization of a line bundle on $\mathscr{M}$, so the left action of $\tilde{f}_\gamma$ and the right action of $\tilde{f}$ agree modulo h. $\square$

The following two propositions are bimodule analogues of Corollary 3.9 and Proposition 3.10. Since their proofs are essentially identical, we omit them.

[17] We note that all quantizations of $\mathscr{M}$ are isomorphic as sheaves of algebras, but they are not isomorphic as sheaves of $\pi^{-1}\mathfrak{S}_{H^2(\mathfrak{M};\mathbb{C})}$-algebras.

Proposition 6.24 *Let $\mathfrak{M}$ and $\mathfrak{M}'$ be two conical symplectic resolutions of the same cone. Fix elements $\lambda, \gamma \in H^2(\mathfrak{M}; \mathbb{C}) \cong H^2(\mathfrak{M}'; \mathbb{C})$, where γ is the Euler class of a line bundle on $\mathfrak{M}$ or its strict transform on $\mathfrak{M}'$. The isomorphism of rings in Corollary 3.9 induces an isomorphism of bimodules ${}_{\lambda+\gamma}T_\lambda \cong {}_{\lambda+\gamma}T'_\lambda$.*

Proposition 6.25 *For any $\lambda, \gamma \in H^2(\mathfrak{M}; \mathbb{C}) \cong H^2(\mathfrak{M}'; \mathbb{C})$, where γ is the Euler class of a line bundle on $\mathfrak{M}$, and any $w \in W$, the isomorphisms of Proposition 3.10 induce isomorphisms of bimodules ${}_{\lambda+\gamma}T_\lambda \cong {}_{w\cdot(\lambda+\gamma)}T_{w\cdot\lambda}$.*

We would like to have an analogue of Proposition 3.6, as well, though an extra hypothesis is needed. The following proposition gives a natural map from ${}_{\lambda+\gamma}T_\lambda$ to $\Gamma_{\mathbb{S}}(\mathfrak{M}; {}_{\lambda+\gamma}\mathcal{T}'_\lambda)$, and gives a sufficient (though not necessary) condition for it to be an isomorphism. (Note that it is always injective.)

Proposition 6.26 *There is a natural map from the bimodule ${}_{\lambda+\gamma}T_\lambda$ to $\Gamma_{\mathbb{S}}(\mathfrak{M}; {}_{\lambda+\gamma}\mathcal{T}'_\lambda)$. If $H^1(\mathfrak{M}; {}_{\lambda+\gamma}\mathcal{T}'_\lambda) = 0$, then this map is an isomorphism.*

Proof: The pullback of ${}_\gamma\mathscr{T}'_0$ along the map $\Delta \to H^2(\mathfrak{M}; \mathbb{C}) \times \Delta$ given by $h \mapsto (h\lambda, h)$ is a quantization of $\mathcal{L}$. By the uniqueness of the quantized line bundles constructed in Proposition 5.2, this pullback is isomorphic to ${}_{\lambda+\gamma}\mathcal{T}'_\lambda$. Since ${}_{\lambda+\gamma}T_\lambda$ is obtained from ${}_\gamma\mathscr{T}'_0$ by first taking sections and then specializing, this defines the required map.

Now suppose that $H^1(\mathfrak{M}; {}_{\lambda+\gamma}\mathcal{T}'_\lambda) = 0$. To prove that our map is surjective, we factor the pullback into two steps. Choose $\nu \in H^2(\mathfrak{M}; \mathbb{C})$ with $\mathscr{M}_\nu(\infty)$ affine. Let ${}_{\lambda+\gamma}\mathscr{T}'^{(\nu)}_\lambda$ be the bimodule on $\mathscr{M}_\nu$ obtained by pulling ${}_\gamma\mathscr{T}'_0$ back along the map $\mathbb{A}^1 \times \Delta \to H^2(\mathfrak{M}; \mathbb{C}) \times \Delta$ taking (t, h) to $(t\nu + h\lambda, h)$. Thus ${}_{\lambda+\gamma}\mathcal{T}'_\lambda$ is obtained from this sheaf by pulling back further by the map $\Delta \to \mathbb{A}^1 \times \Delta$ given by $h \mapsto (0, h)$. Let ${}_{\lambda+\gamma}O_\lambda := \Gamma_{\mathbb{S}}(\mathscr{M}_\nu; {}_{\lambda+\gamma}\mathscr{T}'^{(\nu)}_\lambda)$. To show that our map is surjective, it will suffice to show that

1. the map from $\Gamma_{\mathbb{S}}(\mathscr{M}; {}_\gamma\mathscr{T}'_0)$ to ${}_{\lambda+\gamma}O_\lambda$ is surjective, and
2. the map from ${}_{\lambda+\gamma}O_\lambda$ to $\Gamma_{\mathbb{S}}(\mathfrak{M}; {}_{\lambda+\gamma}\mathcal{T}'_\lambda)$ is surjective.

Consider the variety $\mathscr{N} := \operatorname{Spec} \mathbb{C}[\mathscr{M}]$ from Section 2.2, along with the related variety $\mathscr{N}_\nu := \operatorname{Spec} \mathbb{C}[\mathscr{M}_\nu] \subset \mathscr{N}$. Let $\mathscr{N}^{\mathrm{sm}}$ and $\mathscr{N}^{\mathrm{sm}}_\nu$ be their smooth loci; since the affinization maps for $\mathscr{M}$ and $\mathscr{M}_\nu$ are isomorphisms over the smooth loci, we may regard $\mathscr{N}^{\mathrm{sm}}$ as a subvariety of $\mathscr{M}$ and $\mathscr{N}^{\mathrm{sm}}_\nu$ as a subvariety of $\mathscr{M}_\nu$.

Let ${}_\gamma\mathscr{S}'_0$ be the sheaf on $\mathscr{N}$ obtained from ${}_\gamma\mathscr{T}'_0$ by first restricting it to $\mathscr{N}^{\mathrm{sm}}$ and then pushing it forward to $\mathscr{N}$; since the complement of $\mathscr{N}^{\mathrm{sm}}$ in $\mathscr{M}$ has codimension at least 2, we have

$$\Gamma_{\mathbb{S}}(\mathscr{M}; {}_\gamma\mathscr{T}'_0) \cong \Gamma_{\mathbb{S}}(\mathscr{N}; {}_\gamma\mathscr{S}'_0).$$

Similarly, we define a sheaf ${}_{\lambda+\gamma}\mathscr{S}'^{(\nu)}_{\lambda}$ on $\mathscr{N}_\nu$ obtained from ${}_{\lambda+\gamma}\mathscr{T}'^{(\nu)}_{\lambda}$ by first restricting it to $\mathscr{N}_\nu^{\mathrm{sm}}$ and then pushing it forward to $\mathscr{N}_\nu$, and we have

$$ {}_{\lambda+\gamma}O_\lambda \cong \Gamma_{\mathbb{S}}(\mathscr{N}_\nu; {}_{\lambda+\gamma}\mathscr{S}'^{(\nu)}_{\lambda}). $$

To see that the map from $\Gamma_{\mathbb{S}}(\mathscr{M}; {}_\gamma\mathscr{T}'_0)$ to ${}_{\lambda+\gamma}O_\lambda$ is surjective, it suffices to check that the associated graded is surjective. When we pass to the associated graded, we obtain a map between spaces of sections of two coherent sheaves on $\mathscr{N}$, namely the classical limits $\overline{{}_\gamma\mathscr{S}'_0}$ and $\overline{{}_{\lambda+\gamma}\mathscr{S}'^{(\nu)}_{\lambda}}$. By definition, the restriction of $\overline{{}_{\lambda+\gamma}\mathscr{S}'^{(\nu)}_{\lambda}}$ to $\mathscr{N}_\nu^{\mathrm{sm}}$ is a quotient of the restriction of $\overline{{}_\gamma\mathscr{S}'_0}$ to $\mathscr{N}^{\mathrm{sm}}$. Since the singular locus has codimension 3 on both $\mathscr{N}$ and $\mathscr{N}_\nu$, the induced map between pushforward sheaves is surjective, and since $\mathscr{N}$ is affine, the same is true of the sections.

We now turn to the second surjectivity statement. Consider the exact sequence

$$ 0 \longrightarrow {}_{\lambda+\gamma}\mathscr{T}'^{(\nu)}_{\lambda} \xrightarrow{h^{-1}t} {}_{\lambda+\gamma}\mathscr{T}'^{(\nu)}_{\lambda} \longrightarrow {}_{\lambda+\gamma}\mathcal{T}'_{\lambda} \longrightarrow 0 $$

of sheaves on $\mathscr{M}_\nu$ and its associated long exact sequence

$$ \begin{aligned} 0 \longrightarrow {}_{\lambda+\gamma}O_\lambda \xrightarrow{h^{-1}t} {}_{\lambda+\gamma}O_\lambda \longrightarrow \Gamma_{\mathbb{S}}(\mathfrak{M}; {}_{\lambda+\gamma}\mathcal{T}'_\lambda) \longrightarrow H^1(\mathscr{M}_\nu; {}_{\lambda+\gamma}\mathscr{T}'^{(\nu)}_{\lambda})^{\mathbb{S}} \\ \xrightarrow{h^{-1}t} H^1(\mathscr{M}_\nu; {}_{\lambda+\gamma}\mathscr{T}'^{(\nu)}_{\lambda})^{\mathbb{S}} \longrightarrow H^1(\mathfrak{M}; {}_{\lambda+\gamma}\mathcal{T}'_\lambda)^{\mathbb{S}} \longrightarrow \cdots . \end{aligned} $$

The surjectivity statement that we need is equivalent (by exactness) to injectivity of the action of $h^{-1}t$ on $H^1(\mathscr{M}_\nu; {}_{\lambda+\gamma}\mathscr{T}'^{(\nu)}_{\lambda})^{\mathbb{S}}$.

Since the generic fiber of $\mathscr{M}_\nu$ is affine, $H^1(\mathscr{M}_\nu; {}_{\lambda+\gamma}\mathscr{T}'^{(\nu)}_{\lambda})^{\mathbb{S}}$ is supported on the fiber over 0. This bimodule is Harish-Chandra, so its localization has Lagrangian support in $\mathfrak{M} \times \mathfrak{M}$. Applying Lemma 4.15, we see that $h^{-1}t$ satisfies a polynomial equation on $H^1(\mathscr{M}_\nu; {}_{\lambda+\gamma}\mathscr{T}'^{(\nu)}_{\lambda})^{\mathbb{S}}$, so the bimodule is the sum of finitely many generalized eigenspaces for $h^{-1}t$ and $h^{-1}t$ acts with finite length. In particular, if 0 is a root of this minimal polynomial, the map $h^{-1}t$ is not surjective (since its stable image is a proper summand), and thus $H^1(\mathfrak{M}; {}_{\lambda+\gamma}\mathcal{T}_\lambda)$ is not 0. This is impossible by assumption, so 0 cannot be a root. Thus, $h^{-1}t$ does act invertibly, so the desired map is surjective. □

The following proposition says that derived tensor product with a twisting bimodule does not change the characteristic cycle of the localization. Let N be an object of $D^b(A_\lambda\text{-mod})$, so that $\mathbb{L}\mathrm{Loc}({}_{\lambda+\gamma}T_\lambda \overset{L}{\otimes} N)$ is an object of $D^b(\mathcal{D}_{\lambda+\gamma}\text{-mod})$.

Proposition 6.27 *Assume derived localization holds at λ and $\lambda+\gamma$. Then we have that*

$$ \mathrm{CC}(\mathbb{L}\mathrm{Loc}(N)) = \mathrm{CC}(\mathbb{L}\mathrm{Loc}({}_{\lambda+\gamma}T_\lambda \overset{L}{\otimes} N)). $$

Proof: As in the proof of Proposition 6.26, choose $\nu \in H^2(\mathfrak{M};\mathbb{C})$ such that $\mathscr{M}_\nu(\infty)$ is affine, and consider the sheaf ${}_{\lambda+\gamma}\mathscr{T}'^{(\nu)}_\lambda$. At any point p of $\mathbb{A}^1$, the derived functor of base change to the fiber $\pi^{-1}(p\nu)$ over p sends ${}_{\lambda+\gamma}\mathscr{T}'^{(\nu)}_\lambda$ to the derived localization $\mathbb{L}\mathrm{Loc}({}_{\lambda+\gamma}O_\lambda/(t-p))$ as a module over a quantization of $\mathscr{M}_\nu \times \mathscr{M}_\nu$, since the module ${}_{\lambda+\gamma}O_\lambda$ is flat over $\mathbb{A}^1$.

If p is not 0, then the fiber is affine, and $\mathbb{L}\mathrm{Loc}({}_{\lambda+\gamma}O_\lambda/(t-p))$ is a line bundle on the diagonal in $\pi^{-1}(p\nu) \times \pi^{-1}(p\nu)$. In particular, the class $\mathrm{CC}({}_{\lambda+\gamma}\mathscr{T}'^{(\nu)}_\lambda)$ thus must be the class of the diagonal over every non-zero point in $\mathbb{A}^1$. By Lemma 6.14, we thus have that

$$\mathrm{CC}(\mathbb{L}\mathrm{Loc}({}_{\lambda+\gamma}T_\lambda)) = \mathrm{CC}({}_{\lambda+\gamma}\mathscr{T}'^{(\nu)}_\lambda|_{\pi^{-1}(0)}) = [\mathfrak{M}_\Delta].$$

By Proposition 6.15, the characteristic cycle map intertwines derived tensor product with convolution. Since convolution with the diagonal is trivial, this implies the desired equality. □

We conclude this section by computing these bimodules explicitly in the case where $\mathfrak{M}$ is a symplectic quotient of a vector space, as in Example 2.2. Let G be a connected reductive algebraic group acting on a vector space V with flat moment map $\mu : T^*V \to \mathfrak{g}^*$; let $\mathfrak{M}$ be the symplectic quotient of T^*V at a generic character θ of G, and suppose that the Kirwan map $\mathsf{K} : \chi(\mathfrak{g}) \to H^2(\mathfrak{M};\mathbb{C})$ is an isomorphism. Let A_{T^*V} be the section ring of the unique quantization of T^*V; this is isomorphic to the ring of differential operators on V. Fix a quantized moment map $\eta : U(\mathfrak{g}) \to A_{T^*V}$ and an element $\xi \in \chi(\mathfrak{g})$, and let $\mathcal{D}_\xi$ be the associated quantization of $\mathfrak{M}$ (Section 3.4) with section ring $A = \Gamma_{\mathbb{S}}(\mathcal{D})$. By Proposition 3.13, we have $A \cong \mathrm{End}_{A_{T^*V}}(Y_\xi)$.

Fix a second character ξ' such that $\xi' - \xi$ integrates to a character of G, and consider the $A' - A$ bimodule

$$\mathrm{Hom}(Y_{\xi'}, Y_\xi) \cong \Big(A_{T^*V}/A_{T^*V} \cdot \langle \eta(x) - \xi(x) \mid x \in \mathfrak{g}\rangle\Big)^{\xi'-\xi}. \tag{26}$$

By Proposition 5.4, we have a natural map from $\mathrm{Hom}(Y_{\xi'}, Y_\xi)$ to ${}_{\mathsf{K}(\xi')}T_{\mathsf{K}(\xi)}$. This map is is always injective but it need not be an isomorphism; the restriction to the semistable locus can can cause new sections to appear.

Lemma 6.28 *If $\xi' = \xi + m\theta$ for $m \gg 0$, then the map from $\mathrm{Hom}(Y_{\xi'}, Y_\xi)$ to ${}_{\mathsf{K}(\xi')}T_{\mathsf{K}(\xi)}$ is an isomorphism.*

Proof: The associated graded of this map is the natural map from $\mathbb{C}[\mu^{-1}(0)]_{m\theta}$ to $\Gamma(\mathfrak{M}, \mathcal{L}_{m\theta})$, where the subscript in the source indicated the $\mathbb{S}$-weight space. This map is an isomorphism for sufficiently large m, thus so is our original map. □

Remark 6.29 We note that, by Corollary 3.9 and Proposition 6.24, the source and target of the map in Lemma 6.28 (along with the map itself) are independent of the choice of conical symplectic resolution. Thus Lemma 6.28 simply says that our map is an isomorphism when ξ and ξ' are sufficiently far apart in any generic direction.

6.4 Twisting functors

By Theorem 2.19, the set I of isomorphism classes of conical symplectic resolutions of $\mathfrak{M}_0$ is finite. For each $i \in I$, let $\mathfrak{M}_i$ be a representative resolution. By Remark 2.20, the chambers of the hyperplane arrangement $\mathcal{H}$ are in canonical bijection with $I \times W$, where W is the Weyl group from Section 2.2. For each pair (i, w), let $\Pi_{i,w} \subset P_{\mathbb{R}}$ be the set of parameters λ in the corresponding chamber of $\mathcal{H}$ with the additional property that localization holds at $w\lambda$ on $\mathfrak{M}_i$ and derived localization holds at $w'\lambda$ and $-w'\lambda$ on $\mathfrak{M}_{i'}$ for all pairs (i', w'). Let

$$\Pi := \bigcup_{I\times W} \Pi_{i,w} \subset P_{\mathbb{R}}.$$

Lemma 6.30 *If $w\eta$ is an ample class on $\mathfrak{M}_i$, then for any λ, the class $\lambda + k\eta$ lies in $\Pi_{i,w}$ for all but finitely many $k \in \mathbb{Z}_{\geq 0}$.*

Proof: Recall from Remark 2.20 that the chamber of $\mathcal{H}$ indexed by (i, w) is equal to the w translate of the ample cone of $\mathfrak{M}_i$. Since $w\eta$ is ample on $\mathfrak{M}_i$, so is $w(\lambda + k\eta)$ when k is sufficiently large. The fact that localization holds at $w(\lambda+k\eta)$ for large k follows from Corollary 5.17, and Theorem 4.17 shows the required derived localization statements. The fact that there are only finitely many elements of I shows that only finitely many k need to be removed. $\square$

Let A_λ be the invariant section ring of the quantization with period λ. (Note that, by Corollary 3.9, the ring A_λ does not depend on the choice of resolution of $\mathfrak{M}_0$.) For any pair of elements $\lambda, \lambda' \in H^2(\mathfrak{M}; \mathbb{C})$ that differ by an integral class, let

$$\Phi^{\lambda',\lambda} \colon D(A_\lambda\text{-Mod}) \to D(A_{\lambda'}\text{-Mod}) \tag{27}$$

be the functor obtained by derived tensor product with the bimodule ${}_{\lambda'}T_\lambda$. For any $\lambda \in \Pi$ and $w \in W$, let

$$\Phi^\lambda_w \colon D(A_{w\lambda}\text{-Mod}) \to D(A_\lambda\text{-Mod}) \tag{28}$$

be the equivalence obtained from the isomorphism of Proposition 3.10. Note that the compatibility in the statement of Proposition 3.10 implies that the composition $\Phi^{w\lambda}_{w^{-1}} \circ \Phi^\lambda_w$ is naturally isomorphic to the identity functor.

Proposition 6.31 *Suppose that $\lambda' \in \Pi_{i,w}$. Then the functor $\Phi^{\lambda',\lambda}$ is naturally isomorphic to the composition*

$$D(A_\lambda\text{-Mod}) \xrightarrow{\Phi^\lambda_w} D(A_{w\lambda}\text{-Mod}) \xrightarrow{\mathbb{L}\mathrm{Loc}_i} D(\mathcal{D}_{w\lambda}\text{-Mod})$$
$$\xrightarrow{{}_{w\lambda'}\mathcal{T}'_{w\lambda}\otimes -} D(\mathcal{D}_{w\lambda'}\text{-Mod}) \xrightarrow{\mathbb{R}\Gamma_{\mathbb{S},i}} D(A_{w\lambda'}\text{-Mod}) \xrightarrow{\Phi^{w\lambda'}_{w^{-1}}} D(A_{\lambda'}\text{-Mod}),$$

where the subscript i on $\mathbb{R}\Gamma_{\mathbb{S}}$ and $\mathbb{L}\mathrm{Loc}$ refers to the fact that we are using the resolution $\mathfrak{M}_i$.

Proof: Since $\lambda' \in \Pi_{i,w}$, localization holds at $w\lambda'$, which implies that the higher cohomology of ${}_{w\lambda'}\mathcal{T}'_{w\lambda}$ is trivial. Then Proposition 6.26 tells us that ${}_{w\lambda'}T_{w\lambda} \cong \mathbb{R}\Gamma_{\mathbb{S},i}({}_{w\lambda'}\mathcal{T}'_{w\lambda})$, and therefore that ${}_{w\lambda'}\mathcal{T}'_{w\lambda} \cong \mathbb{L}\mathrm{Loc}_i({}_{w\lambda'}T_{w\lambda})$. The proposition follows immediately using Proposition 6.25. $\square$

Corollary 6.32 *For all $\lambda \in \Pi$ the functor $\Phi^{\lambda',\lambda}$ induces a functor $D^b(A_\lambda\text{-mod}) \to D^b(A_{\lambda'}\text{-mod})$. If $\lambda' \in \Pi$ as well, then this functor is an equivalence.*

Proof: The functor $\mathbb{L}\mathrm{Loc}_i$ induces an equivalence $D^b(A_{w\lambda}\text{-mod}) \to D^b(\mathcal{D}_{w\lambda}\text{-mod})$ as discussed in Remark 4.14. The functor ${}_{w\lambda'}\mathcal{T}'_{w\lambda} \otimes -$ is an equivalence of abelian categories with inverse ${}_{w\lambda}\mathcal{T}'_{w\lambda'} \otimes -$ by the uniqueness part of Proposition 5.2. The functor $\mathbb{R}\Gamma_{\mathbb{S},i}$ induces a functor $D^b(\mathcal{D}_{w\lambda'}\text{-mod}) \to D^b(A_{w\lambda'}\text{-mod})$ by Proposition 4.12, which is also an equivalence if $\lambda' \in \Pi$. $\square$

Corollary 6.33 *If λ and λ' lie in the same chamber of $\mathcal{H}$, then $\Phi^{\lambda,\lambda'} \circ \Phi^{\lambda',\lambda}$ is naturally isomorphic to the identity functor.*

Proof: This follows similarly from Propositions 5.2 and 6.31. $\square$

Fixing a particular $\lambda \in \Pi$, we define **twisting functors** to be the group of endofunctors of $D(A_\lambda\text{-Mod})$ (or of the full subcategory $D^b(A_\lambda\text{-mod})$) obtained by composing functors of the form (27) and (28) and their inverses, and we define **pure twisting functors** to be the subgroup obtained using only functors of the form (27) and their inverses. Note that Corollary 6.33 implies that any such composition that never leaves the chamber in which λ lives is trivial. However, when one crosses a wall and then crosses back, one can and does obtain something nontrivial (see Proposition 6.38 for the case of the Springer resolution).

For Lemma 6.34 we adopt the notational convention, introduced in Section 4.3, whereby we fix $\eta \in H^2(\mathfrak{M}; \mathbb{Z})$ and $\lambda \in H^2(\mathfrak{M}; \mathbb{C})$ and use k in a subscript or superscript in place of $\lambda + k\eta$.

Lemma 6.34 *Suppose that $w\eta$ is very ample on $\mathfrak{M}_i$. Then for any natural numbers $k_\ell > k_{\ell-1} > \cdots > k_1 \geq k_0$, there is a natural isomorphism of functors*

$$\Phi^{k_\ell,k_0} \simeq \Phi^{k_\ell,k_{\ell-1}} \circ \cdots \circ \Phi^{k_1,k_0}.$$

Proof: Let $\mathcal{L}$ be the line bundle on $\mathfrak{M}_i$ with Euler class η. For any $k' > k$, the higher cohomology of $\mathcal{L}^{k'-k}$ vanishes. Therefore the higher cohomology of ${}_{k'}\mathcal{T}'_k$ vanishes as well. By the same argument that we used in the proof of Proposition 6.31, Proposition 6.26 tells us that

$$\begin{aligned}{}_{k_\ell}T_{k_0} &\cong \mathbb{R}\Gamma_{\mathbb{S}}({}_{k_\ell}\mathcal{T}_{k_0}) \cong \mathbb{R}\Gamma_{\mathbb{S}}({}_{k_\ell}\mathcal{T}_{k_{\ell-1}} \otimes_{\mathcal{D}_{k_{\ell-1}}} \cdots \otimes_{\mathcal{D}_{k_1}} {}_{k_1}\mathcal{T}_{k_0}) \\ &\cong \mathbb{R}\Gamma_{\mathbb{S}}({}_{k_\ell}\mathcal{T}_{k_{\ell-1}}) \overset{L}{\otimes}_{A_{k_{\ell-1}}} \cdots \overset{L}{\otimes}_{A_{k_1}} \mathbb{R}\Gamma_{\mathbb{S}}({}_{k_1}\mathcal{T}_{k_0}) \cong {}_{k_\ell}T_{k_{\ell-1}} \overset{L}{\otimes}_{A_{k_{\ell-1}}} \cdots \overset{L}{\otimes}_{A_{k_1}} {}_{k_1}T_{k_0}\end{aligned}$$

as desired. Since $\Phi^{k_\ell,k_0} = {}_{k_\ell}T_{k_0} \overset{L}{\otimes}_{A_{k_0}} -$, the isomorphism follows. □

Let

$$E := H^2(\mathfrak{M};\mathbb{C}) \smallsetminus \bigcup_{H\in\mathcal{H}} H_{\mathbb{C}}$$

be the complement of the complexification of $\mathcal{H}$. The main theorem of this section says that the fundamental group of E/W acts on our category by twisting functors.

Theorem 6.35 *For any $\lambda \in \Pi$, there is a natural homomorphism from $\pi_1(E/W, [\lambda])$ to the group of twisting functors on $D(A_\lambda\text{-Mod})$. The subgroup $\pi_1(E,\lambda)$ maps to the group of pure twisting functors.*

Proof: For each element $(i,w) \in I \times W$, choose an integral class $\eta_{i,w}$ such that $w\eta_{i,w}$ is ample on $\mathfrak{M}_i$. By Lemma 6.30, we may choose a natural number $k_{i,w}$ such that $\lambda_{i,w} := \lambda + k_{i,w}\eta_{i,w}$ lies in $\Pi_{i,w}$. The **Deligne groupoid** of $\mathcal{H}$ is the full sub-groupoid of the fundamental groupoid of E with objects $\{\lambda_{i,w} \mid (i,w) \in I \times W\}$. Note that different choices would lead to a canonically isomorphic groupoid; the only important thing is that we have chosen one representative of each chamber.

The **Deligne quiver** of a real hyperplane arrangement is the quiver with nodes indexed by chambers and arrows in both directions between any two adjacent chambers. Paris [Par93] proves that the Deligne groupoid is isomorphic to the quotient of the fundamental groupoid of the Deligne quiver obtained by identifying any pair of positive paths of minimal length between the same two nodes.[18] Thus, to construct an action of the Deligne groupoid, it is sufficient to first define an action of the Deligne quiver and then check Paris's relations.

Recall that the chambers of $\mathcal{H}$ are in bijection with $I \times W$. We begin by associating the

[18] A path can travel forward or backward along arrows; a positive path is one that always travels forward.

category $D(A_{\lambda_{i,w}}\text{-Mod})$ to the node indexed by (i,w). If the chambers indexed by (i,w) and (j,v) are adjacent, then we assign the functor $\Phi^{\lambda_{j,v},\lambda_{i,w}}$ to the corresponding arrow in the Deligne quiver. We now need to check the relations. Salvetti defines a CW complex which is a W-equivariant homotopy model for the space E. As described in [Sal87, pp. 611-2], the 1-skeleton of this complex is the Deligne quiver, and so the attaching maps of the 2-cells completely describe the relations in the fundamental groupoid. There is one 2-cell for each pair of a codimension 2 face F and an adjacent chamber C, and the attaching map identifies the two minimal positive paths from C to its opposite across F. Thus, we need only check that composition along these paths gives the same functors.

Suppose we are given two such chambers, labeled by (i,w) and (i',w'). Let H be a generic cooriented hyperplane that contains F and bisects both chambers. Figure 1 illustrates a 2-dimensional slice transverse to F, so that F appears as a point and H appears as a line, which in the picture we draw as dotted.

Choose elements μ and ν of $\Pi_{i,w}$ that differ from $\lambda_{i,w}$ by an integral class, with μ on the positive side and ν on the negative side of H. Choose μ' and ν' in $\Pi_{i',w'}$ similarly. Let $\mu = \mu_1, \mu_2, \ldots, \mu_n = \mu'$ be colinear integral representatives of all the chambers on the positive side of H, and let $\nu = \nu_1, \ldots, \nu_\ell = \nu'$ be colinear representatives of all the chambers on the negative side of H. We may arrange these classes such that for all k, $\mu_k - \mu_{k+1}$ and $\nu_k - \nu_{k+1}$ both lie in the chamber indexed by (i,w). Put differently, we may assume that $w\mu_k - w\mu_{k+1}$ and $w\nu_k - w\nu_{k+1}$ are both ample on $\mathfrak{M}_i$. All of this is illustrated in Figure 1.

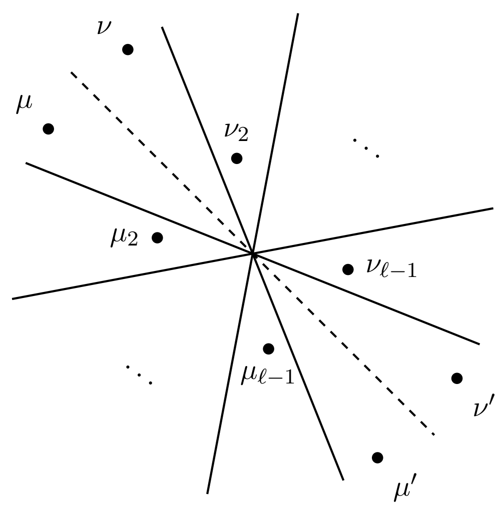

Figure 1: A 2-dimensional slice.

By Corollary 6.33, we may reduce the theorem to checking that the functors

$$\Phi^{\lambda_{i',w'},\mu'} \circ \Phi^{\mu',\mu_{\ell-1}} \circ \cdots \circ \Phi^{\mu_2,\mu} \circ \Phi^{\mu,\lambda_{i,w}} \qquad \text{and} \qquad \Phi^{\lambda_{i',w'},\nu'} \circ \Phi^{\nu',\nu_{\ell-1}} \circ \cdots \circ \Phi^{\nu_2,\nu} \circ \Phi^{\nu,\lambda_{i,w}}$$

from $D(A_{\lambda_{i,w}}\text{-Mod})$ to $D(A_{\lambda_{i',w'}}\text{-Mod})$ are naturally isomorphic. By Corollary 6.33 and Lemma 6.34, both are equivalent to $\Phi^{\lambda_{i',w'},\lambda_{i,w}}$.

We have now established that the Deligne groupoid acts on the derived categories $D(A_{\lambda_{i,w}}\text{-Mod})$ for all $(i,w) \in I \times W$. Specializing to a single parameter, we conclude that $\pi_1(E,\lambda)$ acts on $D(A_\lambda\text{-Mod})$ via pure twisting functors. Furthermore, by Proposition 3.10, we have an action of W on the categories $D(A_{\lambda_{i,w}}\text{-Mod})$ via the functors Φ^λ_w. The uniqueness of the quantizations of line bundles (Proposition 5.2) shows that

$$\Phi^\lambda_w \circ \Phi^{\lambda,\lambda'} \cong \Phi^{w\lambda,w\lambda'} \circ \Phi^{\lambda'}_w,$$

so this action is compatible with the action of W on the Deligne groupoid D, considered as a subgroupoid of the fundamental groupoid. This shows that the semi-direct product $D \rtimes W$ acts on the categories $D(A_{\lambda_{i,w}}\text{-Mod})$. The automorphisms of a point λ in the semi-direct product are isomorphic to $\pi_1(E/W,[\lambda])$. $\square$

Remark 6.36 We have already remarked that $\mathcal{D}$-mod (and therefore A-mod, when localization holds) may be thought of as a twisted algebraic version of the Fukaya category of $\mathfrak{M}$ (Remark 4.3). In this interpretation, we expect the action in Conjecture 6.35 to be given by parallel transport in the universal deformation, along the lines of the construction in [SS06] for Slodowy slices of type A.

Remark 6.37 As in Section 6.1, we may replace $D(A_\lambda\text{-Mod})$ in the statement of Theorem 6.35 with $D^b_{\mathfrak{L}_0}(A_\lambda\text{-mod})$ (see Definition 6.9) for any $\mathbb{S}$-equivariant $\mathfrak{L}_0 \subset \mathfrak{M}_0$, or with the bounded derived category $D^b(C^{\mathfrak{L}_0}_\lambda)$. These categories are related by a realization functor $D^b(C^{\mathfrak{L}_0}_\lambda) \to D^b_{\mathfrak{L}_0}(A_\lambda\text{-mod})$, which may or may not be fully faithful.

If $\mathfrak{M}$ is a hypertoric variety and $\mathfrak{L}$ is as in Example 6.12, we obtain the twisting functors studied in [BLPW10, §6] and [BLPW12, 8.4]. To see this, we need to apply Lemma 6.28 and Remark 6.29, because the functors in [BLPW12, 8.4] are defined using the bimodules in Equation (26).

Recall that BGG category $\mathcal{O}$ is the subcategory of finitely generated $U(\mathfrak{g})$-modules on which $\mathfrak{b}$ acts locally finitely, and $\mathfrak{h}$ acts semi-simply. Let $\mathcal{O}_\lambda$ for a weight λ be the Serre subcategory where the center of $U(\mathfrak{g})$ acts with the same generalized character as on the Verma module with highest weight λ. If $\mathfrak{M} = T^*(G/B)$ and $\mathfrak{L}$ is as in Example 6.12, then for any regular integral weight λ, the category $C^{\mathfrak{L}}_{\lambda+\rho}$ is equivalent to $\mathcal{O}_\lambda$ by Soergel's functor. As discussed above, this means that we have a realization functor $\mathscr{R}_\lambda\colon D^b(\mathcal{O}_\lambda) \cong D^b(C^{\mathfrak{L}}_{\lambda+\rho}) \to D_{\mathfrak{L}}(A_{\lambda+\rho}\text{-mod})$, which is not obviously fully faithful. These functors obviously commute with the translation equivalences between $C^{\mathfrak{L}}_{\lambda+\rho}$ and $C^{\mathfrak{L}}_{\lambda'+\rho}$ where λ,λ' are both dominant and integral; thus the functor $\mathscr{R}_\lambda$ is either fully faithful for all dominant integral λ or for none. The result [BLPW16, 5.13] shows that it must be fully faithful for all λ in an open subset $\mathfrak{U} \subset H^2(G/B)$, so it must be an fully faithful for all dominant λ. Thus, we can consider $D^b(C^{\mathfrak{L}}_{\lambda+\rho})$ as a subcategory of $D_{\mathfrak{L}}(A_{\lambda+\rho}\text{-mod}) \subset D^b(A\text{-mod})$ in this case.

The following result says that this equivalence identifies the functors we call twisting functors with Arkhipov's twisting functors [Ark04, AS03]. More precisely, Arkhipov defines a collection of derived auto-equivalences $\{T_w \mid w \in W\}$ of the category $\mathcal{O}_\lambda$ satisfying the relation $T_w \circ T_{w'} \cong T_{ww'}$ whenever the length of ww' is equal to the sum of the lengths of w and w', which means that these functors generate an action of the generalized braid group. In this case the discriminantal arrangement is equal to the Coxeter arrangement for W, so the fundamental group $\pi_1(E/W, [\lambda])$ is also isomorphic to the generalized braid group.

Proposition 6.38 *Suppose that $\mathfrak{M} = T^*(G/B)$ and let $\mathfrak{L}$ be as in Example 6.12. If $\lambda \in H^2(\mathfrak{M};\mathbb{C})$ is regular, integral, and dominant, then Soergel's equivalence from the block $\mathcal{O}_\lambda$ of BGG category $\mathcal{O}$ to the category $C^{\mathfrak{L}}_{\lambda+\rho}$ intertwines Arhkipov's twisting action on $D^b(\mathcal{O})$ with the twisting action on $D^b(C^{\mathfrak{L}}_{\lambda+\rho}) \subset D(A\text{-Mod})$ from Theorem 6.35.*

Proof: We begin by showing that Arkhipov's twisting functors are uniquely characterized by the following two properties:

- T_w strongly commutes with projective functors [AS03, Lemma 2.1]. That is, for any projective functor F, there is an isomorphism $T_w \circ F \cong F \circ T_w$, and these isomorphisms are compatible with natural transformations of projective functors.
- For all $w \in W$, $T_w V_\lambda \cong V_{w\cdot\lambda}$, where V_λ is the Verma module with highest weight λ.

Indeed, let $\{T'_w \mid w \in W\}$ be any other collection of functors satisfying these conditions. By [BG80, 3.3(iib)], for any irreducible projective object of $\mathcal{O}_\lambda$, there is a projective functor taking V_λ to that object. Since $\mathcal{O}_\lambda$ has enough projectives, for any object N of $\mathcal{O}_\lambda$, there is a complex F_N of projective functors taking V_λ to N. Furthermore, projective functors may be regarded as modules over $\mathfrak{g} \times \mathfrak{g}$ [Bac01], and we have $\mathrm{Hom}_{\mathfrak{g}}(N, N') \cong \mathrm{Hom}_{\mathfrak{g}\times\mathfrak{g}}(F_N, F_{N'})$. We therefore have

$$T'_w N \cong T'_w F_N V_\lambda \cong F_N T'_w V_\lambda \cong F_N V_{w\cdot\lambda} \cong F_N T_w V_\lambda \cong T_w F_N V_\lambda \cong T_w N,$$

and the strong commutativity condition ensures that this induces an isomorphism of functors.

Since λ is dominant, Soergel's equivalence between $\mathcal{O}_\lambda$ and $C^{\mathcal{L}}_{\lambda+\rho}$ is given by composing the functors

$$\mathcal{O}_\lambda \underset{\mathrm{Hom}^{\mathrm{fin}}_{\mathbb{C}}(V_\lambda,-)^\circ}{\overset{(-)^\circ \otimes V_\lambda}{\rightleftarrows}} {}^{1}_{\lambda}\mathscr{H}^\infty_\lambda \underset{\varinjlim_i \mathrm{Hom}^{\mathrm{fin}}_{\mathbb{C}}(V^i_\lambda,-)}{\overset{\varprojlim_i(-\otimes V^i_\lambda)}{\rightleftarrows}} C^{\mathcal{L}}_{\lambda+\rho} \tag{29}$$

where

- ${}^{1}_{\lambda}\mathscr{H}^{\infty}_{\lambda}$ denotes the category of Harish-Chandra bimodules (in the usual sense) for $U(\mathfrak{g})$ with generalized central character λ for both the left and right actions, with the center acting on the left semi-simply,

- $\mathrm{Hom}^{\mathrm{fin}}_{\mathbb{C}}(V_\lambda, N)$ is the Harish-Chandra bimodule of $U(\mathfrak{g})_\Delta$-locally finite $\mathbb{C}$-linear maps $V_\lambda \to N$,

- $(-)^\circ$ denotes the functor on $U(\mathfrak{g})$-$U(\mathfrak{g})$ bimodules which switches the left and right actions, twisting by the antipode of $U(\mathfrak{g})$,

- V^i_λ denotes the length i thickened Verma module $V^i_\lambda := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (U(\mathfrak{h})/\mathfrak{m}^i_\lambda)$, where $\mathfrak{m}_\lambda$ is the kernel of the action of $U(\mathfrak{h})$ on the λ-weight space.

Thus, we need only show that our twisting functors on $D^b(C^{\mathfrak{L}}_{\lambda+\rho})$, transported to $D^b(\mathcal{O}_\lambda)$ via Soergel's equivalence, satisfy these two conditions.

For any element $w \in W$, let $R^w_\lambda := \Phi^{\lambda+\rho}_w({}_{w(\lambda+\rho)}T_{\lambda+\rho})$, where ${}_{w(\lambda+\rho)}T_{\lambda+\rho}$ is regarded as a left $A_{w(\lambda+\rho)}$-module. Consider the twisting functor

$$S_w := \Psi^{\lambda+\rho}_w \circ \Phi^{w(\lambda+\rho),\lambda+\rho} \cong R^w_\lambda \overset{L}{\otimes} -.$$

Under the bi-adjoint equivalences of $C^{\mathcal{L}}_{\lambda+\rho}$ with ${}^{1}_{\lambda}\mathscr{H}^{\infty}_{\lambda}$ described in Equation (29) of Example 6.12, this functor is intertwined with $R^w_\lambda \overset{L}{\otimes} -$, now regarded as a functor on Harish-Chandra bimodules, since tensor product on the left commutes with $\varprojlim(- \otimes V^i_\lambda)$. On the other hand, the equivalence to $\mathcal{O}_\lambda$, described in the same equation, involves exchanging the left and right actions. Thus, any projective functor $F \cong F\big(U(\mathfrak{g})\big) \otimes_{U(\mathfrak{g})} -$ is intertwined with $- \otimes_{U(\mathfrak{g})} F(U(\mathfrak{g}))^\circ \colon {}^{1}_{\lambda}\mathscr{H}^{\infty}_{\lambda} \to {}^{1}_{\lambda}\mathscr{H}^{\infty}_{\lambda}$, which obviously commutes with $R^w_\lambda \otimes -$.

Checking the second condition is an easy geometric calculation. Since λ is dominant and regular, localization holds at λ [BB81]. The localization of V_λ is an object of $D_{\lambda+\rho}$-mod, which we may regard as a twisted D-module by Proposition 4.5. Concretely, it is the restriction of the line bundle $\mathcal{L}_\lambda$ to the open Bruhat cell, where only the action of $\mathfrak{g}$ depends on λ. Tensoring with ${}_{w(\lambda+\rho)}\mathcal{T}'_{\lambda+\rho}$ takes us to the restriction of $\mathcal{L}_{w\cdot\lambda}$ to that cell. The sections of that restriction are exactly the Verma module $V_{w\cdot\lambda}$, since it is generated by a unique U-invariant section of weight $w\cdot\lambda$ (here U is the nilpotent radical of B), and the dimension of weight spaces matches the character of the Verma module. □

We end by analyzing the twisting action of Theorem 6.35 on the level of the Grothendieck group. Assume $\lambda \in \Pi$. Every twisting functor $\Phi \colon D^b(A_\lambda\text{-mod}) \to D^b(A_\lambda\text{-mod})$ is induced by derived tensor product with an algebraic Harish-Chandra bimodule K_Φ; by Proposition

6.8, this implies that the corresponding functor $\mathbb{L}\mathrm{Loc} \circ \Phi \circ \mathbb{R}\Gamma_{\mathbb{S}}\colon D^b(\mathcal{D}_\lambda\text{-mod}) \to D^b(\mathcal{D}_\lambda\text{-mod})$ is induced by convolution with a geometric Harish-Chandra bimodule $F_\Phi \in {}_\lambda\mathbf{HC}^{\mathrm{g}}_\lambda$. By Proposition 6.15, the effect of Φ on characteristic cycles is given by convolution with the characteristic cycle $\mathrm{CC}(F_\Phi)$. Thus we obtain an algebra homomorphism

$$\alpha\colon \mathbb{C}[\pi_1(E/W, [\lambda])] \to H_3^{2\dim \mathfrak{M}}(\mathfrak{M} \times \mathfrak{M}; \mathbb{C}).$$

Proposition 6.39 *The subalgebra $\mathbb{C}[\pi_1(E, \lambda)] \subset \mathbb{C}[\pi_1(E/W, [\lambda])]$ is contained in the kernel of α, thus we obtain an induced homomorphism*

$$\bar{\alpha}\colon \mathbb{C}[W] \to H_3^{2\dim \mathfrak{M}}(\mathfrak{M} \times \mathfrak{M}; \mathbb{C}).$$

Proof: By Proposition 6.27, pure twisting functors preserve characteristic cycles. Since the subalgebra $\mathbb{C}[\pi_1(E, \lambda)] \subset \mathbb{C}[\pi_1(E/W, [\lambda])]$ acts by pure twisting functors, the result follows. □

Remark 6.40 The map $\bar{\alpha}$ also has a direct geometric construction, which precisely matches the one given by Chriss and Ginzburg [CG97, 3.4.1] for $\mathfrak{M} = T^*G/B$. Applying the argument of the proof of Proposition 6.27 to an impure twisting functor shows that the class corresponding to w is a specialization of the graph of the map $w\colon \pi^{-1}(\nu) \to \pi^{-1}(w \cdot \nu)$.

References

[AB02] Michael Atiyah and Roger Bielawski, *Nahm's equations, configuration spaces and flag manifolds*, Bull. Braz. Math. Soc. (N.S.) **33** (2002), no. 2, 157–176.

[ACET] Filippo Ambrosio, Giovanna Carnovale, Francesco Esposito, and Lewis Topley, *Universal filtered quantizations of nilpotent Slodowy slices.*

[Ark04] Sergey Arkhipov, *Algebraic construction of contragradient quasi-Verma modules in positive characteristic*, Representation theory of algebraic groups and quantum groups, Adv. Stud. Pure Math., vol. 40, Math. Soc. Japan, Tokyo, 2004, pp. 27–68.

[AS03] Henning Haahr Andersen and Catharina Stroppel, *Twisting functors on $\mathcal{O}$*, Represent. Theory **7** (2003), 681–699 (electronic).

[AVdBVO89] Maria Jesus Asensio, Michel Van den Bergh, and Freddy Van Oystaeyen, *A new algebraic approach to microlocalization of filtered rings*, Trans. Amer. Math. Soc. **316** (1989), no. 2, 537–553. MR 958890 (90c:16001)

[AW] Marco Andreatta and Jarosław Wiśniewski, *4-dimensional symplectic contractions.*

[Bac01] Erik Backelin, *The Hom-spaces between projective functors*, Represent. Theory **5** (2001), 267–283 (electronic).

[BB81] Alexandre Beĭlinson and Joseph Bernstein, *Localisation de g-modules*, C. R. Acad. Sci. Paris Sér. I Math. **292** (1981), no. 1, 15–18.

[BB93] A. Beĭlinson and J. Bernstein, *A proof of Jantzen conjectures*, I. M. Gel′fand Seminar, Adv. Soviet Math., vol. 16, Amer. Math. Soc., Providence, RI, 1993, pp. 1–50.

[BD00] Roger Bielawski and Andrew S. Dancer, *The geometry and topology of toric hyperkähler manifolds*, Comm. Anal. Geom. **8** (2000), no. 4, 727–760.

[Bea00] Arnaud Beauville, *Symplectic singularities*, Invent. Math. **139** (2000), no. 3, 541–549.

[BG80] J. N. Bernstein and S. I. Gel′fand, *Tensor products of finite- and infinite-dimensional representations of semisimple Lie algebras*, Compositio Math. **41** (1980), no. 2, 245–285.

[BG99] Alexander Beilinson and Victor Ginzburg, *Wall-crossing functors and $\mathscr{D}$-modules*, Represent. Theory **3** (1999), 1–31.

[BK04] Roman Bezrukavnikov and Dmitry Kaledin, *Fedosov quantization in algebraic context*, Mosc. Math. J. **4** (2004), no. 3, 559–592, 782.

[BK12] Gwyn Bellamy and Toshiro Kuwabara, *On deformation quantizations of hypertoric varieties*, 2012, pp. 89–127.

[BLPW10] Tom Braden, Anthony Licata, Nicholas Proudfoot, and Ben Webster, *Gale duality and Koszul duality*, Adv. Math. **225** (2010), no. 4, 2002–2049.

[BLPW12] Tom Braden, Anthony Licata, Nicholas Proudfoot, and Ben Webster, *Hypertoric category $\mathcal{O}$*, Adv. Math. **231** (2012), no. 3-4, 1487–1545.

[BLPW16] ———, *Quantizations of conical symplectic resolutions II: category $\mathcal{O}$ and symplectic duality*, Astérisque (2016), no. 384, 75–179, with an appendix by I. Losev. MR 3594665

[BMO] Alexander Braverman, Davesh Maulik, and Andrei Okounkov, *Quantum cohomology of the springer resolution*, `arXiv:1001.0056`.

[CG97] Neil Chriss and Victor Ginzburg, *Representation theory and complex geometry*, Birkhäuser Boston Inc., Boston, MA, 1997.

[Che10] Hou-Yi Chen, *GAGA for DQ-algebroids*, Rend. Semin. Mat. Univ. Padova **123** (2010), 211–231. MR 2683299 (2011k:32011)

[CL] Sabin Cautis and Aaron Lauda, *Implicit structure in 2-representations of quantum groups*, `arXiv:QA/1111.1431`.

[CW10] Andrei Căldăraru and Simon Willerton, *The Mukai pairing. I. A categorical approach*, New York J. Math. **16** (2010), 61–98.

[DK] Christopher Dodd and Kobi Kremnizer, *A localization theorem for finite W-algebras*, `arXiv:0911.2210`.

[Dur83] Alan H. Durfee, *Neighborhoods of algebraic sets*, Trans. Amer. Math. Soc. **276** (1983), no. 2, 517–530.

[EGGO07] Pavel Etingof, Wee Liang Gan, Victor Ginzburg, and Alexei Oblomkov, *Harish-Chandra homomorphisms and symplectic reflection algebras for wreath-products*, Publ. Math. Inst. Hautes Études Sci. (2007), no. 105, 91–155.

[Fu03] Baohua Fu, *Symplectic resolutions for nilpotent orbits*, Invent. Math. **151** (2003), no. 1, 167–186.

[Gin09] Victor Ginzburg, *Harish-Chandra bimodules for quantized Slodowy slices*, Represent. Theory **13** (2009), 236–271. MR 2515934 (2010g:17009)

[Gor06] Iain Gordon, *A remark on rational Cherednik algebras and differential operators on the cyclic quiver*, Glasg. Math. J. **48** (2006), no. 1, 145–160.

[GS05] I. Gordon and J. T. Stafford, *Rational Cherednik algebras and Hilbert schemes*, Adv. Math. **198** (2005), no. 1, 222–274.

[Hes78] Wim H. Hesselink, *Polarizations in the classical groups*, Math. Z. **160** (1978), no. 3, 217–234.

[HP04] Megumi Harada and Nicholas Proudfoot, *Properties of the residual circle action on a hypertoric variety*, Pacific J. Math. **214** (2004), no. 2, 263–284.

[HTT08] Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki, *D-modules, perverse sheaves, and representation theory*, Progress in Mathematics, vol. 236, Birkhäuser Boston Inc., Boston, MA, 2008, Translated from the 1995 Japanese edition by Takeuchi.

[Jos83] A. Joseph, *On the classification of primitive ideals in the enveloping algebra of a semisimple Lie algebra*, Lie group representations, I (College Park, Md., 1982/1983), Lecture Notes in Math., vol. 1024, Springer, Berlin, 1983, pp. 30–76.

[Kal] Dmitry Kaledin, *Symplectic resolutions: deformations and birational maps*, arXiv:0012008.

[Kal06] ———, *On the coordinate ring of a projective Poisson scheme*, Math. Res. Lett. **13** (2006), no. 1, 99–107.

[Kal08] ———, *Derived equivalences by quantization*, Geom. Funct. Anal. **17** (2008), no. 6, 1968–2004.

[Kal09] ———, *Geometry and topology of symplectic resolutions*, Algebraic geometry—Seattle 2005. Part 2, Proc. Sympos. Pure Math., vol. 80, Amer. Math. Soc., Providence, RI, 2009, pp. 595–628.

[Kas03] Masaki Kashiwara, *D-modules and microlocal calculus*, Translations of Mathematical Monographs, vol. 217, American Mathematical Society, Providence, RI, 2003, Translated from the 2000 Japanese original by Mutsumi Saito, Iwanami Series in Modern Mathematics.

[Kaw02] Yujiro Kawamata, *D-equivalence and K-equivalence*, J. Differential Geom. **61** (2002), no. 1, 147–171.

[KR08] Masaki Kashiwara and Raphaël Rouquier, *Microlocalization of rational Cherednik algebras*, Duke Math. J. **144** (2008), no. 3, 525–573.

[KS12] Masaki Kashiwara and Pierre Schapira, *Deformation quantization modules*, Astérisque (2012), no. 345, xii+147. MR 3012169

[KV02] D. Kaledin and M. Verbitsky, *Period map for non-compact holomorphically symplectic manifolds*, Geom. Funct. Anal. **12** (2002), no. 6, 1265–1295.

[KW07] Anton Kapustin and Edward Witten, *Electric-magnetic duality and the geometric Langlands program*, Commun. Number Theory Phys. **1** (2007), no. 1, 1–236.

[KWWY14] Joel Kamnitzer, Ben Webster, Alex Weekes, and Oded Yacobi, *Yangians and quantizations of slices in the affine Grassmannian*, Algebra Number Theory **8** (2014), no. 4, 857–893. MR 3248988

[Los] Ivan Losev, *Harish-Chandra bimodules over quantized symplectic singularities*, arXiv:1810.07625.

[Los12] Ivan Losev, *Isomorphisms of quantizations via quantization of resolutions*, Adv. Math. **231** (2012), no. 3-4, 1216–1270.

[MN] Kevin McGerty and Thomas Nevins, *Morse decomposition for $\mathcal{D}$-module categories on stacks*, arXiv:1402.7365.

[MN14] Kevin McGerty and Thomas Nevins, *Derived equivalence for quantum symplectic resolutions*, Selecta Math. (N.S.) **20** (2014), no. 2, 675–717. MR 3177930

[Nak94] Hiraku Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*, Duke Math. J. **76** (1994), no. 2, 365–416.

[Nak98] ———, *Quiver varieties and Kac-Moody algebras*, Duke Math. J. **91** (1998), no. 3, 515–560.

[Nak01] ———, *Quiver varieties and finite-dimensional representations of quantum affine algebras*, J. Amer. Math. Soc. **14** (2001), no. 1, 145–238 (electronic).

[Nam] Yoshinori Namikawa, *Poisson deformations and Mori dream spaces*, `arXiv:1305.1698`.

[Nam08] Yoshinori Namikawa, *Flops and Poisson deformations of symplectic varieties*, Publ. Res. Inst. Math. Sci. **44** (2008), no. 2, 259–314.

[Nam10] ———, *Poisson deformations of affine symplectic varieties, II*, Kyoto J. Math. **50** (2010), no. 4, 727–752.

[Nam11] ———, *Poisson deformations of affine symplectic varieties*, Duke Math. J. **156** (2011), no. 1, 51–85.

[NZ09] David Nadler and Eric Zaslow, *Constructible sheaves and the Fukaya category*, J. Amer. Math. Soc. **22** (2009), no. 1, 233–286.

[Par93] Luis Paris, *The covers of a complexified real arrangement of hyperplanes and their fundamental groups*, Topology Appl. **53** (1993), no. 1, 75–103.

[Pre02] Alexander Premet, *Special transverse slices and their enveloping algebras*, Adv. Math. **170** (2002), no. 1, 1–55, With an appendix by Serge Skryabin.

[Pro04] Nicholas Proudfoot, *Hyperkahler analogues of Kahler quotients*, Ph. D. thesis, U.C. Berkeley, 2004, `arXiv:math.AG/0405233`.

[Pro08] ———, *A survey of hypertoric geometry and topology*, Toric Topology, Contemp. Math., vol. 460, Amer. Math. Soc., Providence, RI, 2008, pp. 323–338.

[PW07] Nicholas Proudfoot and Ben Webster, *Intersection cohomology of hypertoric varieties*, J. Algebraic Geom. **16** (2007), no. 1, 39–63.

[Sal87] M. Salvetti, *Topology of the complement of real hyperplanes in* $\mathbf{C}^N$, Invent. Math. **88** (1987), no. 3, 603–618.

[Ser56] Jean-Pierre Serre, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier, Grenoble **6** (1955–1956), 1–42.

[Soe86] Wolfgang Soergel, *Équivalences de certaines catégories de* $\mathfrak{g}$*-modules*, C. R. Acad. Sci. Paris Sér. I Math. **303** (1986), no. 15, 725–728.

[Spa88] N. Spaltenstein, *Resolutions of unbounded complexes*, Compositio Math. **65** (1988), no. 2, 121–154. MR 932640 (89m:18013)

[Spr82] T. A. Springer, *Quelques applications de la cohomologie d'intersection*, Bourbaki Seminar, Vol. 1981/1982, Astérisque, vol. 92, Soc. Math. France, Paris, 1982, pp. 249–273.

[SS06] Paul Seidel and Ivan Smith, *A link invariant from the symplectic geometry of nilpotent slices*, Duke Math. J. **134** (2006), no. 3, 453–514.

[SvdB01] J. T. Stafford and M. van den Bergh, *Noncommutative curves and noncommutative surfaces*, Bull. Amer. Math. Soc. (N.S.) **38** (2001), no. 2, 171–216.

[Web] Ben Webster, *A categorical action on quantized quiver varieties*, `arXiv:1208.5957`.