SINGULAR HODGE THEORY FOR COMBINATORIAL GEOMETRIES

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ABSTRACT. We introduce the intersection cohomology module of a matroid and prove that it satisfies Poincaré duality, the hard Lefschetz theorem, and the Hodge–Riemann relations. As applications, we obtain proofs of Dowling and Wilson's Top-Heavy conjecture and the nonnegativity of the coefficients of Kazhdan–Lusztig polynomials for all matroids.

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1. Introduction

1.1. **Results.** A **matroid** M on a finite set E is a nonempty collection of subsets of E, called **flats** of M, that satisfies the following properties:

- If F_1 and F_2 are flats, then their intersection $F_1 \cap F_2$ is a flat.
- If F is a flat, then any element in $E \backslash F$ is in exactly one flat that is minimal among the flats strictly containing F.

For notational convenience, we assume throughout that M is **loopless**:

- The empty subset of E is a flat.

We write $\mathcal{L}(\mathrm{M})$ for the lattice of all flats of M. Every maximal flag of proper flats of M has the same cardinality $\operatorname{rk} \mathrm{M}$, called the **rank** of M. For any nonnegative integer k, we write $\mathcal{L}^k(\mathrm{M})$ to denote the set of rank k flats of M. A matroid can be equivalently defined in terms of its **independent sets**, **circuits**, or the **rank function**. For background in matroid theory, we refer to [Oxl11] and [Wel76].

Let Γ be a finite group acting on M. By definition, Γ permutes the elements of E in such a way that it sends flats to flats.

Theorem 1.1. The following holds for any $k \leqslant j \leqslant \operatorname{rk} \mathrm{M} - k$.

(1) The cardinality of $\mathcal{L}^k(\mathrm{M})$ is at most the cardinality of $\mathcal{L}^j(\mathrm{M})$.

(2) There is an injective map $\iota\colon \mathcal{L}^k(\mathrm{M}) \to \mathcal{L}^j(\mathrm{M})$ satisfying $F \leqslant \iota(F)$ for all $F \in \mathcal{L}^k(\mathrm{M})$.

(3) There is an injective map $\mathbb{Q}\mathcal{L}^k(\mathrm{M}) \to \mathbb{Q}\mathcal{L}^j(\mathrm{M})$ of permutation representations of Γ.[1]

The first two parts of Theorem 1.1 were conjectured by Dowling and Wilson [DW74, DW75], and have come to be known as the **Top-Heavy conjecture**. Its best known instance is the de Bruijn–Erdős theorem on point-line incidences in projective planes [dBE48]:

> *Every finite set of points E in a projective plane determines at least $|E|$ lines, unless E is contained in a line. In other words, if E is not contained in a line, then the number of lines in the plane containing at least two points in E is at least $|E|$.*

When $\mathcal{L} = \mathcal{L}(\mathrm{M})$ is a Boolean lattice or a projective geometry, Theorem 1.1 is a classical result; see for example [Sta18, Corollary 4.8 and Exercise 4.4]. In these cases, the second statement of

[1] One might hope to combine the last two parts of Theorem 1.1 by asking the map ι to be Γ-equivariant, but this is not possible, even if we drop the condition that $F \leqslant \iota(F)$. For example, when M is the uniform matroid of rank 3 on 4 elements, there is no S_4-equivariant map from $\mathcal{L}^1(\mathrm{M})$ to $\mathcal{L}^2(\mathrm{M})$.

Theorem 1.1 implies that these lattices admit order-matchings

$$\mathcal{L}^0 \to \mathcal{L}^1 \to \cdots \to \mathcal{L}^{\lfloor \frac{\mathrm{rk}\,\mathrm{M}}{2} \rfloor} \leftrightarrow \mathcal{L}^{\lceil \frac{\mathrm{rk}\,\mathrm{M}}{2} \rceil} \leftarrow \cdots \leftarrow \mathcal{L}^{\mathrm{rk}\,\mathrm{M}-1} \leftarrow \mathcal{L}^{\mathrm{rk}\,\mathrm{M}},$$

and hence have the Sperner property:

> *The maximal number of pairwise incomparable subsets of $[n]$ is the maximum among the binomial coefficients $\binom{n}{k}$. Similarly, the maximal number of pairwise incomparable subspaces of $\mathbb{F}_q^n$ is the maximum among the q-binomial coefficients $\binom{n}{k}_q$.*

Other earlier versions of Theorem 1.1, for specific classes of matroids or small values of k, can be found in [Mot51, BK68, Gre70, Mas72, Her73, Kun79, Kun86, Kun93, Kun00]. In [HW17], Theorem 1.1 was proved for matroids realizable over some field. See Section 1.3 for an overview of that proof. Although realizable matroids provide the primary motivation for the definition of a matroid, almost all matroids are not realizable over any field. More precisely, the portion of matroids on the ground set $[n]$ that are realizable over some field goes to zero as n goes to infinity [Nel18].

Our proof of Theorem 1.1 is closely related to Kazhdan–Lusztig theory of matroids, as developed in [EPW16]. For any flat F of M, we define the **localization** of M at F to be the matroid M^F on the ground set F whose flats are the flats of M contained in F. Similarly, we define the **contraction** of M at F to be the matroid M_F on the ground set $E \backslash F$ whose flats are $G \backslash F$ for flats G of M containing F.[2] We also consider the **characteristic polynomial**

$$\chi_{\mathrm{M}}(t) \coloneqq \sum_{I \subseteq E} (-1)^{|I|} t^{\mathrm{crk}\, I},$$

where $\mathrm{crk}\, I$ is the corank of I in M. According to [EPW16, Theorem 2.2], there is a unique way to assign a polynomial $P_{\mathrm{M}}(t)$ to each matroid M, called the **Kazhdan–Lusztig polynomial** of M, subject to the following three conditions:

(a) If the rank of M is zero, then $P_{\mathrm{M}}(t)$ is the constant polynomial 1.

(b) For every matroid M of positive rank, the degree of $P_{\mathrm{M}}(t)$ is strictly less than $\mathrm{rk}\,\mathrm{M}/2$.

(c) For every matroid M, we have $t^{\mathrm{rk}\,\mathrm{M}} P_{\mathrm{M}}(t^{-1}) = \sum_{F \in \mathcal{L}(\mathrm{M})} \chi_{\mathrm{M}^F}(t) \cdot P_{\mathrm{M}_F}(t)$.

Alternatively [BV20, Theorem 2.2], one may define Kazhdan–Lusztig polynomials of matroids by replacing the third condition above with the following condition not involving $\chi_{\mathrm{M}}(t)$:

(c)′ For every matroid M, the polynomial $Z_{\mathrm{M}}(t) \coloneqq \sum_{F \in \mathcal{L}(\mathrm{M})} t^{\mathrm{rk}\, F} P_{\mathrm{M}_F}(t)$ satisfies the identity

$$t^{\mathrm{rk}\,\mathrm{M}} Z_{\mathrm{M}}(t^{-1}) = Z_{\mathrm{M}}(t).$$

[2] In [EPW16], as well as several other references on Kazhdan–Lusztig polynomials of matroids, the localization is denoted M_F and the contraction is denoted M^F. Our notational choice here is consistent with [AHK18] and [BHM$^+$20].

The polynomial $Z_{\mathrm{M}}(t)$, called the ***Z*-polynomial** of M, was introduced in [PXY18] using the first definition of $P_{\mathrm{M}}(t)$, where it was shown to satisfy the displayed identity. The degree of the Z-polynomial of M is exactly the rank of M, and its leading coefficient is 1.

Theorem 1.2. The following holds for any matroid M.

(1) The polynomial $P_{\mathrm{M}}(t)$ has nonnegative coefficients.

(2) The polynomial $Z_{\mathrm{M}}(t)$ is unimodal: The coefficient of t^k in $Z_{\mathrm{M}}(t)$ is less than or equal to the coefficient of t^{k+1} in $Z_{\mathrm{M}}(t)$ for all $k < \operatorname{rk}\mathrm{M}/2$.

The first part of Theorem 1.2 was conjectured in [EPW16, Conjecture 2.3], where it was proved for matroids realizable over some field using l-adic étale intersection cohomology theory. See Section 1.3 for an overview of that proof. For sparse paving matroids, a combinatorial proof of the nonnegativity was given in [LNR20].

Kazhdan–Lusztig polynomials of matroids are special cases of Kazhdan–Lusztig–Stanley polynomials [Sta92, Pro18]. Several important families of Kazhdan–Lusztig–Stanley polynomials turn out to have nonnegative coefficients, including classical Kazhdan–Lusztig polynomials associated with Bruhat intervals [EW14] and g-polynomials of convex polytopes [Kar04, BL05]. For more on this analogy, see Section 1.4.

For a finite group Γ acting on M, one can define the **equivariant Kazhdan–Lusztig polynomial** $P^{\Gamma}_{\mathrm{M}}(t)$ and the **equivariant *Z*-polynomial** $Z^{\Gamma}_{\mathrm{M}}(t)$; see Appendix A for formal definitions. These are polynomials with coefficients in the ring of virtual representations of Γ, with the property that taking dimensions recovers the ordinary polynomials [GPY17, PXY18]. Our proof shows the following strengthening of Theorem 1.2.

Theorem 1.3. The following holds for any matroid M and any finite group Γ acting on M.

(1) The polynomial $P^{\Gamma}_{\mathrm{M}}(t)$ has nonnegative coefficients: The coefficients of $P^{\Gamma}_{\mathrm{M}}(t)$ are isomorphism classes of honest, rather than virtual, representations of Γ.

(2) The polynomial $Z^{\Gamma}_{\mathrm{M}}(t)$ is unimodal: The coefficient of t^k in $Z^{\Gamma}_{\mathrm{M}}(t)$ is isomorphic to a subrepresentation of the coefficient of t^{k+1} in $Z^{\Gamma}_{\mathrm{M}}(t)$ for all $k < \operatorname{rk}\mathrm{M}/2$.

Theorem 1.3 specializes to Theorem 1.2 when we take Γ to be the trivial group. The first part of Theorem 1.3 was conjectured in [GPY17, Conjecture 2.13], where it was proved for matroids that are Γ-equivariantly realizable over some field.[3] For uniform matroids, a combinatorial proof of the equivariant nonnegativity was given in [GPY17, Section 3].

[3] It is much easier to construct matroids that are not Γ-equivariantly realizable than it is to construct matroids that are not realizable. For example, the uniform matroid of rank 2 on 4 elements is realizable over any field with at least three elements, but it is not S_4-equivariantly realizable over any field.

By [GX20, Theorem 1.2], there is a unique way to assign a polynomial $Q_{\mathrm{M}}(t)$ to each matroid M, called the **inverse Kazhdan–Lusztig polynomial** of M, subject to the following three conditions:

(a) If the rank of M is zero, then $Q_{\mathrm{M}}(t)$ is the constant polynomial 1.

(b) For every matroid M of positive rank, the degree of $Q_{\mathrm{M}}(t)$ is strictly less than $\operatorname{rk}\mathrm{M}/2$.

(c) For every matroid M, we have $(-t)^{\operatorname{rk}\mathrm{M}}Q_{\mathrm{M}}(t^{-1}) = \displaystyle\sum_{F\in\mathcal{L}(\mathrm{M})} (-1)^{\operatorname{rk}\mathrm{M}^F} Q_{\mathrm{M}^F}(t)\cdot t^{\operatorname{rk}\mathrm{M}_F}\chi_{\mathrm{M}_F}(t^{-1})$.

We also prove the following result conjectured in [GX20, Conjecture 4.1].

Theorem 1.4. The polynomial $Q_{\mathrm{M}}(t)$ has nonnegative coefficients.

In fact, our proof shows that the coefficients of the **equivariant inverse Kazhdan–Lusztig polynomial** $Q^{\Gamma}_{\mathrm{M}}(t)$ defined in Appendix A are isomorphism classes of honest, rather than virtual, representations of Γ.

1.2. **Proof strategy.** We now provide an outline of the proofs of Theorems 1.1, 1.2, and 1.3. The algebro-geometric motivations for these arguments will appear in Section 1.3.

For any matroid M of rank d, consider the **graded Möbius algebra**

$$\mathrm{H}(\mathrm{M}) \coloneqq \bigoplus_{F\in\mathcal{L}(\mathrm{M})} \mathbb{Q}y_F.$$

The grading is defined by declaring the degree of the element y_F to be $\operatorname{rk} F$, the rank of F in M. The multiplication is defined by the formula

$$y_F y_G \coloneqq \begin{cases} y_{F\vee G} & \text{if } \operatorname{rk} F + \operatorname{rk} G = \operatorname{rk}(F\vee G),\\ 0 & \text{if } \operatorname{rk} F + \operatorname{rk} G > \operatorname{rk}(F\vee G),\end{cases}$$

where $\vee$ stands for the join of flats in the lattice $\mathcal{L}(\mathrm{M})$. Let $\mathrm{CH}(\mathrm{M})$ be the **augmented Chow ring** of M, introduced in [BHM+20]. We will review the definition of $\mathrm{CH}(\mathrm{M})$ in Section 2, but for now it will suffice to know the following three things:

- $\mathrm{CH}(\mathrm{M})$ contains $\mathrm{H}(\mathrm{M})$ as a graded subalgebra [BHM+20, Proposition 2.15].
- $\mathrm{CH}(\mathrm{M})$ is equipped with a **degree isomorphism** $\deg\colon \mathrm{CH}^d(\mathrm{M}) \to \mathbb{Q}$ [BHM+20, Definition 2.12].
- By the Krull–Schmidt theorem, up to isomorphism, there is a unique indecomposable graded $\mathrm{H}(\mathrm{M})$-module direct summand $\mathrm{IH}(\mathrm{M}) \subseteq \mathrm{CH}(\mathrm{M})$ that contains $\mathrm{H}(\mathrm{M})$.[4]

[4]For the Krull–Schmidt theorem, see, for example, [Ati56, Theorem 1]. By [CF82, Corollary 2] or [GG82, Theorem 3.2], the indecomposability in the category of graded $\mathrm{H}(\mathrm{M})$-modules implies the indecomposability in the category of $\mathrm{H}(\mathrm{M})$-modules. Thus, the intersection cohomology of M is an indecomposable module over $\mathrm{H}(\mathrm{M})$.

In this introduction, we temporarily define the **intersection cohomology** of M to be the graded $\mathrm{H}(\mathrm{M})$-module $\mathrm{IH}(\mathrm{M})$. This defines the intersection cohomology of M up to isomorphism of graded $\mathrm{H}(\mathrm{M})$-modules. In Section 3, we will construct a canonical submodule $\mathrm{IH}(\mathrm{M}) \subseteq \mathrm{CH}(\mathrm{M})$ that is preserved by all symmetries of M. The construction of $\mathrm{IH}(\mathrm{M})$ as an explicit submodule of $\mathrm{CH}(\mathrm{M})$, or more generally the construction of the **canonical decomposition** of $\mathrm{CH}(\mathrm{M})$ as a graded $\mathrm{H}(\mathrm{M})$-module, will be essential in our proofs of the main results but not in their statements.

We fix any decomposition of the graded $\mathrm{H}(\mathrm{M})$-module $\mathrm{CH}(\mathrm{M})$ as above, and consider any positive linear combination

$$\ell = \sum_{F \in \mathcal{L}^1(\mathrm{M})} c_F y_F, \quad c_F \text{ is positive for every rank 1 flat } F \text{ of } \mathrm{M}.$$

Our central result is that $\mathrm{IH}(\mathrm{M})$ satisfies the **Kähler package** with respect to $\ell \in \mathrm{H}^1(\mathrm{M})$.

Theorem 1.5. The following holds for any matroid M of rank d.

(1) (Poincaré duality theorem) For every nonnegative $k \leqslant d/2$, the bilinear pairing

$$\mathrm{IH}^k(\mathrm{M}) \times \mathrm{IH}^{d-k}(\mathrm{M}) \longrightarrow \mathbb{Q}, \qquad (\eta_1, \eta_2) \longmapsto \deg(\eta_1\eta_2)$$

is non-degenerate.

(2) (Hard Lefschetz theorem) For every nonnegative $k \leqslant d/2$, the multiplication map

$$\mathrm{IH}^k(\mathrm{M}) \longrightarrow \mathrm{IH}^{d-k}(\mathrm{M}), \qquad \eta \longmapsto \ell^{d-2k}\eta$$

is an isomorphism.

(3) (Hodge–Riemann relations) For every nonnegative $k \leqslant d/2$, the bilinear form

$$\mathrm{IH}^k(\mathrm{M}) \times \mathrm{IH}^k(\mathrm{M}) \longrightarrow \mathbb{Q}, \qquad (\eta_1, \eta_2) \longmapsto (-1)^k \deg(\ell^{d-2k}\eta_1\eta_2)$$

is positive definite on the kernel of multiplication by ℓ^{d-2k+1}.

We now show how Theorem 1.5 implies Theorem 1.1.

Proof of Theorem 1.1, assuming Theorem 1.5. It follows from the hard Lefschetz theorem that the multiplication map $\ell^{j-k}\colon \mathrm{IH}^k(\mathrm{M}) \to \mathrm{IH}^j(\mathrm{M})$ is injective for $j \leqslant d-k$. Since $\mathrm{H}(\mathrm{M}) \subseteq \mathrm{CH}(\mathrm{M})$, we have $\mathrm{H}(\mathrm{M}) \subseteq \mathrm{IH}(\mathrm{M})$. After restricting the multiplication map to the $\mathrm{H}(\mathrm{M})$-submodule $\mathrm{H}(\mathrm{M}) \subseteq \mathrm{IH}(\mathrm{M})$, we obtain an injection

$$\ell^{j-k}\colon \ \mathrm{H}^k(\mathrm{M}) \longrightarrow \mathrm{H}^j(\mathrm{M}).$$

Taking ℓ to be the sum of all y_F over the rank 1 flats F, we obtain part (3). If we write this injection as a matrix in terms of the natural bases, the matrix is supported on the pairs satisfying $F \leqslant G$. Part (2) follows from the existence of a nonzero term in a maximal minor for this matrix. Clearly, part (1) follows from either part (2) or part (3). ☐

The following propositions will be key ingredients in the proof of Theorem 1.2. We write $\mathfrak{m}$ for the graded maximal ideal of $\mathrm{H}(\mathrm{M})$, write $\mathbb{Q}$ for the one-dimensional graded $\mathrm{H}(\mathrm{M})$-module in degree zero, and write $\mathrm{IH}(\mathrm{M})_\varnothing$ for the graded vector space

$$\mathrm{IH}(\mathrm{M}) \otimes_{\mathrm{H}(\mathrm{M})} \mathbb{Q} \cong \mathrm{IH}(\mathrm{M})/\mathfrak{m}\,\mathrm{IH}(\mathrm{M}).$$

Proposition 1.6. For every matroid M of positive rank, $\mathrm{IH}(\mathrm{M})_\varnothing$ vanishes in degrees $\geqslant \mathrm{rk}\,\mathrm{M}/2$.

Proposition 1.7. For all nonnegative k, there is a canonical graded vector space isomorphism

$$\mathfrak{m}^k\,\mathrm{IH}(\mathrm{M})/\mathfrak{m}^{k+1}\,\mathrm{IH}(\mathrm{M}) \cong \bigoplus_{F\in\mathcal{L}^k(\mathrm{M})} \mathrm{IH}(\mathrm{M}_F)_\varnothing[-k].$$

For the content of the word "canonical" in Proposition 1.7, we refer to the explicit construction of the isomorphism in Section 12.3. For a geometric description in the realizable case, see Section 1.3. When a finite group Γ acts on M, it acts on the intersection cohomology of M, and the isomorphism is that of Γ-representations

$$\mathfrak{m}^k\,\mathrm{IH}(\mathrm{M})/\mathfrak{m}^{k+1}\,\mathrm{IH}(\mathrm{M}) \cong \bigoplus_{F\in\mathcal{L}^k(\mathrm{M})} \frac{|\Gamma_F|}{|\Gamma|}\,\mathrm{Ind}_{\Gamma_F}^{\Gamma}\,\mathrm{IH}(\mathrm{M}_F)_\varnothing[-k],$$

where $\Gamma_F \subseteq \Gamma$ is the subgroup of elements fixing F.[5]

Proofs of Theorems 1.2 and 1.3, assuming Theorem 1.5 and Propositions 1.6 and 1.7. We define polynomials

$$\tilde{P}_\mathrm{M}(t) \coloneqq \sum_{k\geqslant 0} \dim\left(\mathrm{IH}^k(\mathrm{M})_\varnothing\right) t^k \ \text{ and } \ \tilde{Z}_\mathrm{M}(t) \coloneqq \sum_{k\geqslant 0} \dim\left(\mathrm{IH}^k(\mathrm{M})\right) t^k.$$

We argue $\tilde{P}_\mathrm{M}(t) = P_\mathrm{M}(t)$ and $\tilde{Z}_\mathrm{M}(t) = Z_\mathrm{M}(t)$ by induction on the rank of M. The statement is clear when the rank is zero, so assume that M has positive rank and that the statement holds for matroids of strictly smaller rank. Taking Poincaré polynomials of the graded vector spaces in Proposition 1.7 and summing over all k, we get

$$\tilde{Z}_\mathrm{M}(t) = \sum_{F\in\mathcal{L}(\mathrm{M})} t^{\mathrm{rk}\,F}\,\tilde{P}_{\mathrm{M}_F}(t).$$

When combined with our inductive hypothesis, the above gives

$$\tilde{Z}_\mathrm{M}(t) = \tilde{P}_\mathrm{M}(t) + \sum_{F\neq\varnothing} t^{\mathrm{rk}\,F} P_{\mathrm{M}_F}(t).$$

On the other hand, by Theorem 1.5 and Proposition 1.6, we have

$$\tilde{Z}_\mathrm{M}(t) = t^{\mathrm{rk}\,\mathrm{M}}\tilde{Z}_\mathrm{M}(t^{-1}) \ \text{ and } \ \deg \tilde{P}_\mathrm{M}(t) < \mathrm{rk}\,\mathrm{M}/2$$

[5] One may eliminate the fraction $|\Gamma_F|/|\Gamma|$ at the cost of choosing one representative of each Γ-orbit in $\mathcal{L}^k(\mathrm{M})$.

The desired identities now follow from the second definition of Kazhdan–Lusztig polynomials of matroids given above [BV20, Theorem 2.2].

The nonnegativity of the coefficients of $P_{\mathrm{M}}(t)$ is immediate from the fact that it is the Poincaré polynomial of a graded vector space. The unimodality of $Z_{\mathrm{M}}(t)$ follows from the hard Lefschetz theorem for $\mathrm{IH}(\mathrm{M})$. All of the steps of this argument still hold when interpreted equivariantly with respect to any group of symmetries of M by Lemma A.1, Definition A.3, and Corollary A.5. □

We record the numerical identities for $P_{\mathrm{M}}(t)$ and $Z_{\mathrm{M}}(t)$ obtained in the above proof.

Theorem 1.8. For any matroid M, we have

$$P_{\mathrm{M}}(t) = \sum_{k \geqslant 0} \dim \left(\mathrm{IH}^k(\mathrm{M})_\varnothing\right) t^k \ \text{ and } \ Z_{\mathrm{M}}(t) = \sum_{k \geqslant 0} \dim \left(\mathrm{IH}^k(\mathrm{M})\right) t^k.$$

When a finite group Γ acts on M, the analogous identities hold for $P^\Gamma_{\mathrm{M}}(t)$ and $Z^\Gamma_{\mathrm{M}}(t)$.

Remark 1.9. The explicit construction of $\mathrm{IH}(\mathrm{M})$ as a submodule of $\mathrm{CH}(\mathrm{M})$ appears in Section 3, but the fact that it is an indecomposable summand of $\mathrm{CH}(\mathrm{M})$ is not established until much later. It follows from Proposition 6.5, which can only be applied after we have proved Theorem 1.5. See Remark 6.1 for why this is the case.

Remark 1.10. The astute reader will note that the only part of Theorem 1.5 that appears in the applications is the hard Lefschetz theorem. However, we know of no way to prove the hard Lefschetz theorem by itself. Instead, we roll all three statements up into a grand induction. See Remark 1.14 for more on this philosophy.

Remark 1.11. We have not yet commented on our strategy for proving Theorem 1.4. This proof will also rely on Theorem 1.5, and will proceed by interpreting $Q_{\mathrm{M}}(t)$ as the graded multiplicity of the trivial graded $\mathrm{H}(\mathrm{M})$-module in a complex of $\mathrm{H}(\mathrm{M})$-modules called the **small Rouquier complex**. See Sections 4.2 and 8.6 for more details.

1.3. **The realizable case.** We now give the geometric motivation for the statements in Sections 1.1 and 1.2, and in particular review the proofs of Theorems 1.1 and 1.2 for realizable matroids.

Let V be a vector space of dimension d over a field $\mathbb{F}$, let E be a finite set, and let $\sigma\colon E \to V^\vee$ be a map whose image spans the dual vector space $V^\vee$. The collection of subsets $S \subseteq E$ for which σ is injective on S and $\sigma(S)$ is a linearly independent set in $V^\vee$ forms the independent sets of a matroid M of rank d. Any matroid which arises in this way is called **realizable** over $\mathbb{F}$, and σ is called a **realization** of M over $\mathbb{F}$.[6] We continue to assume that M is loopless. In terms of the realization σ, this means that the image of σ does not contain the zero vector.

[6]When a finite group Γ acts on M, we say that M is **Γ-equivariantly realizable** over $\mathbb{F}$ if there is a Γ-equivariant map $\sigma\colon E \to V^\vee$ for some representation V of Γ over $\mathbb{F}$.

For any flat F of M, let $V_F \subseteq V$ be the subspace perpendicular to $\{\sigma(e)\}_{e\in F}$, and let V^F be the quotient space V/V_F. Then we have canonical maps

$$\sigma^F\colon F \to (V^F)^\vee \text{ and } \sigma_F\colon E\backslash F \to (V_F)^\vee$$

realizing the localization M^F and the contraction M_F, respectively.

Consider the linear map $V \to \mathbb{F}^E$ whose e-th coordinate is given by $\sigma(e)$. The assumption that the image of σ spans $V^\vee$ implies that this map is injective. The decomposition $\mathbb{P}^1_{\mathbb{F}} = \mathbb{F} \sqcup \{\infty\}$ gives an embedding of $\mathbb{F}^E$ into $(\mathbb{P}^1_{\mathbb{F}})^E$, and we let $Y \subseteq (\mathbb{P}^1_{\mathbb{F}})^E$ denote the closure of the image of V. This projective variety is called the **Schubert variety** of σ. The terminology is chosen to suggest that Y has many similarities to classical Schubert varieties. It has a stratification by affine spaces, whose strata are the orbits of the additive group V on Y, indexed by flats of M. For any flat F of M, let

$$U^F \coloneqq \{p \in Y \mid p_e = \infty \text{ if and only if } e \notin F\}.$$

For example, U^E is the vector space V and $U^\varnothing$ is the point ∞^E. More generally, U^F is isomorphic to V^F, and these subvarieties form a stratification of Y with U^F contained in the closure of U^G if and only if F is contained in G [PXY18, Lemmas 7.5 and 7.6].

The Schubert variety Y is singular, and it admits a canonical resolution X called the **augmented wonderful variety**, obtained by first blowing up the point $U^\varnothing$, then the proper transforms of the closures of U^F for all rank 1 flats F, and so on. A different description of X as an iterated blow-up of a projective space appears in [BHM$^+$20, Section 2.4].

For the remainder of this section, we will assume for simplicity that $\mathbb{F} = \mathbb{C}$; see Remark 1.12 for a discussion of what happens over other fields. The rings and modules introduced in Section 1.2 have the following interpretations in terms of the varieties X and Y. The graded Möbius algebra $\mathrm{H}(\mathrm{M})$ is isomorphic to the rational cohomology ring $\mathrm{H}^\bullet(Y)$ [HW17, Theorem 14], and the augmented Chow ring $\mathrm{CH}(\mathrm{M})$ is isomorphic to the rational Chow ring of X, or equivalently to the rational cohomology ring $\mathrm{H}^\bullet(X)$. By applying the decomposition theorem to the map from X to Y, we find that the intersection cohomology $\mathrm{IH}^\bullet(Y)$ is isomorphic as a graded $\mathrm{H}^\bullet(Y)$-module to a direct summand of $\mathrm{H}^\bullet(X)$.[7] A slight extension of an argument of Ginzburg [Gin91] shows that $\mathrm{IH}^\bullet(Y)$ is indecomposable as an $\mathrm{H}^\bullet(Y)$-module, which implies that it coincides with our module $\mathrm{IH}(\mathrm{M})$.[8] Theorem 1.5 is a standard result in Hodge theory for singular projective varieties.

[7] All of these cohomology rings and intersection cohomology groups of varieties vanish in odd degree, and our isomorphisms double degree. So $\mathrm{H}^1(\mathrm{M}) \cong \mathrm{H}^2(Y)$, $\mathrm{CH}^1(\mathrm{M}) \cong \mathrm{H}^2(X)$, $\mathrm{IH}^1(\mathrm{M}) \cong \mathrm{IH}^2(Y)$, and so on.

[8] To be precise, two hypotheses of [Gin91] are not satisfied by Y: it is not the closure of a Białynicki-Birula cell for a torus action on a smooth projective variety, and the natural torus which acts is one-dimensional, so it is not possible to find an attracting cocharacter at each fixed point. However, each fixed point has an affine neighborhood with an attracting action of the multiplicative group, and this is enough.

For each flat F of M, let $\mathrm{H}^\bullet(\mathrm{IC}_{Y,F})$ denote the cohomology of the stalk of the intersection cohomology complex IC_Y at a point in U^F. The restriction map on global sections from $\mathrm{IH}^\bullet(Y)$ to $\mathrm{H}^\bullet(\mathrm{IC}_{Y,\varnothing})$ descends to $\mathrm{IH}^\bullet(Y)_\varnothing$, and another application of the result of [Gin91] implies that the induced map from $\mathrm{IH}^\bullet(Y)_\varnothing$ to $\mathrm{H}^\bullet(\mathrm{IC}_{Y,\varnothing})$ is an isomorphism. A fundamental property of the intersection cohomology sheaf IC_Y is that, if the dimension d of Y is positive, then the stalk cohomology group $\mathrm{H}^{2k}(\mathrm{IC}_{Y,F})$ vanishes for $k \geqslant d$. This proves Proposition 1.6 in the realizable case.

Let Y_F be the Schubert variety associated with the realization σ_F of M_F. We have a canonical inclusion $Y_F \to Y$, which is a normally nonsingular slice to the stratum U^F. Thus it induces an isomorphism from $\mathrm{H}^\bullet(\mathrm{IC}_{Y,F})$ to $\mathrm{H}^\bullet(\mathrm{IC}_{Y_F,\varnothing})$, see [Pro18, Proposition 4.11]. Let $j_F\colon U^F \to Y$ denote the inclusion of the stratum U^F. Our stratification of Y induces a spectral sequence with

$$E_1^{p,q} = \bigoplus_{F \in \mathcal{L}^p(\mathrm{M})} \mathrm{H}_c^{p+q}(j_F^* \,\mathrm{IC}_Y)$$

that converges to $\mathrm{IH}^\bullet(Y)$. The summands of $E_1^{p,q}$ satisfy

$$\mathrm{H}_c^{p+q}(j_F^*\,\mathrm{IC}_Y) \cong \Big(\mathrm{H}^\bullet(\mathrm{IC}_{Y,F}) \otimes \mathrm{H}_c^\bullet(U_F)\Big)^{p+q} \cong \Big(\mathrm{H}^\bullet(\mathrm{IC}_{Y,F})[-2p]\Big)^{p+q} \cong \mathrm{H}^{q-p}(\mathrm{IC}_{Y_F,\varnothing}).$$

Since $\mathrm{H}^\bullet(\mathrm{IC}_{Y_F,\varnothing})$ vanishes in odd degree, our spectral sequence degenerates at the E_1 page [PXY18, Section 7]. This means that $\mathrm{IH}^\bullet(Y)$ vanishes in odd degree, and that the degree $2k$ part of the graded vector space $\mathfrak{m}^p\,\mathrm{IH}^\bullet(Y)/\mathfrak{m}^{p+1}\,\mathrm{IH}^\bullet(Y)$ is isomorphic to

$$E_\infty^{p,2k-p} = E_1^{p,2k-p} \cong \bigoplus_{F \in \mathcal{L}^p(\mathrm{M})} \mathrm{H}^{2(k-p)}(\mathrm{IC}_{Y_F,\varnothing}) \cong \bigoplus_{F \in \mathcal{L}^p(\mathrm{M})} \mathrm{IH}^{2(k-p)}(Y_F)_\varnothing.$$

This proves Proposition 1.7 in the realizable case.

Remark 1.12. If the field $\mathbb{F}$ is not equal to the complex numbers, then we can mimic all of the geometric arguments in this section using l-adic étale cohomology for some prime l not equal to the characteristic of $\mathbb{F}$. In this setting there is no geometric analogue of the Hodge–Riemann relations, so Hodge theory does not give us the full Kähler package of Theorem 1.5. It is interesting to note that Theorem 1.5 gives us a truly new result for matroids that are realizable only in positive characteristic. Namely, it says that there is a rational form for the l-adic étale intersection cohomology of the Schubert variety for which the Hodge–Riemann relations hold. We suspect that $\mathrm{IH}(\mathrm{M})$ is a Chow analogue of the intersection cohomology of Y.

Remark 1.13. If one wants to write down a maximally streamlined proof of Theorem 1.1 for realizable matroids, it is not necessary to know that $\mathrm{H}^\bullet(Y)$ is isomorphic to the graded Möbius algebra of M, and it is not necessary to consider the augmented wonderful variety X or the augmented Chow ring of M. One considers $\mathrm{IH}^\bullet(Y)$ as a module over $\mathrm{H}^\bullet(Y)$ and applies the same argument outlined in Section 1.2. The statements that $\mathrm{IH}^\bullet(Y)$ contains $\mathrm{H}^\bullet(Y)$ as a submodule, that $\mathrm{H}^\bullet(Y)$ has a basis indexed by flats, and that the matrix for the multiplication by a power of an ample class in this basis is supported on pairs $F \leqslant G$ follow from [BE09, Theorem 2.1, Theorem 3.1, and Lemma 5.1]. For the proof of Theorem 1.2, we need to know that the cohomology groups $\mathrm{H}^\bullet(\mathrm{IC}_{Y,F})$ vanish in odd degree in order to conclude that the spectral sequence degenerates. To see this, we can either embed $\mathrm{IH}^\bullet(Y)$ in $\mathrm{H}^\bullet(X)$ as in the text above, or we can rely on an inductive argument as in [Pro18, Theorem 3.6].

1.4. **Kazhdan–Lusztig–Stanley polynomials.** In this section, we will discuss two antecedents to our work in the context of Kazhdan–Lusztig–Stanley theory. Let P be a locally finite ranked poset. For all $x \leqslant y \in P$, let $r_{xy} \coloneqq \operatorname{rk} y - \operatorname{rk} x$. A ***P*-kernel** is a collection of polynomials $\kappa_{xy}(t) \in \mathbb{Z}[t]$ for each $x \leqslant y \in P$ satisfying the following conditions:

- For all $x \in P$, $\kappa_{xx}(t) = 1$.
- For all $x \leqslant y \in P$, $\deg \kappa_{xy}(t) \leqslant r_{xy}$.
- For all $x < z \in P$, $\displaystyle\sum_{x \leqslant y \leqslant z} t^{r_{xy}} \kappa_{xy}(t^{-1}) \kappa_{yz}(t) = 0$.

Given such a collection of polynomials, Stanley [Sta92] showed that there exists a unique collection of polynomials $f_{xy}(t) \in \mathbb{Z}[t]$ for each $x \leqslant y \in P$ satisfying the following conditions:

- For all $x \in P$, $f_{xx}(t) = 1$.
- For all $x < y \in P$, $\deg f_{xy}(t) < r_{xy}/2$.
- For all $x \leqslant z \in P$, $t^{r_{xz}} f_{xz}(t^{-1}) = \displaystyle\sum_{x \leqslant y \leqslant z} \kappa_{xy}(t) f_{yz}(t)$.

The polynomials $f_{xy}(t)$ are called **Kazhdan–Lusztig–Stanley polynomials**.

The first motivation for this construction comes from classical Kazhdan–Lusztig polynomials. If we take the poset to be a Coxeter group W equipped with the Bruhat order and the W-kernel to be the R-polynomials $R_{xy}(t)$, then the polynomials $f_{xy}(t)$ are called **Kazhdan–Lusztig polynomials**. These polynomials were introduced by Kazhdan and Lusztig in [KL79], where they were conjectured to have nonnegative coefficients. This was proved for Weyl groups in [KL80] by interpreting $f_{xy}(t)$ as the Poincaré polynomial for a stalk of the intersection cohomology sheaf of a classical Schubert variety. For arbitrary Coxeter groups, the conjecture remained open for 34 years before it was proved by Elias and Williamson [EW14], who used Soergel bimodules as a combinatorial replacement for intersection cohomology groups of classical Schubert varieties.

The second motivation for this definition comes from convex polytopes. Let Δ be a convex polytope, and let P be the poset of faces of Δ, ordered by reverse inclusion and ranked by codimension, with the convention that the codimension of the empty face is $\dim \Delta + 1$. This poset is Eulerian, which means that the polynomials $(t-1)^{r_{xy}}$ form a P-kernel. The polynomial $g_\Delta(t) \coloneqq f_{\Delta\varnothing}(t)$ is called the ***g*-polynomial** of Δ. When Δ is rational, this polynomial can be shown to have nonnegative coefficients by interpreting it as the Poincaré polynomial for a stalk of the intersection cohomology sheaf of a toric variety [DL91, Fie91]. For arbitrary convex polytopes, nonnegativity of the coefficients of the g-polynomial was proved 13 years later by Karu [Kar04], who used the theory of combinatorial intersection cohomology of fans [BBFK02, BL03, Bra06] as a replacement for intersection cohomology groups of toric varieties.

In our setting, we consider the ranked poset $\mathcal{L}(\mathrm{M})$ along with the $\mathcal{L}(\mathrm{M})$-kernel consisting of the characteristic polynomials $\chi_{FG}(t) := \chi_{\mathrm{M}^G_F}(t)$, and we find that $f_{\varnothing E}(t)$ is equal to the Kazhdan–Lusztig polynomial $P_{\mathrm{M}}(t)$. When M is realizable, this polynomial can be shown to have nonnegative coefficients by interpreting it as the Poincaré polynomial for a stalk of the intersection cohomology sheaf of the Schubert variety Y, as explained in Section 1.3. Theorem 1.2 is obtained by using $\mathrm{IH}(\mathrm{M})$ as a replacement for the intersection cohomology group of Y.

Remark 1.14. It is reasonable to ask to what extent these three nonnegativity results can be unified. In the geometric setting (Weyl groups, rational polytopes, realizable matroids), it is possible to write down a general theorem that has each of these results as a special case [Pro18, Theorem 3.6]. However, the problem of finding algebraic or combinatorial replacements for the intersection cohomology groups of stratified algebraic varieties is not one for which we have a general solution. Each of the three theories described above involves numerous details that are unique to that specific case. The one insight that we can take away is that, while the hard Lefschetz theorem is typically the main statement needed for applications, it is always necessary to prove Poincaré duality, the hard Lefschetz theorem, and the Hodge–Riemann relations together as a single package.

Remark 1.15. The analogue of Theorem 1.1 for Weyl groups appears in [BE09], and for general Coxeter groups (using Soergel bimodules) in [MS20]. There is no analogous result for convex polytopes because toric varieties associated with non-simple polytopes do not in general admit stratifications by affine spaces.

Remark 1.16. For a locally finite poset P, consider the incidence algebra

$$I(P) := \prod_{x \leqslant y \in P} \mathbb{Z}[t], \quad \text{where } (uv)_{xz}(t) := \sum_{x \leqslant y \leqslant z} u_{xy}(t) v_{yz}(t) \text{ for } u, v \in I(P).$$

An element $h \in I(P)$ has an inverse, left or right, if and only if $h_{xx}(t) = \pm 1$ for all $x \in P$. In this case, the left and right inverses are unique and they coincide [Pro18, Lemma 2.1]. In terms of the incidence algebra, the inverse Kazhdan–Lusztig polynomial of M can be interpreted as

$$Q_{\mathrm{M}}(t) = (-1)^{\operatorname{rk} \mathrm{M}} f^{-1}_{\varnothing E}(t),$$

where f is the Kazhdan–Lusztig polynomial viewed as an element of $I(\mathcal{L}(\mathrm{M}))$. We note that the analogous constructions for finite Coxeter groups and convex polytopes do not give us anything new. Specifically, for a finite Coxeter group, we have

$$(-1)^{r_{xy}} f_{xy}^{-1}(t) = f_{(w_0 y)(w_0 x)}(t),$$

where $w_0 \in W$ is the longest word [Pro18, Example 2.12]. For a convex polytope, we have

$$(-1)^{\dim \Delta + 1} f_{\Delta\varnothing}^{-1}(t) = g_{\Delta^*}(t),$$

where Δ^* is the dual polytope of Δ [Pro18, Example 2.14]. The explanation for these statements is that the corresponding P-kernels are **alternating** [Pro18, Proposition 2.11], which means that $(-t)^{r_{xy}}\kappa_{xy}(t^{-1}) = \kappa_{xy}(t)$. The same is not true for characteristic polynomials, which is why inverse Kazhdan–Lusztig polynomials of matroids are fundamentally different from ordinary Kazhdan–Lusztig polynomials of matroids.

1.5. **Outline.** In Section 2, we recall the definitions of the Chow ring and the augmented Chow ring of a matroid, then we review properties established in [BHM$^+$20] of various pushforward and pullback maps between these rings. In Section 3, we define the intersection cohomology modules of matroids, explain how these modules behave under the pullback and pushforward maps, and define the host of statements that make up our main inductive proof.

With all the key players defined, we provide Section 4 as a guide to the inductive proof of the main theorem of the paper, Theorem 3.17. No definitions or proofs are given here, and the section is meant only to provide intuition for the structure of the proof. This section may be skipped, but we hope that the reader benefits from flipping back to this section to "see what the authors were thinking" as they read the rest of the paper.

The proof of the main theorem begins in Section 5 and continues for the remainder of the paper. We use Sections 5 and 6 to establish some general results about modules over the graded Möbius algebra, and in particular about the intersection cohomology modules. The results in Section 5 are not inductive in nature and are established outside of the inductive loop. Section 7 studies the Poincaré pairings on various $\mathrm{H}(\mathrm{M})$-submodules of $\mathrm{CH}(\mathrm{M})$ and how they behave under linear-algebraic operations such as tensor products. Section 8 is dedicated to introducing and studying the so-called Rouquier complexes; as in [EW14], we use these to prove a version of weak Lefschetz, which for us is a certain vanishing condition for the socles of our intersection cohomology modules. In Section 8.6, we explain how Theorem 3.17 can be used to deduce Theorem 1.4. Sections 9 and 10 use the semi-small decomposition developed in [BHM$^+$20] to perform an induction involving the deletion $\mathrm{M}\backslash i$ of a single element i from M. Section 11 explores how the hard Lefschetz theorem and Hodge–Riemann relations behave when deforming Lefschetz operators. Section 12 puts all of the results from the previous sections together to finish the inductive proof of Theorem 3.17, from which Theorems 1.1, 1.2, and 1.5 follow. Finally, the appendix establishes the framework needed to deduce Theorem 1.3 as well as the equivariant part of Theorem 1.1.

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2. The Chow ring and the augmented Chow ring of a matroid

For the remainder of this paper, we write d for the rank of M and n for the cardinality of E. We continue to assume that M is a loopless matroid on E. Under this assumption, n is positive if and only if d is positive.

2.1. **Definitions of the rings.** We recall the definitions of the Chow ring of a matroid introduced in [FY04] and the augmented Chow ring of a matroid introduced in [BHM+20]. To each matroid M on E, we assign two polynomial rings with rational coefficients

$$\underline{S}_{\mathrm{M}} \coloneqq \mathbb{Q}[x_F \,|\, F \text{ is a nonempty proper flat of } \mathrm{M}] \text{ and}$$
$$S_{\mathrm{M}} \coloneqq \mathbb{Q}[x_F \,|\, F \text{ is a proper flat of } \mathrm{M}] \otimes \mathbb{Q}[y_i \,|\, i \text{ is an element of } E].$$

Definition 2.1. The **Chow ring** of M is the quotient algebra

$$\underline{\mathrm{CH}}(\mathrm{M}) \coloneqq \underline{S}_{\mathrm{M}}/(\underline{I}_{\mathrm{M}} + \underline{J}_{\mathrm{M}}),$$

where $\underline{I}_{\mathrm{M}}$ is the ideal generated by the linear forms

$$\sum_{i_1 \in F} x_F - \sum_{i_2 \in F} x_F, \quad \text{for every pair of distinct elements } i_1 \text{ and } i_2 \text{ of } E,$$

and $\underline{J}_{\mathrm{M}}$ is the ideal generated by the quadratic monomials

$$x_{F_1} x_{F_2}, \quad \text{for every pair of incomparable nonempty proper flats } F_1 \text{ and } F_2 \text{ of } \mathrm{M}.$$

When d is positive, the Chow ring of M is the Chow ring of an $(n-1)$-dimensional smooth toric variety defined by a $(d-1)$-dimensional fan $\underline{\Pi}_{\mathrm{M}}$, called the **Bergman fan** of M [FY04, Theorem 3].

Definition 2.2. The **augmented Chow ring** of M is the quotient algebra

$$\mathrm{CH}(\mathrm{M}) \coloneqq S_{\mathrm{M}}/(I_{\mathrm{M}} + J_{\mathrm{M}}),$$

where I_{M} is the ideal generated by the linear forms

$$y_i - \sum_{i \notin F} x_F, \quad \text{for every element } i \text{ of } E,$$

and J_{M} is the ideal generated by the quadratic monomials

$$x_{F_1} x_{F_2}, \quad \text{for every pair of incomparable proper flats } F_1 \text{ and } F_2 \text{ of } \mathrm{M}, \text{ and}$$
$$y_i\, x_F, \quad \text{for every element } i \text{ of } E \text{ and every proper flat } F \text{ of } \mathrm{M} \text{ not containing } i.$$

The augmented Chow ring of M is the Chow ring of an n-dimensional smooth toric variety defined by a d-dimensional fan Π_{M}, called the **augmented Bergman fan** of M [BHM+20, Proposition 2.10]. Note that the Chow ring is isomorphic to the quotient of the augmented Chow ring by the ideal generated by all the elements y_i, and that two elements y_i and y_j are equal if and only if i and j are contained in the same rank 1 flat of M.

By [BHM+20, Proposition 2.15], there is a unique graded algebra homomorphism

$$\mathrm{H}(\mathrm{M}) \longrightarrow \mathrm{CH}(\mathrm{M}), \qquad y_{\bar{i}} \longmapsto y_i,$$

where $\bar{i}$ denotes the unique rank 1 flat of M containing an element i of E, and this homomorphism is injective. Thus, we may identify the graded Möbius algebra with the subalgebra of the augmented Chow ring generated by the y_is. One of the principal goals of this paper is to understand the $\mathrm{H}(\mathrm{M})$-module structure of $\mathrm{CH}(\mathrm{M})$. The Chow ring $\underline{\mathrm{CH}}(\mathrm{M})$ will play an important supporting role.

The description of $\underline{\mathrm{CH}}(\mathrm{M})$ in terms of $\underline{\Pi}_{\mathrm{M}}$ reveals that $\underline{\mathrm{CH}}(\mathrm{M})$ vanishes in degrees $\geqslant d$. Similarly, the description of $\mathrm{CH}(\mathrm{M})$ in terms of Π_{M} reveals that $\mathrm{CH}(\mathrm{M})$ vanishes in degrees $> d$. Furthermore, one can construct distinguished isomorphisms from the graded pieces $\underline{\mathrm{CH}}^{d-1}(\mathrm{M})$ and $\mathrm{CH}^d(\mathrm{M})$ to $\mathbb{Q}$.

Definition 2.3. Let M be a loopless matroid of rank d.

(1) When d is positive, we define the **degree map** for $\underline{\mathrm{CH}}(\mathrm{M})$ to be the unique linear map

$$\underline{\deg}_{\mathrm{M}} \colon \underline{\mathrm{CH}}^{d-1}(\mathrm{M}) \longrightarrow \mathbb{Q}, \qquad \prod_{F \in \underline{\mathscr{F}}} x_F \longmapsto 1,$$

where $\underline{\mathscr{F}}$ is any complete flag of nonempty proper flats of M.

(2) We define the **degree map** for $\mathrm{CH}(\mathrm{M})$ to be the unique linear map

$$\deg_{\mathrm{M}} \colon \mathrm{CH}^d(\mathrm{M}) \longrightarrow \mathbb{Q}, \qquad \prod_{F \in \mathscr{F}} x_F \longmapsto 1,$$

where $\mathscr{F}$ is any complete flag of proper flats of M.

By [BHM+20, Proposition 2.8], these maps are unique, well-defined, and bijective.

2.2. **The pullback and pushforward maps.** In this subsection, we assume that E is nonempty. Before recalling the definitions of the pullback and pushforward maps, we need the Chow classes α, $\underline{\alpha}$, and $\underline{\beta}$, defined as

$$\alpha = \alpha_{\mathrm{M}} \coloneqq \sum_G x_G \in \mathrm{CH}^1(\mathrm{M}),$$

where the sum is over all proper flats G of M, and

$$\underline{\alpha} = \underline{\alpha}_{\mathrm{M}} \coloneqq \sum_{i \in G} x_G \in \underline{\mathrm{CH}}^1(\mathrm{M}),$$

where the sum is over all nonempty proper flats G of M containing a given element i in E, and

$$\underline{\beta} = \underline{\beta}_{\mathrm{M}} := \sum_{i \notin G} x_G \in \underline{\mathrm{CH}}^1(\mathrm{M}),$$

where the sum is over all nonempty proper flats G of M not containing a given element i in E. The linear relations defining $\underline{\mathrm{CH}}(\mathrm{M})$ show that $\underline{\alpha}$ and $\underline{\beta}$ do not depend on the choice of i. Note that the natural map from $\mathrm{CH}(\mathrm{M})$ to $\underline{\mathrm{CH}}(\mathrm{M})$ takes α to $\underline{\alpha}$ and $-x_\varnothing$ to $\underline{\beta}$.

Let F be a proper flat of M. The following definition is motivated by the geometry of augmented Bergman fans [BHM+20, Propositions 2.17 and 2.18].

Definition 2.4. The **pullback** φ^F_{M} is the unique surjective graded algebra homomorphism

$$\mathrm{CH}(\mathrm{M}) \longrightarrow \underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F)$$

that satisfies the following properties:

- If G is a flat properly contained in F, then $\varphi^F_{\mathrm{M}}(x_G) = 1 \otimes x_G$.
- If G is a flat properly containing F, then $\varphi^F_{\mathrm{M}}(x_G) = x_{G \setminus F} \otimes 1$.
- If G is a flat incomparable to F, then $\varphi^F_{\mathrm{M}}(x_G) = 0$.
- If G is the flat F, then $\varphi^F_{\mathrm{M}}(x_F) = -1 \otimes \alpha_{\mathrm{M}^F} - \underline{\beta}_{\mathrm{M}_F} \otimes 1$.

The **pushforward** ψ^F_{M} is the unique degree one linear map

$$\underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F) \longrightarrow \mathrm{CH}(\mathrm{M})$$

that maps the monomial $\prod_{F'} x_{F' \setminus F} \otimes \prod_{F''} x_{F''}$ to the monomial $x_F \prod_{F'} x_{F'} \prod_{F''} x_{F''}$.

Of particular importance will be the pullback $\varphi^\varnothing_{\mathrm{M}}$, which is a surjective graded algebra homomorphism from $\mathrm{CH}(\mathrm{M})$ to $\underline{\mathrm{CH}}(\mathrm{M})$. The following results can be found in [BHM+20, Section 2].

Proposition 2.5. The pullback φ^F_{M} and the pushforward ψ^F_{M} have the following properties:

(1) If i is an element of F, then $\varphi^F_{\mathrm{M}}(y_i) = 1 \otimes y_i$.

(2) If i is not an element of F, then $\varphi^F_{\mathrm{M}}(y_i) = 0$.

(3) The equality $\varphi^F_{\mathrm{M}}(\alpha_{\mathrm{M}}) = \underline{\alpha}_{\mathrm{M}_F} \otimes 1$ holds.

(4) The pushforward ψ^F_{M} is injective.

(5) The pushforward ψ^F_{M} commutes with the degree maps: $\underline{\deg}_{\mathrm{M}_F} \otimes \deg_{\mathrm{M}^F} = \deg_{\mathrm{M}} \circ \psi^F_{\mathrm{M}}$.

(6) The pushforward ψ^F_{M} is a homomorphism of $\mathrm{CH}(\mathrm{M})$-modules:

$$\eta \psi^F_{\mathrm{M}}(\xi) = \psi^F_{\mathrm{M}}\big(\varphi^F_{\mathrm{M}}(\eta)\xi\big) \ \text{ for any } \eta \in \mathrm{CH}(\mathrm{M}) \text{ and } \xi \in \underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F).$$

We use the pullback map to make $\underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F)$ into a module over $\mathrm{CH}(\mathrm{M})$ and $\mathrm{H}(\mathrm{M})$. By part (1) of the above proposition, $\mathrm{H}(\mathrm{M})$ acts only on the second tensor factor.

For later use, we record here the following immediate consequence of Proposition 2.5.

Lemma 2.6. For any $\eta \in \mathrm{CH}(\mathrm{M})$ and $\xi \in \underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F)$, we have

$$\deg_{\mathrm{M}}\left(\eta\psi^F_{\mathrm{M}}(\xi)\right) = \underline{\deg}_{\mathrm{M}_F} \otimes \deg_{\mathrm{M}^F}\left(\varphi^F_{\mathrm{M}}(\eta)\xi\right).$$

Since the pushforward ψ^F_{M} is injective, the statement below shows that the graded $\mathrm{CH}(\mathrm{M})$-module $\underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F)[-1]$ is isomorphic to the principal ideal of x_F in $\mathrm{CH}(\mathrm{M})$.

Proposition 2.7. The composition $\psi^F_{\mathrm{M}} \circ \varphi^F_{\mathrm{M}}\colon\ \mathrm{CH}(\mathrm{M}) \to \mathrm{CH}(\mathrm{M})$ is the multiplication by x_F.

We next introduce the analogous maps for Chow rings (rather than augmented Chow rings). Let F be a nonempty proper flat of M. The following definition is motivated by the geometry of Bergman fans [BHM+20, Propositions 2.20 and 2.21].

Definition 2.8. The **pullback** $\underline{\varphi}^F_{\mathrm{M}}$ is the unique surjective graded algebra homomorphism

$$\underline{\mathrm{CH}}(\mathrm{M}) \longrightarrow \underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F)$$

that satisfies the following properties:

- If G is a flat properly contained in F, then $\underline{\varphi}^F_{\mathrm{M}}(x_G) = 1 \otimes x_G$.
- If G is a flat properly containing F, then $\underline{\varphi}^F_{\mathrm{M}}(x_G) = x_{G\setminus F} \otimes 1$.
- If G is a flat incomparable to F, then $\underline{\varphi}^F_{\mathrm{M}}(x_G) = 0$.
- If G is the flat F, then $\underline{\varphi}^F_{\mathrm{M}}(x_F) = -1 \otimes \underline{\alpha}_{\mathrm{M}^F} - \underline{\beta}_{\mathrm{M}_F} \otimes 1$.

The **pushforward** $\underline{\psi}^F_{\mathrm{M}}$ is the unique degree one linear map

$$\underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F) \longrightarrow \underline{\mathrm{CH}}(\mathrm{M})$$

that maps the monomial $\prod_{F'} x_{F'\setminus F} \otimes \prod_{F''} x_{F''}$ to the monomial $x_F \prod_{F'} x_{F'} \prod_{F''} x_{F''}$.

The following analogue of Proposition 2.5 can be found in [BHM+20, Section 2].

Proposition 2.9. The pullback $\underline{\varphi}^F_{\mathrm{M}}$ and the pushforward $\underline{\psi}^F_{\mathrm{M}}$ have the following properties:

(1) We have $\underline{\varphi}^F_{\mathrm{M}}(\underline{\alpha}_{\mathrm{M}}) = \underline{\alpha}_{\mathrm{M}_F} \otimes 1$ and $\underline{\varphi}^F_{\mathrm{M}}(\underline{\beta}_{\mathrm{M}}) = 1 \otimes \underline{\beta}_{\mathrm{M}^F}$.

(2) The pushforward $\underline{\psi}^F_{\mathrm{M}}$ is injective.

(3) The pushforward $\underline{\psi}^F_{\mathrm{M}}$ commutes with the degree maps: $\underline{\deg}_{\mathrm{M}_F} \otimes \underline{\deg}_{\mathrm{M}^F} = \underline{\deg}_{\mathrm{M}} \circ \underline{\psi}^F_{\mathrm{M}}$.

(4) The pushforward $\underline{\psi}^F_{\mathrm{M}}$ is a homomorphism of $\underline{\mathrm{CH}}(\mathrm{M})$-modules:

$$\eta\underline{\psi}^F_{\mathrm{M}}(\xi) = \underline{\psi}^F_{\mathrm{M}}\left(\underline{\varphi}^F_{\mathrm{M}}(\eta)\xi\right) \text{ for any } \eta \in \underline{\mathrm{CH}}(\mathrm{M}) \text{ and } \xi \in \underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F).$$

The following analogue of Lemma 2.6 immediately follows from Proposition 2.9.

Lemma 2.10. For any $\eta \in \underline{\mathrm{CH}}(\mathrm{M})$ and $\xi \in \underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F)$, we have

$$\underline{\deg}_{\mathrm{M}} \left(\eta \underline{\psi}^F_{\mathrm{M}}(\xi)\right) = \underline{\deg}_{\mathrm{M}_F} \otimes \underline{\deg}_{\mathrm{M}^F} \left(\underline{\varphi}^F_{\mathrm{M}}(\eta)\xi\right).$$

Since the pushforward $\underline{\psi}^F_{\mathrm{M}}$ is injective, the statement below shows that the graded $\underline{\mathrm{CH}}(\mathrm{M})$-module $\underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F)[-1]$ is isomorphic to the principal ideal of x_F in $\underline{\mathrm{CH}}(\mathrm{M})$.

Proposition 2.11. The composition $\underline{\psi}^F_{\mathrm{M}} \circ \underline{\varphi}^F_{\mathrm{M}} \colon \underline{\mathrm{CH}}(\mathrm{M}) \to \underline{\mathrm{CH}}(\mathrm{M})$ is the multiplication by x_F.

Finally, we introduce a third flavor of pullback and pushforward maps, this time relating the augmented Chow ring of M to the augmented Chow ring of M_F for any flat F of M, with no tensor products. The notational difference is that F is now in the subscript rather than the superscript.

Definition 2.12. The **pullback** φ^{M}_F is the unique surjective graded algebra homomorphism

$$\mathrm{CH}(\mathrm{M}) \longrightarrow \mathrm{CH}(\mathrm{M}_F)$$

that satisfies the following properties:

- If G is a proper flat containing F, then $\varphi^{\mathrm{M}}_F(x_G) = x_{G \setminus F}$.
- If G is a proper flat not containing F, then $\varphi^{\mathrm{M}}_F(x_G) = 0$.

The **pushforward** ψ^{M}_F is the unique degree k linear map

$$\mathrm{CH}(\mathrm{M}_F) \longrightarrow \mathrm{CH}(\mathrm{M})$$

that maps the monomial $\prod_{F'} x_{F' \setminus F}$ to the monomial $y_F \prod_{F'} x_{F'}$.

The next results can be found in [BHM+20, Section 2].

Proposition 2.13. The pullback φ^{M}_F and the pushforward ψ^{M}_F have the following properties:

(1) If i is an element of F, then $\varphi^{\mathrm{M}}_F(y_i) = 0$.

(2) If i is not an element of F, then $\varphi^{\mathrm{M}}_F(y_i) = y_i$.

(3) The equality $\varphi^{\mathrm{M}}_F(\alpha_{\mathrm{M}}) = \alpha_{\mathrm{M}_F}$ holds.

(4) The pushforward ψ^{M}_F is injective.

(5) The pushforward ψ^{M}_F commutes with the degree maps: $\deg_{\mathrm{M}_F} = \deg_{\mathrm{M}} \circ \psi^{\mathrm{M}}_F$.

(6) The pushforward ψ^{M}_F is a homomorphism of $\mathrm{CH}(\mathrm{M})$-modules:

$$\eta \psi^{\mathrm{M}}_F(\xi) = \psi^{\mathrm{M}}_F\left(\varphi^{\mathrm{M}}_F(\eta)\xi\right) \text{ for any } \eta \in \mathrm{CH}(\mathrm{M}) \text{ and } \xi \in \mathrm{CH}(\mathrm{M}_F).$$

The following analogue of Lemmas 2.6 and 2.10 follows from Proposition 2.13.

Lemma 2.14. For any $\eta \in \mathrm{CH}(\mathrm{M})$ and $\xi \in \mathrm{CH}(\mathrm{M}_F)$, we have

$$\deg_{\mathrm{M}}\left(\eta \psi^{\mathrm{M}}_F(\xi)\right) = \deg_{\mathrm{M}_F}\left(\varphi^{\mathrm{M}}_F(\eta)\xi\right).$$

Since the pushforward ψ^{M}_F is injective, the statement below shows that the graded $\mathrm{CH}(\mathrm{M})$-module $\mathrm{CH}(\mathrm{M}_F)[-k]$ is isomorphic to the principal ideal of y_F in $\mathrm{CH}(\mathrm{M})$.

Proposition 2.15. The composition $\psi^{\mathrm{M}}_F \circ \varphi^{\mathrm{M}}_F \colon \mathrm{CH}(\mathrm{M}) \to \mathrm{CH}(\mathrm{M})$ is the multiplication by y_F.

2.3. **New lemmas.** Until now, everything that has appeared in Section 2 was proved in [BHM+20]. In this section, we state a few additional lemmas about the pushforward and pullback maps that will be needed in this paper.

The following lemma will be needed for the proof of Proposition 3.7.

Lemma 2.16. Suppose that F and G are incomparable proper flats of M. Then

$$\varphi^G_{\mathrm{M}}\psi^F_{\mathrm{M}} = 0 \ \text{ and } \ \underline{\varphi}^G_{\mathrm{M}}\underline{\psi}^F_{\mathrm{M}} = 0.$$

Proof. We only prove the first equality. The second one follows from the same arguments. By Definition 2.4 and Proposition 2.5, the pushforward ψ^G_{M} is injective and the pullback φ^F_{M} is surjective. Thus, it is sufficient to show $\psi^G_{\mathrm{M}}\varphi^G_{\mathrm{M}}\psi^F_{\mathrm{M}}\varphi^F_{\mathrm{M}} = 0$. Since the compositions $\psi^G_{\mathrm{M}}\varphi^G_{\mathrm{M}}$ and $\psi^F_{\mathrm{M}}\varphi^F_{\mathrm{M}}$ are equal to the multiplications by x_G and x_F respectively (Proposition 2.7), the assertion follows because $x_G x_F = 0$ in $\mathrm{CH}(\mathrm{M})$. □

The next lemma will be used in the proofs of Propositions 7.3, 11.4, 11.7, and 12.2.

Lemma 2.17. Let F be a proper flat of M.

(1) For any $\mu, \nu \in \underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F)$, we have

$$\deg_{\mathrm{M}}\left(\psi^F_{\mathrm{M}}\mu \cdot \psi^F_{\mathrm{M}}\nu\right) = -\underline{\deg}_{\mathrm{M}_F} \otimes \deg_{\mathrm{M}^F}\left((\underline{\beta}_{\mathrm{M}_F} \otimes 1 + 1 \otimes \alpha_{\mathrm{M}^F})\mu\nu\right).$$

(2) When F is nonempty, for any $\mu, \nu \in \underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F)$, we have

$$\underline{\deg}_{\mathrm{M}}\left(\underline{\psi}^F_{\mathrm{M}}\mu \cdot \underline{\psi}^F_{\mathrm{M}}\nu\right) = -\underline{\deg}_{\mathrm{M}_F} \otimes \underline{\deg}_{\mathrm{M}^F}\left((\underline{\beta}_{\mathrm{M}_F} \otimes 1 + 1 \otimes \underline{\alpha}_{\mathrm{M}^F})\mu\nu\right).$$

Proof. We prove only part (1); the proof of part (2) is identical. By Proposition 2.5, we have

$$\deg_{\mathrm{M}}\left(\psi^F_{\mathrm{M}}\mu \cdot \psi^F_{\mathrm{M}}\nu\right) = \underline{\deg}_{\mathrm{M}_F} \otimes \deg_{\mathrm{M}^F}\left(\varphi^F_{\mathrm{M}}\psi^F_{\mathrm{M}}\mu \cdot \nu\right).$$

Since φ^F_{M} is surjective, there exists $\nu' \in \mathrm{CH}(\mathrm{M})$ such that $\varphi^F_{\mathrm{M}}\nu' = \nu$. Then,

$$\varphi^F_{\mathrm{M}}\psi^F_{\mathrm{M}}\mu \cdot \nu = \varphi^F_{\mathrm{M}}\psi^F_{\mathrm{M}}\mu \cdot \varphi^F_{\mathrm{M}}\nu' = \varphi^F_{\mathrm{M}}(\psi^F_{\mathrm{M}}\mu \cdot \nu') = \varphi^F_{\mathrm{M}}\psi^F_{\mathrm{M}}\left(\mu \cdot \varphi^F_{\mathrm{M}}(\nu')\right) = \varphi^F_{\mathrm{M}}\psi^F_{\mathrm{M}}(\mu\nu).$$

Combining the above two equations, and applying Proposition 2.5 again, we have

$$\deg_{\mathrm{M}}\left(\psi^F_{\mathrm{M}}\mu \cdot \psi^F_{\mathrm{M}}\nu\right) = \underline{\deg}_{\mathrm{M}_F} \otimes \deg_{\mathrm{M}^F}\left(\varphi^F_{\mathrm{M}}\psi^F_{\mathrm{M}}(\mu\nu)\right) = \deg_{\mathrm{M}}\left(\psi^F_{\mathrm{M}}\varphi^F_{\mathrm{M}}\psi^F_{\mathrm{M}}(\mu\nu)\right).$$

Recall that $\varphi^F_{\mathrm{M}}(x_F) = -\underline{\beta}_{\mathrm{M}_F} \otimes 1 - 1 \otimes \alpha_{\mathrm{M}^F}$, and therefore

$$\psi^F_{\mathrm{M}}\varphi^F_{\mathrm{M}}\psi^F_{\mathrm{M}}(\mu\nu) = x_F\psi^F_{\mathrm{M}}(\mu\nu) = \psi^F_{\mathrm{M}}\big(\varphi^F_{\mathrm{M}}(x_F)\mu\nu\big) = -\psi^F_{\mathrm{M}}\big((\underline{\beta}_{\mathrm{M}_F} \otimes 1 + 1 \otimes \alpha_{\mathrm{M}^F})\mu\nu\big).$$

This implies that

$$\begin{aligned}\deg_{\mathrm{M}}\left(\psi^F_{\mathrm{M}}\mu \cdot \psi^F_{\mathrm{M}}\nu\right) &= -\deg_{\mathrm{M}}\Big(\psi^F_{\mathrm{M}}\big((\underline{\beta}_{\mathrm{M}_F} \otimes 1 + 1 \otimes \alpha_{\mathrm{M}^F})\mu\nu\big)\Big)\\ &= -\underline{\deg}_{\mathrm{M}_F} \otimes \deg_{\mathrm{M}^F}\big((\underline{\beta}_{\mathrm{M}_F} \otimes 1 + 1 \otimes \alpha_{\mathrm{M}^F})\mu\nu\big).\end{aligned}$$ □

For later use, we collect here useful commutative diagrams involving the pullback and the pushforward maps.

Lemma 2.18. Let F be a proper flat of M.

(1) The following diagram commutes:

$$\begin{tikzcd}
\underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F) \arrow[r, "\mathrm{id} \otimes \varphi^{\varnothing}_{\mathrm{M}^F}"] \arrow[d, "\psi^F_{\mathrm{M}}"] & \underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F) \arrow[r, "\mathrm{id} \otimes \psi^{\varnothing}_{\mathrm{M}^F}"] \arrow[d, "\underline{\psi}^F_{\mathrm{M}}"] & \underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F) \arrow[d, "\psi^F_{\mathrm{M}}"] \\
\mathrm{CH}(\mathrm{M}) \arrow[r, "\varphi^{\varnothing}_{\mathrm{M}}"] & \underline{\mathrm{CH}}(\mathrm{M}) \arrow[r, "\psi^{\varnothing}_{\mathrm{M}}"] & \mathrm{CH}(\mathrm{M}).
\end{tikzcd}$$

(2) More generally, for any flat $G < F$, the following diagram commutes:

$$\begin{tikzcd}
\underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F) \arrow[r, "\mathrm{id} \otimes \varphi^{G}_{\mathrm{M}^F}"] \arrow[d, "\psi^F_{\mathrm{M}}"] & \underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F_G) \otimes \mathrm{CH}(\mathrm{M}^G) \arrow[r, "\mathrm{id} \otimes \psi^{G}_{\mathrm{M}^F}"] \arrow[d, "\underline{\psi}^{F \setminus G}_{\mathrm{M}_G} \otimes \mathrm{id}"] & \underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F) \arrow[d, "\psi^F_{\mathrm{M}}"] \\
\mathrm{CH}(\mathrm{M}) \arrow[r, "\varphi^{G}_{\mathrm{M}}"] & \underline{\mathrm{CH}}(\mathrm{M}_G) \otimes \mathrm{CH}(\mathrm{M}^G) \arrow[r, "\psi^{G}_{\mathrm{M}}"] & \mathrm{CH}(\mathrm{M}).
\end{tikzcd}$$

(3) For any nonempty flat $G < F$, the following diagram commutes:

$$\begin{tikzcd}
\underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F) \arrow[r, "\mathrm{id} \otimes \underline{\varphi}^{G}_{\mathrm{M}^F}"] \arrow[d, "\underline{\psi}^F_{\mathrm{M}}"] & \underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F_G) \otimes \underline{\mathrm{CH}}(\mathrm{M}^G) \arrow[r, "\mathrm{id} \otimes \underline{\psi}^{G}_{\mathrm{M}^F}"] \arrow[d, "\underline{\psi}^{F \setminus G}_{\mathrm{M}_G} \otimes \mathrm{id}"] & \underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F) \arrow[d, "\underline{\psi}^F_{\mathrm{M}}"] \\
\underline{\mathrm{CH}}(\mathrm{M}) \arrow[r, "\underline{\varphi}^{G}_{\mathrm{M}}"] & \underline{\mathrm{CH}}(\mathrm{M}_G) \otimes \underline{\mathrm{CH}}(\mathrm{M}^G) \arrow[r, "\underline{\psi}^{G}_{\mathrm{M}}"] & \underline{\mathrm{CH}}(\mathrm{M}).
\end{tikzcd}$$

(4) For any flat $G \leqslant F$, the following diagram commutes:

$$\begin{tikzcd}
\underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F) \arrow[r, "\mathrm{id} \otimes \varphi^{\mathrm{M}^F}_{G}"] \arrow[d, "\psi^F_{\mathrm{M}}"] & \underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F_G) \arrow[r, "\mathrm{id} \otimes \psi^{\mathrm{M}^F}_{G}"] \arrow[d, "\psi^{F \setminus G}_{\mathrm{M}_G}"] & \underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F) \arrow[d, "\psi^F_{\mathrm{M}}"] \\
\mathrm{CH}(\mathrm{M}) \arrow[r, "\varphi^{\mathrm{M}}_{G}"] & \mathrm{CH}(\mathrm{M}_G) \arrow[r, "\psi^{\mathrm{M}}_{G}"] & \mathrm{CH}(\mathrm{M}).
\end{tikzcd}$$

(5) For any nonempty flat $G < F$, the following diagram commutes:

$$\begin{tikzcd}
\underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F) \arrow[rr, "\psi^F_{\mathrm{M}}"] \arrow[d, "\mathrm{id} \otimes \varphi^G_{\mathrm{M}^F}"] & & \mathrm{CH}(\mathrm{M}) \arrow[d, "\varphi^G_{\mathrm{M}}"] \\
\underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F_G) \otimes \mathrm{CH}(\mathrm{M}^G) \arrow[rr, "\underline{\psi}^{F \setminus G}_{\mathrm{M}_G} \otimes \mathrm{id}"] & & \underline{\mathrm{CH}}(\mathrm{M}_G) \otimes \mathrm{CH}(\mathrm{M}^G).
\end{tikzcd}$$

(6) For any nonempty flat $G < F$, the following diagram commutes:

$$\begin{tikzcd}
\underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F) \arrow[rr, "\underline{\psi}^F_{\mathrm{M}}"] \arrow[d, "\mathrm{id} \otimes \underline{\varphi}^G_{\mathrm{M}^F}"] & & \underline{\mathrm{CH}}(\mathrm{M}) \arrow[d, "\underline{\varphi}^G_{\mathrm{M}}"] \\
\underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F_G) \otimes \underline{\mathrm{CH}}(\mathrm{M}^G) \arrow[rr, "\underline{\psi}^{F \setminus G}_{\mathrm{M}_G} \otimes \mathrm{id}"] & & \underline{\mathrm{CH}}(\mathrm{M}_G) \otimes \underline{\mathrm{CH}}(\mathrm{M}^G).
\end{tikzcd}$$

We omit the proof, which is a straightforward computation.

2.4. **Hodge theory of the Chow ring and the augmented Chow ring.** Let $\underline{\mathcal{K}}(\mathrm{M})$ be the open cone in $\underline{\mathrm{CH}}^1(\mathrm{M})$ consisting of strictly convex piecewise linear functions on the Bergman fan $\underline{\Pi}_{\mathrm{M}}$, and let $\mathcal{K}(\mathrm{M})$ of $\mathrm{CH}(\mathrm{M})$ be the open cone in $\mathrm{CH}^1(\mathrm{M})$ consisting of strictly convex piecewise linear functions on the augmented Bergman fan Π_{M}. See [BHM+20, Section 2] for definitions of the Bergman fan $\underline{\Pi}_{\mathrm{M}}$, the augmented Bergman fan Π_{M}, and the convexity of piecewise linear functions on them. Ultimately, the only properties of $\underline{\mathcal{K}}(\mathrm{M})$ and $\mathcal{K}(\mathrm{M})$ that we will use in this paper is that they are nonempty. This fact, along with Theorems 2.19 and 2.20 and Proposition 7.10, will be used to deduce that $\underline{\mathrm{CH}}(\mathrm{M})$ and $\mathrm{CH}(\mathrm{M})$ satisfy the Hancock condition of Section 7.3.

The following results are proved in [BHM+20].

Theorem 2.19. Let M be a matroid on E, and let ℓ be any element of $\mathcal{K}(\mathrm{M})$.

(1) (Poincaré duality theorem) For every nonnegative integer $k \leqslant d/2$, the bilinear pairing

$$\mathrm{CH}^k(\mathrm{M}) \times \mathrm{CH}^{d-k}(\mathrm{M}) \longrightarrow \mathbb{Q}, \quad (\eta_1, \eta_2) \longmapsto \deg_{\mathrm{M}}(\eta_1 \eta_2)$$

is non-degenerate.

(2) (Hard Lefschetz theorem) For every nonnegative integer $k \leqslant d/2$, the multiplication map

$$\mathrm{CH}^k(\mathrm{M}) \longrightarrow \mathrm{CH}^{d-k}(\mathrm{M}), \quad \eta \longmapsto \ell^{d-2k} \eta$$

is an isomorphism.

(3) (Hodge–Riemann relations) For every nonnegative integer $k \leqslant d/2$, the bilinear form

$$\mathrm{CH}^k(\mathrm{M}) \times \mathrm{CH}^k(\mathrm{M}) \longrightarrow \mathbb{Q}, \quad (\eta_1, \eta_2) \longmapsto (-1)^k \deg_{\mathrm{M}}(\ell^{d-2k} \eta_1 \eta_2)$$

is positive definite on the kernel of the multiplication by ℓ^{d-2k+1}.

Theorem 2.20. Let $\underline{\ell}$ be any element of $\underline{\mathcal{K}}(\mathrm{M})$.

(1) (Poincaré duality theorem) For every nonnegative integer $k < d/2$, the bilinear pairing

$$\underline{\mathrm{CH}}^k(\mathrm{M}) \times \underline{\mathrm{CH}}^{d-k-1}(\mathrm{M}) \longrightarrow \mathbb{Q}, \quad (\eta_1, \eta_2) \longmapsto \underline{\deg}_{\mathrm{M}}(\eta_1\eta_2)$$

is non-degenerate.

(2) (Hard Lefschetz theorem) For every nonnegative integer $k < d/2$, the multiplication map

$$\underline{\mathrm{CH}}^k(\mathrm{M}) \longrightarrow \underline{\mathrm{CH}}^{d-k-1}(\mathrm{M}), \quad \eta \longmapsto \underline{\ell}^{d-2k-1}\eta$$

is an isomorphism.

(3) (Hodge–Riemann relations) For every nonnegative integer $k < d/2$, the bilinear form

$$\underline{\mathrm{CH}}^k(\mathrm{M}) \times \underline{\mathrm{CH}}^k(\mathrm{M}) \longrightarrow \mathbb{Q}, \quad (\eta_1, \eta_2) \longmapsto (-1)^k\underline{\deg}_{\mathrm{M}}(\underline{\ell}^{d-2k-1}\eta_1\eta_2)$$

is positive definite on the kernel of the multiplication by $\underline{\ell}^{d-2k}$.

Theorem 2.20 was first proved as the main result of [AHK18].

3. The intersection cohomology of a matroid

The purpose of this section is to define the $\mathrm{H}(\mathrm{M})$-module $\mathrm{IH}(\mathrm{M})$ along with various related objects, and to state the litany of results that will be proved in our inductive argument.

3.1. Definition of the intersection cohomology modules.

Let $\underline{\mathrm{H}}(\mathrm{M})$ be the subalgebra of $\underline{\mathrm{CH}}(\mathrm{M})$ generated by $\underline{\beta}$. For any subspace V of $\underline{\mathrm{CH}}(\mathrm{M})$, we set

$$V^\perp := \Big\{\eta \in \underline{\mathrm{CH}}(\mathrm{M}) \mid \underline{\deg}_{\mathrm{M}}(v\eta) = 0 \text{ for all } v \in V\Big\}.$$

Note that V is an $\underline{\mathrm{H}}(\mathrm{M})$-submodule if and only if $V^\perp$ is an $\underline{\mathrm{H}}(\mathrm{M})$-submodule.

We recursively construct the subspaces $\underline{\mathrm{IH}}(\mathrm{M})$ and $\underline{\mathrm{J}}(\mathrm{M})$ of $\underline{\mathrm{CH}}(\mathrm{M})$ as follows. Proposition 2.9 shows that $\underline{\psi}^F_{\mathrm{M}}\underline{\mathrm{J}}(\mathrm{M}_F) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F)$ is an $\underline{\mathrm{H}}(\mathrm{M})$-submodule of $\underline{\mathrm{CH}}(\mathrm{M})$ for every nonempty proper flat F.

Definition 3.1. Let M be a loopless matroid of positive rank d.

(1) We define the $\underline{\mathrm{H}}(\mathrm{M})$-submodule $\underline{\mathrm{IH}}(\mathrm{M})$ of $\underline{\mathrm{CH}}(\mathrm{M})$ by

$$\underline{\mathrm{IH}}(\mathrm{M}) := \left(\sum_{\varnothing < F < E} \underline{\psi}^F_{\mathrm{M}}\underline{\mathrm{J}}(\mathrm{M}_F) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F)\right)^\perp,$$

where the sum is over all nonempty proper flats F of M.

(2) We define the graded subspace $\underline{\mathrm{J}}(\mathrm{M})$ of $\underline{\mathrm{CH}}(\mathrm{M})$ by setting

$$\underline{\mathrm{J}}^k(\mathrm{M}) \coloneqq \begin{cases} \underline{\mathrm{IH}}^k(\mathrm{M}) & \text{if } k \leqslant (d-2)/2, \\ \underline{\beta}^{2k-d+2}\, \underline{\mathrm{IH}}^{d-k-2}(\mathrm{M}) & \text{if } k \geqslant (d-2)/2. \end{cases}$$

For example, when M is a rank 1 matroid, we have

$$\underline{\mathrm{CH}}(\mathrm{M}) = \underline{\mathrm{IH}}(\mathrm{M}) = \mathbb{Q} \ \text{ and } \ \underline{\mathrm{J}}(\mathrm{M}) = 0,$$

and when M is a rank 2 matroid, we have

$$\underline{\mathrm{CH}}(\mathrm{M}) = \underline{\mathrm{IH}}(\mathrm{M}) = \mathbb{Q} \oplus \mathbb{Q}\underline{\beta} \ \text{ and } \ \underline{\mathrm{J}}(\mathrm{M}) = \mathbb{Q}.$$

In Section 12, we will prove that $\underline{\mathrm{IH}}(\mathrm{M})$ satisfies the hard Lefschetz theorem with respect to $\underline{\beta}$: For every nonnegative integer $k < d/2$, the multiplication map

$$\underline{\mathrm{IH}}^k(\mathrm{M}) \longrightarrow \underline{\mathrm{IH}}^{d-k-1}(\mathrm{M}), \quad \eta \longmapsto \underline{\beta}^{d-2k-1}\eta$$

is an isomorphism. Equivalently, $\underline{\mathrm{IH}}(\mathrm{M})$ is the unique representation of the Lie algebra

$$\mathfrak{sl}_2 = \mathbb{Q}\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

such that the first matrix acts via multiplication by $\underline{\beta}$ and the second matrix acts on $\underline{\mathrm{IH}}^k(\mathrm{M})$ via multiplication by $2k-d+1$. In terms of the $\mathfrak{sl}_2$-action, we have

$$\underline{\mathrm{J}}(\mathrm{M}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot \underline{\mathrm{IH}}(\mathrm{M}).$$

Let i be an element of E, and let $\underline{\mathrm{H}}_i(\mathrm{M})$ be the subalgebra of $\underline{\mathrm{CH}}(\mathrm{M})$ generated by $\underline{\beta}$ and $x_{\{i\}}$.

Convention 3.2. We take $x_{\{i\}} = 0$ when $\{i\}$ is not a flat.

As before, V is an $\underline{\mathrm{H}}_i(\mathrm{M})$-submodule if and only if $V^\perp$ is an $\underline{\mathrm{H}}_i(\mathrm{M})$-submodule. Proposition 2.9 shows that $\underline{\psi}^F_{\mathrm{M}}\underline{\mathrm{J}}(\mathrm{M}_F) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F)$ is an $\underline{\mathrm{H}}_i(\mathrm{M})$-submodule of $\underline{\mathrm{CH}}(\mathrm{M})$ for every nonempty proper flat F different from $\{i\}$. The following extension of $\underline{\mathrm{IH}}(\mathrm{M})$ will play a central role in our inductive argument.

Definition 3.3. We define the $\underline{\mathrm{H}}_i(\mathrm{M})$-submodule $\underline{\mathrm{IH}}_i(\mathrm{M})$ of $\underline{\mathrm{CH}}(\mathrm{M})$ by

$$\underline{\mathrm{IH}}_i(\mathrm{M}) \coloneqq \left(\sum_{F \neq \{i\}} \underline{\psi}^F_{\mathrm{M}}\, \underline{\mathrm{J}}(\mathrm{M}_F) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F) \right)^{\perp},$$

where the sum is over all nonempty proper flats F of M different from $\{i\}$.[9]

[9] Our convention gives $\underline{\mathrm{IH}}_i(\mathrm{M}) = \underline{\mathrm{IH}}(\mathrm{M})$ when $\{i\}$ is not a flat.

We now consider the graded algebras

$$\mathrm{H}(\mathrm{M}) \coloneqq \text{the subalgebra of } \mathrm{CH}(\mathrm{M}) \text{ generated by } y_i \text{ for } i \in E\text{, and}$$
$$\mathrm{H}_\circ(\mathrm{M}) \coloneqq \text{the subalgebra of } \mathrm{CH}(\mathrm{M}) \text{ generated by } y_i \text{ for } i \in E \text{ and } x_\varnothing.$$

If E is the empty set, then $x_\varnothing$ does not exist, and we do not define $\mathrm{H}_\circ(\mathrm{M})$. As mentioned before, the subalgebra $\mathrm{H}(\mathrm{M})$ can be identified with the graded Möbius algebra of M defined in the introduction [BHM+20, Proposition 2.15]. For a subspace V of $\mathrm{CH}(\mathrm{M})$, we set

$$V^\perp \coloneqq \Big\{\eta \in \mathrm{CH}(\mathrm{M}) \mid \deg_{\mathrm{M}}(v\eta) = 0 \ \text{ for all } v \in V\Big\}.$$

Proposition 2.5 shows that $\psi^F_{\mathrm{M}}\underline{\mathrm{J}}(\mathrm{M}_F)\otimes\mathrm{CH}(\mathrm{M}^F)$ is an $\mathrm{H}(\mathrm{M})$-submodule of $\mathrm{CH}(\mathrm{M})$ for every proper flat F of M. When F is nonempty, $\psi^F_{\mathrm{M}}\underline{\mathrm{J}}(\mathrm{M}_F)\otimes\mathrm{CH}(\mathrm{M}^F)$ is in fact an $\mathrm{H}_\circ(\mathrm{M})$-submodule of $\mathrm{CH}(\mathrm{M})$.

Definition 3.4. Let M be a loopless matroid.

(1) We define the $\mathrm{H}(\mathrm{M})$-submodule $\mathrm{IH}(\mathrm{M})$ of $\mathrm{CH}(\mathrm{M})$ by

$$\mathrm{IH}(\mathrm{M}) \coloneqq \left(\sum_{F<E} \psi^F_{\mathrm{M}}\underline{\mathrm{J}}(\mathrm{M}_F)\otimes\mathrm{CH}(\mathrm{M}^F)\right)^\perp,$$

where the sum is over all proper flats F of M.

(2) If E is nonempty, we define the $\mathrm{H}_\circ(\mathrm{M})$-submodule $\mathrm{IH}_\circ(\mathrm{M})$ of $\mathrm{CH}(\mathrm{M})$ by

$$\mathrm{IH}_\circ(\mathrm{M}) \coloneqq \left(\sum_{\varnothing<F<E} \psi^F_{\mathrm{M}}\underline{\mathrm{J}}(\mathrm{M}_F)\otimes\mathrm{CH}(\mathrm{M}^F)\right)^\perp,$$

where the sum is over all nonempty proper flats F of M.

We now state some basic properties of the pullbacks and pushforwards for the subspaces we have defined.

Lemma 3.5. For any nonempty proper flat F of M, we have

$$\varphi^F_{\mathrm{M}}\,\mathrm{IH}_\circ(\mathrm{M}) \subseteq \underline{\mathrm{IH}}(\mathrm{M}_F)\otimes\mathrm{CH}(\mathrm{M}^F) \ \text{ and } \ \underline{\varphi}^F_{\mathrm{M}}\underline{\mathrm{IH}}(\mathrm{M}) \subseteq \underline{\mathrm{IH}}(\mathrm{M}_F)\otimes\underline{\mathrm{CH}}(\mathrm{M}^F).$$

Proof. We prove the second inclusion. The first inclusion follows from the same argument.

We need to show that, for any nonempty proper flat G of M properly containing F,

$$\underline{\varphi}^F_{\mathrm{M}}\underline{\mathrm{IH}}(\mathrm{M}) \text{ is orthogonal to } \big(\underline{\psi}^{G\setminus F}_{\mathrm{M}_F}\underline{\mathrm{J}}(\mathrm{M}_G)\otimes\underline{\mathrm{CH}}(\mathrm{M}^G_F)\big)\otimes\underline{\mathrm{CH}}(\mathrm{M}^F) \text{ in } \underline{\mathrm{CH}}(\mathrm{M}_F)\otimes\underline{\mathrm{CH}}(\mathrm{M}^F).$$

By Lemma 2.10, the above is equivalent to the statement that

$$\underline{\mathrm{IH}}(\mathrm{M}) \text{ is orthogonal to } \underline{\psi}^F_{\mathrm{M}}\big(\underline{\psi}^{G\setminus F}_{\mathrm{M}_F}\underline{\mathrm{J}}(\mathrm{M}_G)\otimes\underline{\mathrm{CH}}(\mathrm{M}^G_F)\big)\otimes\underline{\mathrm{CH}}(\mathrm{M}^F) \text{ in } \underline{\mathrm{CH}}(\mathrm{M}).$$

This follows from Lemma 2.18 (3) and the orthogonality between $\underline{\mathrm{IH}}(\mathrm{M})$ and $\underline{\psi}^G_{\mathrm{M}}\underline{\mathrm{J}}(\mathrm{M}_G)\otimes\underline{\mathrm{CH}}(\mathrm{M}^G)$ in $\underline{\mathrm{CH}}(\mathrm{M})$. $\square$

Lemma 3.6. The following holds for any loopless matroid M.

(1) For any nonempty proper flat F of M, we have $\varphi_F^{\mathrm{M}} \, \mathrm{IH}_\circ(\mathrm{M}) \subseteq \mathrm{IH}(\mathrm{M}_F)$.

(2) For any proper flat $G \leqslant F$ of M, we have $\varphi_G^{\mathrm{M}} \psi_{\mathrm{M}}^F \underline{\mathrm{J}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F) = \psi_{\mathrm{M}_G}^{F \setminus G} \underline{\mathrm{J}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F_G)$.

Proof. For the first part, it suffices to show that for any flat G containing F,

$$\varphi_F^{\mathrm{M}} \, \mathrm{IH}_\circ(\mathrm{M}) \text{ and } \psi_{\mathrm{M}_F}^{G \setminus F} \big(\underline{\mathrm{J}}(\mathrm{M}_G) \otimes \mathrm{CH}(\mathrm{M}^G_F)\big) \text{ are orthogonal in } \mathrm{CH}(\mathrm{M}_F).$$

By Lemma 2.14, this is equivalent to the statement that

$$\mathrm{IH}_\circ(\mathrm{M}) \text{ and } \psi_F^{\mathrm{M}} \psi_{\mathrm{M}_F}^{G \setminus F} \big(\underline{\mathrm{J}}(\mathrm{M}_G) \otimes \mathrm{CH}(\mathrm{M}^G_F)\big) \text{ are orthogonal in } \mathrm{CH}(\mathrm{M}).$$

By Lemma 2.18 (4), we have

$$\psi_F^{\mathrm{M}} \psi_{\mathrm{M}_F}^{G \setminus F} \big(\underline{\mathrm{J}}(\mathrm{M}_G) \otimes \mathrm{CH}(\mathrm{M}^G_F)\big) = \psi_{\mathrm{M}}^G \big(\underline{\mathrm{J}}(\mathrm{M}_G) \otimes \psi_F^{\mathrm{M}^G} \, \mathrm{CH}(\mathrm{M}^G_F)\big).$$

Since $\mathrm{IH}_\circ(\mathrm{M})$ is orthogonal to $\psi_{\mathrm{M}}^G \big(\underline{\mathrm{J}}(\mathrm{M}_G) \otimes \mathrm{CH}(\mathrm{M}^G)\big)$ by construction, the subspaces $\mathrm{IH}_\circ(\mathrm{M})$ and $\psi_F^{\mathrm{M}} \psi_{\mathrm{M}_F}^{G \setminus F} \big(\underline{\mathrm{J}}(\mathrm{M}_G) \otimes \mathrm{CH}(\mathrm{M}^G_F)\big)$ are orthogonal in $\mathrm{CH}(\mathrm{M})$.

The second statement follows from the surjectivity of $\varphi_G^{\mathrm{M}^F}$ and Lemma 2.18 (4), more specifically the commutativity of the square on the left. $\square$

Proposition 3.7. The graded linear subspaces

$$\underline{\psi}_{\mathrm{M}}^F \underline{\mathrm{J}}(\mathrm{M}_F) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F) \subseteq \underline{\mathrm{CH}}(\mathrm{M}),$$

where F varies through all nonempty proper flats of M, are mutually orthogonal in $\underline{\mathrm{CH}}(\mathrm{M})$. Similarly, the graded linear subspaces

$$\psi_{\mathrm{M}}^F \underline{\mathrm{J}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F) \subseteq \mathrm{CH}(\mathrm{M}),$$

where F varies through all proper flats of M, are mutually orthogonal in $\mathrm{CH}(\mathrm{M})$.

Proof. We only prove the second statement. The first statement follows from the same arguments.

Let F and G be distinct nonempty proper flats. We want to show that $\psi_{\mathrm{M}}^F \underline{\mathrm{J}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F)$ is orthogonal to $\psi_{\mathrm{M}}^G \underline{\mathrm{J}}(\mathrm{M}_G) \otimes \mathrm{CH}(\mathrm{M}^G)$ in $\mathrm{CH}(\mathrm{M})$. By Lemma 2.6, this is equivalent to showing that

$$\varphi_{\mathrm{M}}^G \psi_{\mathrm{M}}^F \underline{\mathrm{J}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F) \text{ is orthogonal to } \underline{\mathrm{J}}(\mathrm{M}_G) \otimes \mathrm{CH}(\mathrm{M}^G) \text{ in } \underline{\mathrm{CH}}(\mathrm{M}_G) \otimes \mathrm{CH}(\mathrm{M}^G).$$

If F and G are incomparable, this follows from Lemma 2.16, so we may assume without loss of generality that $G < F$. By Lemma 2.18 (5), the subspace $\varphi_{\mathrm{M}}^G \psi_{\mathrm{M}}^F \underline{\mathrm{J}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F)$ is equal to

$$\Big(\underline{\psi}_{\mathrm{M}_G}^{F \setminus G} \otimes \mathrm{id}\Big) \circ \big(\mathrm{id} \otimes \varphi_{\mathrm{M}^F}^G\big) \big(\underline{\mathrm{J}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F)\big).$$

By Lemma 2.10 for the matroid M_G, the statement that this subspace is orthogonal to $\underline{\mathrm{J}}(\mathrm{M}_G) \otimes \mathrm{CH}(\mathrm{M}^G)$ is equivalent to the statement that

$$\left(\mathrm{id} \otimes \varphi^G_{\mathrm{M}^F}\right) \left(\underline{\mathrm{J}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F)\right) \text{ is orthogonal to } \left(\underline{\varphi}^{F\setminus G}_{\mathrm{M}_G} \otimes \mathrm{id}\right) \left(\underline{\mathrm{J}}(\mathrm{M}_G) \otimes \mathrm{CH}(\mathrm{M}^G)\right).$$

For this, it is sufficient to show that

$$\underline{\mathrm{J}}(\mathrm{M}_F) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F_G) \text{ is orthogonal to } \underline{\varphi}^{F\setminus G}_{\mathrm{M}_G} \underline{\mathrm{J}}(\mathrm{M}_G) \text{ in } \underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F_G).$$

By Lemma 2.10, the orthogonality between the above two subspaces is equivalent to

$$\text{the orthogonality of } \underline{\psi}^{F\setminus G}_{\mathrm{M}_G} \underline{\mathrm{J}}(\mathrm{M}_F) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F_G) \text{ and } \underline{\mathrm{J}}(\mathrm{M}_G) \text{ in } \underline{\mathrm{CH}}(\mathrm{M}_G).$$

This follows from the orthogonality between $\underline{\psi}^{F\setminus G}_{\mathrm{M}_G} \underline{\mathrm{J}}(\mathrm{M}_F) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F_G)$ and $\underline{\mathrm{IH}}(\mathrm{M}_G)$, and the fact that $\underline{\mathrm{J}}(\mathrm{M}_G) \subseteq \underline{\mathrm{IH}}(\mathrm{M}_G)$. □

3.2. **The statements.** Let $\mathrm{N} = \bigoplus_{k \geqslant 0} \mathrm{N}^i$ be a graded $\mathbb{Q}$-vector space endowed with a bilinear form

$$\langle -, - \rangle \colon \mathrm{N} \times \mathrm{N} \to \mathbb{Q}$$

and a linear operator $\mathrm{L} \colon \mathrm{N} \to \mathrm{N}$ of degree 1 that satisfies $\langle \mathrm{L}(\eta), \xi \rangle = \langle \eta, \mathrm{L}(\xi) \rangle$ for all $\eta, \xi \in \mathrm{N}$.

Definition 3.8. Using the notation above, we define three properties for N.

(1) We say that N satisfies **Poincaré duality of degree d** if the bilinear form $\langle -, - \rangle$ is non-degenerate, and for $\eta \in \mathrm{N}^j$ and $\xi \in \mathrm{N}^k$, the pairing $\langle \eta, \xi \rangle$ is nonzero only when $j + k = d$.

(2) We say that N satisfies the **hard Lefschetz theorem of degree d** if the linear map

$$\mathrm{L}^{d-2k} \colon \mathrm{N}^k \to \mathrm{N}^{d-k}$$

is an isomorphism for all $k \leqslant d/2$.

(3) We say that N satisfies the **Hodge–Riemann relations of degree d** if the restriction of

$$\mathrm{N}^k \times \mathrm{N}^k \longrightarrow \mathbb{Q}, \qquad (\eta, \xi) \longmapsto (-1)^k \langle \mathrm{L}^{d-2k}(\eta), \xi \rangle$$

to the kernel of $\mathrm{L}^{d-2k+1} \colon \mathrm{N}^k \to \mathrm{N}^{d-k+1}$ is positive definite for all $k \leqslant d/2$. Elements of $\ker \mathrm{L}^{d-2k+1}$ are called **primitive classes**.

We now define the central statements that appear in the induction.

First, the augmented Chow ring admits canonical decompositions into $\mathrm{H}(\mathrm{M})$-modules, and the Chow ring admits canonical decompositions into $\underline{\mathrm{H}}(\mathrm{M})$-modules.

Definition 3.9 (Canonical decompositions)**.** Let i be an element of the ground set E.

$\mathsf{CD}(\mathrm{M})$: We have the direct sum decomposition

$$\mathrm{CH}(\mathrm{M}) = \mathrm{IH}(\mathrm{M}) \oplus \bigoplus_{F<E} \psi^F_{\mathrm{M}} \underline{\mathrm{J}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F),$$

where the sum is over all proper flats F of M.

$\mathsf{CD}_\circ(\mathrm{M})$: We have the direct sum decomposition

$$\mathrm{CH}(\mathrm{M}) = \mathrm{IH}_\circ(\mathrm{M}) \oplus \bigoplus_{\varnothing<F<E} \psi^F_{\mathrm{M}} \underline{\mathrm{J}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F),$$

where the sum is over all nonempty proper flats F of M.

$\underline{\mathsf{CD}}(\mathrm{M})$: We have the direct sum decomposition

$$\underline{\mathrm{CH}}(\mathrm{M}) = \underline{\mathrm{IH}}(\mathrm{M}) \oplus \bigoplus_{\varnothing<F<E} \underline{\psi}^F_{\mathrm{M}} \underline{\mathrm{J}}(\mathrm{M}_F) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F),$$

where the sum is over all nonempty proper flats F of M.

$\underline{\mathsf{CD}}_i(\mathrm{M})$: We have the direct sum decomposition

$$\underline{\mathrm{CH}}(\mathrm{M}) = \underline{\mathrm{IH}}_i(\mathrm{M}) \oplus \bigoplus_{F \neq \{i\}} \underline{\psi}^F_{\mathrm{M}} \underline{\mathrm{J}}(\mathrm{M}_F) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F),$$

where the sum is over all nonempty proper flats F of M different from $\{i\}$.

Convention 3.10. We will use a superscript to denote that the decompositions hold in certain degrees. For example, $\mathsf{CD}^{\leqslant k}(\mathrm{M})$ means that the direct sum decomposition holds in degrees less than or equal to k.

Remark 3.11. Let V and W be finite-dimensional $\mathbb{Q}$-vector spaces with subspaces $V_1 \subseteq V$ and $W_1 \subseteq W$. Given a non-degenerate pairing $V \times W \to \mathbb{Q}$, we can define the orthogonal subspaces $W_1^\perp \subseteq V$ and $V_1^\perp \subseteq W$. It is straightforward to check that $W = W_1 \oplus V_1^\perp$ if and only if $V = V_1 \oplus W_1^\perp$. Applying this fact repeatedly, we have

$$\mathsf{CD}^k(\mathrm{M}) \Longleftrightarrow \mathsf{CD}^{d-k}(\mathrm{M}), \quad \mathsf{CD}(\mathrm{M}) \Longleftrightarrow \mathsf{CD}^{\leqslant \frac{d}{2}}(\mathrm{M}), \quad \text{and} \quad \mathsf{CD}_\circ(\mathrm{M}) \Longleftrightarrow \mathsf{CD}_\circ^{\leqslant \frac{d}{2}}(\mathrm{M}).$$

Similarly, we have $\underline{\mathsf{CD}}(\mathrm{M}) \Longleftrightarrow \underline{\mathsf{CD}}^{\leqslant \frac{d-1}{2}}(\mathrm{M})$ and $\underline{\mathsf{CD}}_i(\mathrm{M}) \Longleftrightarrow \underline{\mathsf{CD}}_i^{\leqslant \frac{d-1}{2}}(\mathrm{M})$.

Remark 3.12. Let V be a finite-dimensional $\mathbb{Q}$-vector space equipped with a non-degenerate symmetric bilinear form and let $W \subset V$ be a subspace. Then the restriction of the form to W is non-degenerate if and only if $V = W \oplus W^\perp$. In light of Remark 3.11, this implies that

$$\mathsf{CD}^k(\mathrm{M}) \Longleftrightarrow \mathsf{PD}^k(\mathrm{M}), \quad \mathsf{CD}^k_\circ(\mathrm{M}) \Longleftrightarrow \mathsf{PD}^k_\circ(\mathrm{M}), \quad \text{and} \quad \underline{\mathsf{CD}}^k(\mathrm{M}) \Longleftrightarrow \underline{\mathsf{PD}}^k(\mathrm{M}).$$

Let R be a graded $\mathbb{Q}$-algebra that is generated in positive degree, and let $\mathfrak{m} \subseteq R$ denote the unique graded maximal ideal. For any graded R-module N, the **socle** of N is the graded submodule

$$\mathrm{soc}(\mathrm{N}) \coloneqq \{n \in \mathrm{N} \mid \mathfrak{m} \cdot n = 0\}.$$

The next conditions assert that the socles of the intersection cohomology modules defined in Section 3.1 vanish in low degrees. As before, the symbol d stands for the rank of the matroid M.

Definition 3.13 (No socle conditions)**.**

$\mathsf{NS}(\mathrm{M})$: The socle of the $\mathrm{H}(\mathrm{M})$-module $\mathrm{IH}(\mathrm{M})$ vanishes in degrees less than or equal to $d/2$.

$\mathsf{NS}_\circ(\mathrm{M})$: The socle of the $\mathrm{H}_\circ(\mathrm{M})$-module $\mathrm{IH}_\circ(\mathrm{M})$ vanishes in degrees less than or equal to $d/2$.

$\underline{\mathsf{NS}}(\mathrm{M})$: The socle of the $\underline{\mathrm{H}}(\mathrm{M})$-module $\underline{\mathrm{IH}}(\mathrm{M})$ vanishes in degrees less than or equal to $(d-2)/2$.

In particular, for even d, the no socle condition for $\mathrm{IH}(\mathrm{M})$ says that the socle of the $\mathrm{H}(\mathrm{M})$-module $\mathrm{IH}(\mathrm{M})$ is concentrated in degrees strictly larger than the middle degree $d/2$. On the other hand, for an odd number d, the socle of the $\underline{\mathrm{H}}(\mathrm{M})$-module $\underline{\mathrm{IH}}(\mathrm{M})$ may be nonzero in the middle degree $(d-1)/2$.

Recall that we have Poincaré pairings on $\mathrm{CH}(\mathrm{M})$ and $\underline{\mathrm{CH}}(\mathrm{M})$ defined by

$$\langle \eta, \xi \rangle_{\mathrm{CH}(\mathrm{M})} \coloneqq \deg_{\mathrm{M}}(\eta\, \xi) \quad \text{and} \quad \langle \underline{\eta}, \underline{\xi} \rangle_{\underline{\mathrm{CH}}(\mathrm{M})} \coloneqq \underline{\deg}_{\mathrm{M}}(\underline{\eta}\, \underline{\xi}).$$

Moreover, with respect to the above bilinear forms, $\mathrm{CH}(\mathrm{M})$ satisfies Poincaré duality of degree d and $\underline{\mathrm{CH}}(\mathrm{M})$ satisfies Poincaré duality of degree $d-1$, by Theorems 2.19 and 2.20.

Definition 3.14 (Poincaré dualities)**.**

$\mathsf{PD}(\mathrm{M})$: The graded vector space $\mathrm{IH}(\mathrm{M})$ satisfies Poincaré duality of degree d with respect to the Poincaré pairing on $\mathrm{CH}(\mathrm{M})$.

$\mathsf{PD}_\circ(\mathrm{M})$: The graded vector space $\mathrm{IH}_\circ(\mathrm{M})$ satisfies Poincaré duality of degree d with respect to the Poincaré pairing on $\mathrm{CH}(\mathrm{M})$.

$\underline{\mathsf{PD}}(\mathrm{M})$: The graded vector space $\underline{\mathrm{IH}}(\mathrm{M})$ satisfies Poincaré duality of degree $d-1$ with respect to the Poincaré pairing on $\underline{\mathrm{CH}}(\mathrm{M})$.

Definition 3.15 (Hard Lefschetz theorems)**.**

$\mathsf{HL}(\mathrm{M})$: For any positive linear combination $y = \sum_{j \in E} c_j y_j$, the graded vector space $\mathrm{IH}(\mathrm{M})$ satisfies the hard Lefschetz theorem of degree d with respect to multiplication by y.

$\mathsf{HL}_\circ(\mathrm{M})$: For any positive linear combination $y = \sum_{j \in E} c_j y_j$, there is a positive ϵ such that the graded vector space $\mathrm{IH}_\circ(\mathrm{M})$ satisfies the hard Lefschetz theorem of degree d with respect to multiplication by $y - \epsilon x_\varnothing$.

$\mathsf{HL}_i(\mathrm{M})$: For any positive linear combination $y' = \sum_{j\in E\setminus i} c_j y_j$, the graded vector space $\mathrm{IH}(\mathrm{M})$ satisfies the hard Lefschetz theorem of degree d with respect to multiplication by y'.

$\underline{\mathsf{HL}}(\mathrm{M})$: The graded vector space $\underline{\mathrm{IH}}(\mathrm{M})$ satisfies the hard Lefschetz theorem of degree $d-1$ with respect to multiplication by $\underline{\beta}$.

$\underline{\mathsf{HL}}_i(\mathrm{M})$: The graded vector space $\underline{\mathrm{IH}}_i(\mathrm{M})$ satisfies the hard Lefschetz theorem of degree $d-1$ with respect to multiplication by $\underline{\beta} - x_{\{i\}}$. Here we recall our convention that $x_{\{i\}} = 0$ if $\{i\}$ is not a flat.

Definition 3.16 (Hodge–Riemann relations)**.**

$\mathsf{HR}(\mathrm{M})$: For any positive linear combination $y = \sum_{j\in E} c_j y_j$, the graded vector space $\mathrm{IH}(\mathrm{M})$ satisfies the Hodge–Riemann relations of degree d with respect to the Poincaré pairing on $\mathrm{CH}(\mathrm{M})$ and the multiplication by y.

$\mathsf{HR}_\circ(\mathrm{M})$: For any positive linear combination $y = \sum_{j\in E} c_j y_j$, there is a positive ϵ such that the graded vector space $\mathrm{IH}_\circ(\mathrm{M})$ satisfies the Hodge–Riemann relations of degree d with respect to the Poincaré pairing on $\mathrm{CH}(\mathrm{M})$ and the multiplication by $y - \epsilon x_\varnothing$.

$\mathsf{HR}_i(\mathrm{M})$: For any positive linear combination $y' = \sum_{j\in E\setminus i} c_j y_j$, the graded vector space $\mathrm{IH}(\mathrm{M})$ satisfies the Hodge–Riemann relations of degree d with respect to the Poincaré pairing on $\mathrm{CH}(\mathrm{M})$ and the multiplication by y'.

$\underline{\mathsf{HR}}(\mathrm{M})$: The graded vector space $\underline{\mathrm{IH}}(\mathrm{M})$ satisfies the Hodge–Riemann relations of degree $d-1$ with respect to the Poincaré pairing on $\underline{\mathrm{CH}}(\mathrm{M})$ and the multiplication by $\underline{\beta}$.

$\underline{\mathsf{HR}}_i(\mathrm{M})$: The graded vector space $\underline{\mathrm{IH}}_i(\mathrm{M})$ satisfies the Hodge–Riemann relations of degree $d-1$ with respect to the Poincaré pairing on $\underline{\mathrm{CH}}(\mathrm{M})$ and the multiplication by $\underline{\beta} - x_{\{i\}}$.

As before, we will use a superscript to denote that the conditions hold in certain degrees. For example, $\mathsf{PD}^k(\mathrm{M})$ means the Poincaré pairing on $\mathrm{CH}(\mathrm{M})$ induces a non-degenerate pairing between $\mathrm{IH}^k(\mathrm{M})$ and $\mathrm{IH}^{d-k}(\mathrm{M})$, and $\mathsf{HL}^k(\mathrm{M})$ means the hard Lefschetz map from $\mathrm{IH}^k(\mathrm{M})$ to $\mathrm{IH}^{d-k}(\mathrm{M})$ is an isomorphism.

Now we state the main result of this paper, which will be proved using induction on the cardinality of the ground set E.

Theorem 3.17. *Let* M *be a loopless matroid on* E*. If* E *is nonempty, the following statements hold:*

$$
\begin{array}{lllll}
\mathsf{CD}(\mathrm{M}), & \mathsf{NS}(\mathrm{M}), & \mathsf{PD}(\mathrm{M}), & \mathsf{HL}(\mathrm{M}), & \mathsf{HR}(\mathrm{M}), \\
\mathsf{CD}_\circ(\mathrm{M}), & \mathsf{NS}_\circ(\mathrm{M}), & \mathsf{PD}_\circ(\mathrm{M}), & \mathsf{HL}_\circ(\mathrm{M}), & \mathsf{HR}_\circ(\mathrm{M}), \\
\underline{\mathsf{CD}}(\mathrm{M}), & \underline{\mathsf{NS}}(\mathrm{M}), & \underline{\mathsf{PD}}(\mathrm{M}), & \underline{\mathsf{HL}}(\mathrm{M}), & \underline{\mathsf{HR}}(\mathrm{M}).
\end{array}
$$

As a byproduct, we will also prove the statements $\mathsf{HL}_i(\mathrm{M})$, $\mathsf{HR}_i(\mathrm{M})$, $\underline{\mathsf{HL}}_i(\mathrm{M})$, and $\underline{\mathsf{HR}}_i(\mathrm{M})$. However, we will not use these statements in our applications, and we do not need them in the inductive hypothesis.

Remark 3.18. If E is the empty set, the statements $\mathsf{CD}(\mathrm{M})$, $\mathsf{PD}(\mathrm{M})$, $\mathsf{HL}(\mathrm{M})$, and $\mathsf{HR}(\mathrm{M})$ hold tautologically. The statement $\mathsf{NS}(\mathrm{M})$ fails, as we have $\mathrm{H}(\mathrm{M}) = \mathrm{CH}(\mathrm{M}) = \mathrm{IH}(\mathrm{M}) = \mathbb{Q}$, so the socle is nonvanishing in degree 0. This is directly related to the fact that the Kazhdan–Lusztig polynomial of the rank zero matroid has larger than expected degree. The remaining statements do not make sense because $\mathrm{IH}_\circ(\mathrm{M})$ and $\underline{\mathrm{IH}}(\mathrm{M})$ are not defined when E is empty.

4. Guide to the proof

The proof of our main result, Theorem 3.17, is a complex induction involving all of the statements introduced in the previous section. A more or less complete diagram of the steps of the induction appears in Figure 1. The purpose of this section is to highlight the main steps in the proof, to explain what these steps mean in the geometric setting when M is realizable, and to make some comparisons with the structure of the proofs of Karu [Kar04] and Elias–Williamson [EW14].

We hope that readers will benefit from flipping back to this section frequently as they read the rest of the paper. However, this section is not needed for establishing the results in this paper; it is included only to communicate the overall structure and geometric insight behind the main ingredients of the proof. It may be skipped in full by readers who would like to stick to a purely formal treatment.

4.1. **Canonical decomposition.** As discussed in Section 1.3, when the matroid M is realizable, $\mathrm{CH}(\mathrm{M})$ is the cohomology ring of a resolution X of the Schubert variety Y. Thus, the decomposition theorem suggests that $\mathrm{CH}(\mathrm{M})$ should be a direct sum of indecomposable graded $\mathrm{H}(\mathrm{M})$-submodules, each of which is isomorphic to a shift of $\mathrm{IH}(\mathrm{M}^F)$ for some flat F.[10] In our proof, we obtain such a decomposition as a consequence of the coarser decomposition $\mathsf{CD}(\mathrm{M})$ (Definition 3.9). The summand in $\mathsf{CD}(\mathrm{M})$ indexed by the proper flat F is isomorphic as an $\mathrm{H}(\mathrm{M})$-module to a direct sum of shifts of copies of $\mathrm{CH}(\mathrm{M}^F)$, so it can be further decomposed using the same formula. Iterating this, one can obtain a decomposition of $\mathrm{CH}(\mathrm{M})$ into shifted copies of $\mathrm{IH}(\mathrm{M}^F)$ for various flats F.

The decomposition $\mathsf{CD}(\mathrm{M})$ has several properties which make proving it easier than proving the full decomposition into irreducible modules directly. First of all, $\mathsf{CD}(\mathrm{M})$ is canonical, since the definition of $\underline{\mathrm{J}}(\mathrm{M})$ does not involve any choices (Definition 3.1). Second, the summands are

[10] There is a surjection $\mathrm{H}(\mathrm{M}) \to \mathrm{H}(\mathrm{M}^F)$ defined by setting $y_G = 0$ unless $G \leqslant F$, so $\mathrm{IH}(\mathrm{M}^F)$ is naturally an $\mathrm{H}(\mathrm{M})$-module.

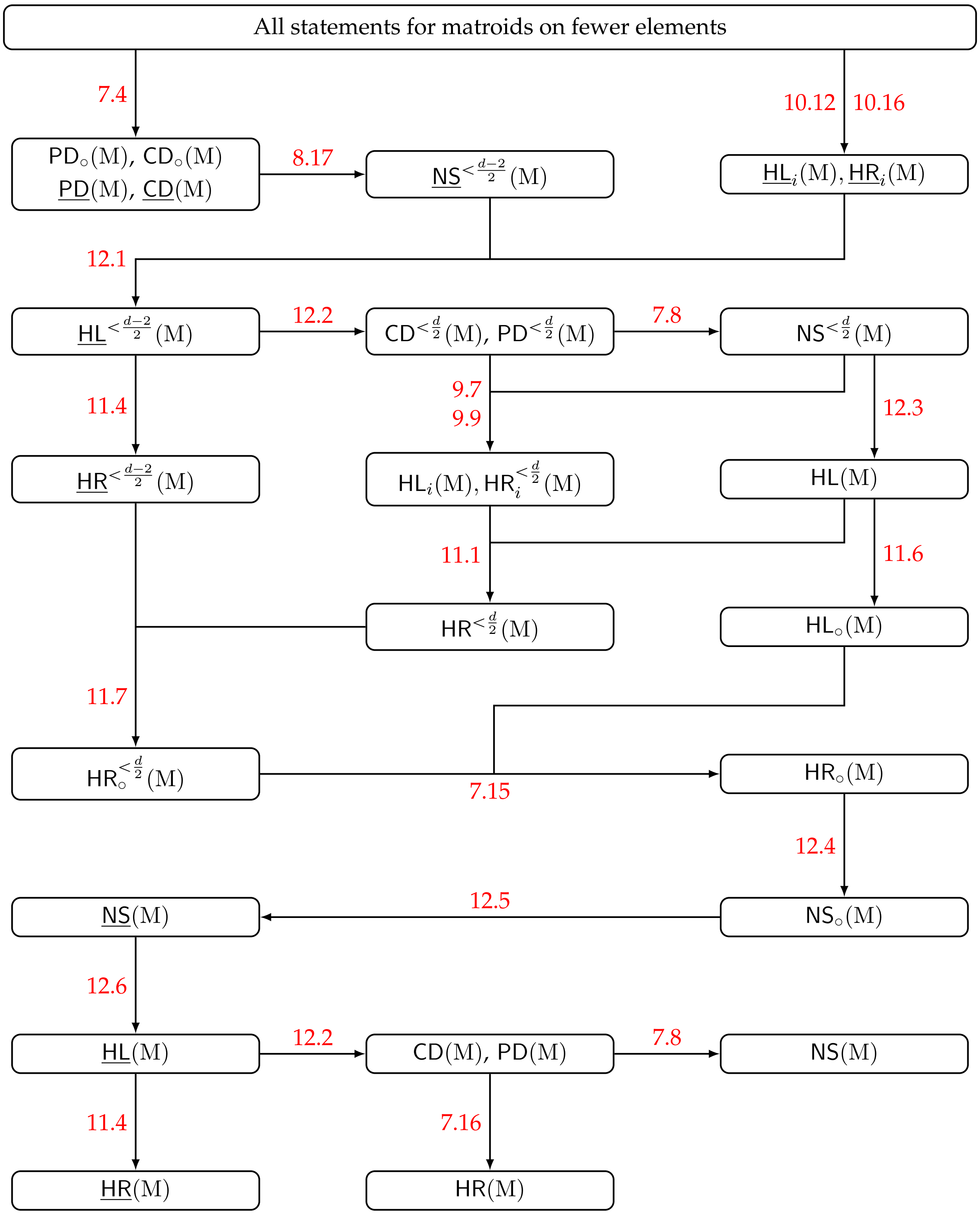

FIGURE 1. Diagram of the proof

orthogonal to each other with respect to the Poincaré pairing on $\mathrm{CH}(\mathrm{M})$ (Proposition 3.7), and in fact, we define $\mathrm{IH}(\mathrm{M})$ to be the perpendicular space to the other summands (Definition 3.4).

The problem then is to show that the terms actually do form a direct sum. It turns out that, if we assume inductively that all our results hold for the matroids M_F with F a nonempty flat, then the weaker decomposition $\mathsf{CD}_\circ(\mathrm{M})$ (Definition 3.9) follows by a simple formal argument (Corollary 7.4). Thus, we need to show that we have defined $\underline{\mathrm{J}}(\mathrm{M}) \subseteq \underline{\mathrm{IH}}(\mathrm{M})$ such that $\psi^{\varnothing}_{\mathrm{M}}(\underline{\mathrm{J}}(\mathrm{M}))$ and its perpendicular space inside $\mathrm{IH}_\circ(\mathrm{M})$ form a direct sum. We show in Proposition 12.2 that this is a consequence of the hard Lefschetz property $\underline{\mathsf{HL}}(\mathrm{M})$ for $\underline{\mathrm{IH}}(\mathrm{M})$.

Let us explain the motivation for the definition of $\underline{\mathrm{J}}(\mathrm{M})$, using two functors which are defined in greater generality in Section 5.2. For a graded $\mathrm{H}(\mathrm{M})$-module N, its **costalk** $\mathrm{N}_{[\varnothing]}$ is the socle of N, the submodule of elements annihilated by the maximal ideal $\mathfrak{m}$ generated by all y_F. Dually, its **stalk** $\mathrm{N}_\varnothing$ is the cosocle of N, or in other words the quotient $\mathrm{N}/\mathfrak{m}\mathrm{N}$. There is a natural transformation $\mathrm{N}_{[\varnothing]} \to \mathrm{N}_\varnothing$ from the costalk to the stalk. If N is contained in the image of $\psi^{\varnothing}_{\mathrm{M}}$, which is the ideal generated by $x_\varnothing$, then $\mathfrak{m}$ annihilates N, and hence $\mathrm{N}_{[\varnothing]} = \mathrm{N} = \mathrm{N}_\varnothing$. Thus, if N is a direct summand of $\mathrm{IH}_\circ(\mathrm{M})$, then the composition $\mathrm{N}_{[\varnothing]} \to \mathrm{IH}_\circ(\mathrm{M})_{[\varnothing]} \to \mathrm{IH}_\circ(\mathrm{M})_\varnothing$ must be injective.

On the other hand, the costalk-stalk maps for the modules $\mathrm{CH}(\mathrm{M})$ and $\mathrm{IH}_\circ(\mathrm{M})$ have nice descriptions. The stalk $\mathrm{CH}(\mathrm{M})_\varnothing$ is isomorphic to $\underline{\mathrm{CH}}(\mathrm{M})$, and using the push-pull maps $\varphi^{\varnothing}_{\mathrm{M}}$ and $\psi^{\varnothing}_{\mathrm{M}}$ along with the fact that the ideals $\langle x_\varnothing \rangle$ and $\mathfrak{m}$ are each others' annihilators (Lemma 5.2), it follows that the costalk is isomorphic to $\underline{\mathrm{CH}}(\mathrm{M})[-1]$, and under these isomorphisms the costalk-stalk map is identified with multiplication by $\underline{\beta}$. In Corollaries 7.6 and 7.7, we show that we have a similar picture for the direct summand $\mathrm{IH}_\circ(\mathrm{M})$: The costalk-stalk map for this module can be identified with the multiplication

$$\underline{\mathrm{IH}}(\mathrm{M})[-1] \xrightarrow{\cdot\underline{\beta}} \underline{\mathrm{IH}}(\mathrm{M}).$$

Once we know $\underline{\mathsf{HL}}(\mathrm{M})$, we see that the largest possible subspace of $\underline{\mathrm{IH}}(\mathrm{M})$ on which multiplication by $\underline{\beta}$ acts injectively, and so could produce a direct summand, is the span of all classes $\underline{\beta}^j a$, where a is a primitive class in $\underline{\mathrm{IH}}^k(\mathrm{M})$ and j is strictly less than $d-1-2k$. This is precisely our space $\underline{\mathrm{J}}(\mathrm{M})$. It also follows that the stalk $\mathrm{IH}(\mathrm{M})_\varnothing$ is the quotient $\underline{\mathrm{IH}}(\mathrm{M})/\underline{\beta}\cdot\underline{\mathrm{IH}}(\mathrm{M})$, which allows us to conclude the condition $\underline{\mathsf{NS}}(\mathrm{M})$, or equivalently, Proposition 1.6.

Remark 4.1. Let us explain the geometry behind these definitions and statements when M is realizable as in Section 1.3. Recall that the augmented wonderful variety X is obtained from the Schubert variety Y by blowing up the proper transforms of the closures $\overline{U^F}$ of strata U^F in order of increasing dimension, and in particular the exceptional divisor has a component D_F for any proper flat F. The map ψ^F_{M} of Definition 2.4 is the Gysin pushforward for the divisor D_F. The divisor $D_\varnothing$ is the fiber of $X \to Y$ over the point stratum $U^\varnothing$; it is the **wonderful variety** of [DCP95], and we denote it here by $\underline{X}_\varnothing$. Its cohomology ring is identified with $\underline{\mathrm{CH}}(\mathrm{M})$, and the restriction $\mathrm{H}^\bullet(X) \to \mathrm{H}^\bullet(D_\varnothing)$ is identified with the pullback $\varphi^{\varnothing}_{\mathrm{M}}\colon\ \mathrm{CH}(\mathrm{M}) \to \underline{\mathrm{CH}}(\mathrm{M})$ of Definition 2.4.

When F is not the empty flat, the divisor D_F is isomorphic to the product $\underline{X}_F \times X^F$, where $\underline{X}_F$ is the fiber of the resolution X_F of the Schubert variety Y_F over the point stratum, and X^F is the resolution of $\overline{U^F}$. The fact that D_F is a product gives the tensor product structure on the domain of ψ^F_{M}, and it explains why we are able to prove a "decomposition theorem" whose summands are copies of $\mathrm{CH}(\mathrm{M}^F)$ rather than smaller intersection cohomology spaces.

The resolution $X \to Y$ factors through $Y_\circ$, the blow-up of Y at the point stratum $U^\varnothing$. The cohomology class of the exceptional divisor pulls back to the element $x_\varnothing$ in $\mathrm{CH}(\mathrm{M})$, and the decomposition $\mathsf{CD}_\circ(\mathrm{M})$ reflects the decomposition theorem applied to the map $X \to Y_\circ$. In particular, $\mathrm{IH}_\circ(\mathrm{M})$ is isomorphic to the intersection cohomology of $Y_\circ$, and its quotient $\underline{\mathrm{IH}}(\mathrm{M})$ is isomorphic to the intersection cohomology of the exceptional fiber $\underline{Y}$, whose resolution is $D_\varnothing = \underline{X}_\varnothing$. Although the whole cohomology ring of $\underline{Y}$ does not seem to be combinatorially accessible, we know that the ample class given by its normal bundle in $Y_\circ$ is $\underline{\beta}$. The fact that the hard Lefschetz theorem for $\underline{\beta}$ acting on $\mathrm{IH}^\bullet(\underline{Y})$ implies the decomposition theorem for $Y_\circ \to Y$ is a well-known phenomenon thanks to the work of de Cataldo and Migliorini [dCM09].

The variety $\underline{Y}$ can be viewed as a "local" counterpart to Y, since the singularity of Y at the point stratum is the affine cone over the projective variety $\underline{Y}$. One of the reasons for the complexity of our inductive argument is the need to prove statements in both the "local" and "global" setting: we prove a canonical decomposition $\underline{\mathsf{CD}}(\mathrm{M})$ of $\underline{\mathrm{CH}}(\mathrm{M})$ analogous to $\mathsf{CD}(\mathrm{M})$, we prove the Hodge–Riemann relations $\underline{\mathsf{HR}}(\mathrm{M})$ for $\underline{\mathrm{IH}}(\mathrm{M})$, and so on. This is in contrast to Karu's proof for the combinatorial intersection cohomology of fans [Kar04], where an important role is played by the fact that any affine toric variety is a (weighted) cone over a projective toric variety of dimension one less.

4.2. **Rouquier complexes.** As an intermediate step to proving $\underline{\mathsf{HL}}(\mathrm{M})$, we prove the weaker statement $\underline{\mathsf{NS}}(\mathrm{M})$ (Definition 3.13). When d is even, the statement that there is no socle in degree exactly $(d-2)/2$ is equivalent to hard Lefschetz in that degree, since $\underline{\mathrm{IH}}^{\frac{d-2}{2}}(\mathrm{M})$ and $\underline{\mathrm{IH}}^{\frac{d}{2}}(\mathrm{M})$ have the same dimension by Poincaré duality. The no socle condition in this middle degree requires a more elaborate argument (Section 4.6), and our first step is to prove that $\underline{\mathrm{IH}}(\mathrm{M})$ has no socle in degrees strictly less than $(d-2)/2$ (Corollary 8.17).

We do this in Section 8 by constructing a complex $\bar{\mathrm{C}}^\bullet_\circ(\mathrm{M})$ of graded $\mathrm{H}_\circ(\mathrm{M})$-modules, which we call the (small reduced) **Rouquier complex**. It has the following properties:

(1) $\bar{\mathrm{C}}^k_\circ(\mathrm{M})$ vanishes for $k < 0$ or $k > d$.

(2) $\bar{\mathrm{C}}^0_\circ(\mathrm{M})$ is isomorphic to $\mathrm{IH}_\circ(\mathrm{M})$ (Theorem 8.16 (2)).

(3) For any $1 \leqslant k \leqslant d$, $\bar{\mathrm{C}}^k_\circ(\mathrm{M})$ is isomorphic to a direct sum of modules of the form $\mathrm{IH}_\circ(\mathrm{M}^F)[(k - \operatorname{crk} F)/2]$, where F is a nonempty proper flat such that $\operatorname{crk} F - k$ is nonnegative and even (Theorem 8.16 (2)).

(4) For any k, the cohomology $\mathrm{H}^k(\bar{\mathrm{C}}^\bullet_\circ(\mathrm{M})_\varnothing)$ of the stalk complex vanishes except in degree $d-1-k$ (Proposition 8.13).

These properties imply in particular that the differential $\bar{\mathrm{C}}^0_\circ(\mathrm{M})_\varnothing \to \bar{\mathrm{C}}^1_\circ(\mathrm{M})_\varnothing$ of the stalk complex is injective except in degree $d-1$, that its source is isomorphic to $\underline{\mathrm{IH}}(\mathrm{M})$, and that its target is isomorphic to a direct sum of $\mathbb{Q}[\underline{\beta}]$-modules of the form $\underline{\mathrm{IH}}(\mathrm{M}^F)[(1-\operatorname{crk} F)/2]$. Since the flats F are proper, the matroids M^F have smaller ground sets, and we can assume inductively that we have proved the hard Lefschetz property for the $\underline{\beta}$-action on each module $\underline{\mathrm{IH}}(\mathrm{M}^F)$ occurring in $\bar{\mathrm{C}}^1_\circ(\mathrm{M})_\varnothing$. Together with property (3), which restricts the shifts that can occur, this implies that $\bar{\mathrm{C}}^1_\circ(\mathrm{M})_\varnothing$ has no $\underline{\beta}$-socle in degrees strictly less than $(d-2)/2$, and since property (4) says the differential is injective in those degrees, the same holds for $\bar{\mathrm{C}}^0_\circ(\mathrm{M})_\varnothing$, which by (2) is isomorphic to $\underline{\mathrm{IH}}(\mathrm{M})$.

We only use the first differential of the complex $\bar{\mathrm{C}}^\bullet_\circ(\mathrm{M})$ in our proof, but it is no more difficult for us to construct the whole complex, and its existence should be of independent interest. This complex is analogous to the Rouquier complex of Soergel bimodules which appears in the proof of Elias and Williamson, and like that complex it models a certain "Verma-type" perverse sheaf in the realizable case. In our setting, the perverse sheaf lives on the blow-up variety $Y_\circ$. This variety has a natural stratification $Y_\circ = \coprod_{F\neq\varnothing} U^F_\circ$ with the property that the image of $\overline{U^F_\circ}$ in Y is $\overline{U^F}$. If $j\colon U^E_\circ \to Y_\circ$ denotes the inclusion, then the pure modules $\bar{\mathrm{C}}^k_\circ(\mathrm{M})_\varnothing$ are the cohomology groups of the associated graded sheaves of the weight filtration on the mixed perverse sheaf $j_!\underline{\mathbb{Q}}_{U^E_\circ}$, and the differentials are given by the extensions between successive graded pieces. Since the associated graded sheaves are pure, they are direct sums of shifts of intersection cohomology sheaves, and the fact that $j_!\underline{\mathbb{Q}}_{U^E_\circ}$ is perverse (up to a shift) implies the restriction on shifts of the summands in property (3).

We find the complex $\bar{\mathrm{C}}^\bullet_\circ(\mathrm{M})$ as a quasi-isomorphic subcomplex of a larger complex $\mathrm{C}^\bullet_\circ(\mathrm{M})$ which we call the big reduced Rouquier complex. The big reduced complex is easier to define (Section 8.1): we put $\mathrm{C}^0_\circ(\mathrm{M}) \coloneqq \mathrm{CH}(\mathrm{M})$, and for positive i, we put

$$\mathrm{C}^i_\circ(\mathrm{M}) \coloneqq \bigoplus_{\varnothing<F_1<\cdots<F_i<E} x_{F_1}\cdots x_{F_i}\,\mathrm{CH}(\mathrm{M})[i].$$

The entries of the differential are multiplication by monomials x_F, up to sign. The $\mathrm{H}_\circ(\mathrm{M})$-modules in this complex are **pure** (Corollary 8.4 (2)), meaning that they are isomorphic to direct sums of shifts of modules $\mathrm{IH}_\circ(\mathrm{M}^F)$. The small reduced complex is the quasi-isomorphic complex obtained by canceling all summands which are mapped isomorphically to a summand in the next degree (Lemma 8.14).

While the small reduced Rouquier complex represents a sheaf on $Y_\circ$, the big reduced Rouquier complex reflects the geometry of the resolution $p\colon X \to Y_\circ$. The open set $U \coloneqq p^{-1}(U^E_\circ)$ is the complement of the union D of the divisors D_F for $F \neq \varnothing$, and the sheaf $j_!\underline{\mathbb{Q}}_{U^E_\circ}$ is isomorphic to $p_!j_{U!}\underline{\mathbb{Q}}_U$, where $j_U\colon U \to X$ is the inclusion. The divisor D has normal crossings, so $j_{U!}\underline{\mathbb{Q}}_U$ has a filtration whose i-th graded piece is (up to a shift) the direct sum of constant sheaves on all i-fold intersections of divisors D_F. The cohomology of this graded piece is the module $\mathrm{C}^i_\circ(\mathrm{M})$.

We also construct nonreduced variants: the small Rouquier complex $\bar{\mathrm{C}}^\bullet(\mathrm{M})$ is a complex of graded $\mathrm{H}(\mathrm{M})$-modules, whose i-th entry is a sum of modules $\mathrm{IH}(\mathrm{M}^F)[(i-\operatorname{crk} F)/2]$, where now F is allowed to be any flat, including $\varnothing$ (Theorem 8.16 (1)). As in the reduced case, the small Rouquier complex is quasi-isomorphic to a larger complex $\mathrm{C}^\bullet(\mathrm{M})$, called the big Rouquier complex, which is simpler to define. These complexes represent the extension of $\mathbb{Q}_{U^E}$ by zero on Y. In some ways they are more natural than the reduced complexes, but proving that $\bar{\mathrm{C}}^\bullet(\mathrm{M})$ satisfies the analogue of the perversity property (3) requires a number of properties, including the full canonical decomposition $\mathsf{CD}(\mathrm{M})$, which have not been proved until the full induction is complete.

4.3. **Hard Lefschetz for $\mathbf{IH}(\mathbf{M})$.** The proof of the statement $\mathsf{HL}(\mathrm{M})$ (Definition 3.15) follows a standard argument similar to one which appears in [Kar04] and [EW14], using restriction to divisors to deduce the hard Lefschetz theorem from the Hodge–Riemann relations for smaller matroids (Proposition 12.3). The basic structure we use is a factorization of multiplication by the degree k monomial y_F as the composition of the maps φ^{M}_F and ψ^{M}_F (Proposition 2.15). We take a class $\ell = \sum_{F\in\mathcal{L}^1(\mathrm{M})} c_F y_F$ with positive c_F as in the statement of Theorem 1.5. If we have a class $\eta \in \mathrm{IH}^k(\mathrm{M})$ for $k < d/2$ for which $\ell^{d-2k}\eta = 0$, applying φ^{M}_F for any $F \in \mathcal{L}^1(\mathrm{M})$ gives

$$\varphi^{\mathrm{M}}_F(\ell)^{d-2k} \cdot \varphi^{\mathrm{M}}_F(\eta) = 0.$$

Since $\operatorname{rk} \mathrm{M}_F = d-1$, this says that $\varphi^{\mathrm{M}}_F(\eta)$ is a primitive class in $\mathrm{IH}^k(\mathrm{M}_F)$ with respect to the class $\ell' \coloneqq \varphi^{\mathrm{M}}_F(\ell)$. This class satisfies the hypotheses of Theorem 1.5 for the matroid M_F, so we can assume inductively that the Hodge–Riemann relations hold. By Proposition 2.13 and Lemma 3.6 (1), we have

$$0 = \deg_{\mathrm{M}}(\ell^{d-2k}\eta^2) = \sum_F c_F \deg_{\mathrm{M}_F}((\ell')^{d-2k-1}\varphi^{\mathrm{M}}_F(\eta)^2).$$

Since the c_F are all positive, the Hodge–Riemann relations for M_F imply that all of the summands have the same sign, and so they all must vanish. Since the Hodge–Riemann forms are non-degenerate, we must have $\varphi^{\mathrm{M}}_F(\eta) = 0$ for every F, and so η is annihilated by every y_F. In other words, η is in the socle of the $\mathrm{H}(\mathrm{M})$-module $\mathrm{IH}(\mathrm{M})$. However, we show in Proposition 7.8 that the socle of $\mathrm{IH}(\mathrm{M})$ vanishes in any degree less than or equal to $d/2$ for which the canonical decomposition $\mathsf{CD}(\mathrm{M})$ holds. At this point in the induction, we only know this decomposition outside of the middle degree $d/2$, but this is enough.

4.4. **Deletion induction for $\mathbf{IH}(\mathbf{M})$.** An important step of our argument is deducing the Hodge–Riemann relations $\mathsf{HR}(\mathrm{M})$ and $\underline{\mathsf{HR}}(\mathrm{M})$ (Definition 3.16), except possibly in the middle degree (postponed until Section 4.6), by inductively using the Hodge–Riemann relations for matroids on smaller sets. The arguments for $\mathrm{IH}(\mathrm{M})$ and $\underline{\mathrm{IH}}(\mathrm{M})$ are somewhat parallel, but the case of $\mathrm{IH}(\mathrm{M})$ is simpler, so we begin with it even though it appears later in the structure of the whole proof.

This step uses the relation between M and the **deletion** $\mathrm{M}\backslash i$. This is a matroid on the set $E\backslash i$ whose independent sets are the independent sets of M which do not contain i. We will assume that i is not a coloop of M, which means that there is at least one basis which does not contain i, and therefore that M and $\mathrm{M}\backslash i$ have the same rank. If all elements of E are coloops, then M is a Boolean matroid. This is the base case of our induction; we prove Theorem 3.17 in this case by a direct calculation in Section 12.2. For simplicity, we assume in this section and in Section 4.5 that all of the rank one flats are singletons, and in particular that $\{i\}$ is a flat.

We have a homomorphism $\theta_i^{\mathrm{M}}\colon\ \mathrm{CH}(\mathrm{M}\backslash i) \to \mathrm{CH}(\mathrm{M})$ which takes y_j to y_j for each $j \neq i$, and so it sends $\mathrm{H}(\mathrm{M}\backslash i)$ injectively to $\mathrm{H}(\mathrm{M})$ (Section 9.1). This map plays a major role in the semi-small decomposition of $\mathrm{CH}(\mathrm{M})$ obtained in [BHM+20]. In Section 9, we prove the following result, modulo the technical issue described in Remark 4.2 below.

Theorem. As an $\mathrm{H}(\mathrm{M}\backslash i)$-module, $\mathrm{IH}(\mathrm{M})$ is isomorphic to a direct sum of modules of the form

$$\mathrm{IH}((\mathrm{M}\backslash i)^G)[-(\operatorname{crk} G)/2], \tag{$*$}$$

where G is a flat of $\mathrm{M}\backslash i$ of even corank.

Remark 4.2. At the stage of the induction that this argument appears, we only know the canonical decomposition $\mathsf{CD}(\mathrm{M})$ holds in degrees outside of the middle degree when d is even. So we actually prove the theorem above for a modified module $\widetilde{\mathrm{IH}}(\mathrm{M})$, defined in Section 9.2, which we can prove is a direct summand of $\mathrm{CH}(\mathrm{M})$ (Lemma 9.3). It equals $\mathrm{IH}(\mathrm{M})$ except in the middle degree $d/2$, where it equals $\mathrm{IH}_\circ^{\frac{d}{2}}(\mathrm{M})$. Because of this, the argument below only gives the Hodge–Riemann relations for $\mathrm{IH}(\mathrm{M})$ in degrees strictly less than $d/2$. We need a separate argument later to handle the middle degree, which we highlight in Section 4.6. The theorem as stated is true, but it can only be proved after the entire induction is finished.

Because $\mathrm{M}\backslash i$ has a smaller ground set, we can inductively assume that all of our statements hold for all of the matroids $(\mathrm{M}\backslash i)^G$ in the theorem. In particular, $\mathrm{IH}((\mathrm{M}\backslash i)^G)$ satisfies hard Lefschetz and the Hodge–Riemann relations for any positive linear combination $\ell' = \sum_{j\neq i} c_j y_j \in \mathrm{H}(\mathrm{M}\backslash i)$. The shift by $-(\operatorname{crk} G)/2$ in the summand ($*$) ensures that each summand is centered at the same middle degree as $\mathrm{IH}(\mathrm{M})$, so our theorem shows that $\mathrm{IH}(\mathrm{M})$ satisfies hard Lefschetz for the class ℓ'. That is, $\mathsf{HL}_i(\mathrm{M})$ holds (Corollary 9.7). By keeping careful track of how the Poincaré pairing restricts to the summand ($*$) (Lemma 9.8), we can also deduce that the Hodge–Riemann inequalities hold for ℓ'. That is, the statement $\mathsf{HR}_i^{<\frac{d}{2}}(\mathrm{M})$ also holds (Corollary 9.9).

Next we use a standard deformation argument to pass from the special class ℓ' to a class $\ell = \ell' + c_i y_i$ with positive c_i. We have already shown $\mathsf{HL}(\mathrm{M})$, $\mathsf{HL}_i(\mathrm{M})$, and $\mathsf{HR}_i^{<\frac{d}{2}}(\mathrm{M})$; that is, $\mathrm{IH}(\mathrm{M})$

satisfies hard Lefschetz for both ℓ and ℓ', and the Hodge–Riemann relations hold for ℓ'. But for a continuous family of classes all of which satisfy hard Lefschetz, the signature of the associated pairings cannot change, so the Hodge–Riemann relations for ℓ' imply them for ℓ. Hence, we have deduced the statement $\mathsf{HR}^{<\frac{d}{2}}(\mathrm{M})$ (Proposition 11.1).

Remark 4.3. When M is realizable, the theorem above follows from a study of the properties of a map $q\colon Y \to Y'$, where Y and Y' are the Schubert varieties corresponding to M and $\mathrm{M}\backslash i$, respectively. This map is obtained by restricting the projection $(\mathbb{P}^1)^E \to (\mathbb{P}^1)^{E\backslash i}$, and it is compatible with the stratifications: for each $F \in \mathcal{L}(\mathrm{M})$, the restriction of q to U^F has image $U^{F\backslash i}$. The resulting map $U^F \to U^{F\backslash i}$ is an isomorphism if $\mathrm{rk}_\mathrm{M} F = \mathrm{rk}_{\mathrm{M}\backslash i}(F\backslash i)$, or a fiber bundle with $\mathbb{P}^1$ fibers if $\mathrm{rk}_\mathrm{M} F = \mathrm{rk}_{\mathrm{M}\backslash i}(F\backslash i) + 1$. An easy argument shows that $q_* \mathrm{IC}_Y$ is perverse, and by the decomposition theorem, it is semisimple. These two properties together give the theorem. We point to [BV20, Section 1.1] for more geometric insight in this direction.

In order to prove the theorem for all matroids, we must prove the analogous properties in our algebraic setting: that $\mathrm{IH}(\mathrm{M})$ is pure as an $\mathrm{H}(\mathrm{M}\backslash i)$-module, meaning that it is a direct sum of shifts of modules $\mathrm{IH}((\mathrm{M}\backslash i)^G)$, and that it is perverse, meaning that the shifts of the summands are as in ($*$). Purity follows from the fact that $\mathrm{CH}(\mathrm{M})$ is a direct sum of $\mathrm{CH}(\mathrm{M}\backslash i)$-modules of the form $\mathrm{CH}((\mathrm{M}\backslash i)^F)[k]$ for various flats $F \in \mathcal{L}(\mathrm{M}\backslash i)$ and $k \in \mathbb{Z}$. This was proved in [BHM+20]; we recall this result in Section 9.1.

To show that $\mathrm{IH}(\mathrm{M})$ is perverse as an $\mathrm{H}(\mathrm{M}\backslash i)$-module, we imitate the proof from the geometric case. We define **stalk** and **costalk** functors (Section 5.2)

$$(-)_F, (-)_{[F]}\colon\ \mathrm{H}(\mathrm{M})\text{-mod} \to \mathbb{Q}\text{-mod},\ \ F \in \mathcal{L}(\mathrm{M})$$

generalizing the case $F = \varnothing$ discussed previously. We show that Poincaré duality (Lemma 9.3) together with the no socle condition $\mathsf{NS}(\mathrm{M})$ implies that the stalks and costalks of $\mathrm{IH}(\mathrm{M})$ satisfy the degree restrictions expected for intersection cohomology (Proposition 6.3). When $\mathrm{IH}(\mathrm{M})$ is considered as an $\mathrm{H}(\mathrm{M}\backslash i)$-module, the (co)stalk at $F \in \mathcal{L}(\mathrm{M}\backslash i)$ is the sum of the (co)stalks at the flats $\{F, F \cup i\} \cap \mathcal{L}(\mathrm{M})$ (the short exact sequence (6)). This implies (Lemma 9.5) that the degree restrictions on $\mathrm{H}(\mathrm{M}\backslash i)$ (co)stalks are relaxed by one from the ones for $\mathrm{IH}(\mathrm{M}\backslash i)$, which shows that each summand must appear with the correct "perverse" shift.

Remark 4.4. The map $q\colon Y \to Y'$ resembles a map which naturally appears in the inductive computation of intersection cohomology of Schubert varieties, and which motivates a key step of the proof of Elias–Williamson. Given a Schubert variety X_{ys} with $ys > y$ and s a simple reflection, there is a map from a $\mathbb{P}^1$-bundle over the smaller Schubert variety X_y to X_{ys}. Like the map q, it is compatible with the stratification by cells, and the fibers are either points or rational curves, so the pushforward of the IC sheaf is a perverse sum of IC sheaves. However, the roles of the source and target in the two situations are different. In our case, the base Y' is a simpler variety which we can assume inductively that we already understand. In contrast, the Schubert variety map uses inductive knowledge about X_y to deduce results about the base X_{ys}.

4.5. **Deletion induction for $\underline{\mathrm{IH}}(\mathrm{M})$.** In Section 10, we use a similar argument to deduce hard Lefschetz and the Hodge–Riemann relations for $\underline{\mathrm{IH}}(\mathrm{M})$ from the same statements for matroids on smaller ground sets. There is one significant difficulty, however. We would like to decompose $\underline{\mathrm{IH}}(\mathrm{M})$ as a direct sum of terms of the form

$$\underline{\mathrm{IH}}((\mathrm{M}\backslash i)^G)[-(\operatorname{crk} G)/2], \tag{$**$}$$

but these are not modules over the same ring. The operators $\underline{\beta}_{\mathrm{M}}$ and $\underline{\beta}_{\mathrm{M}\backslash i}$ which act on these spaces are the images of $-x_\varnothing$ in $\mathrm{CH}(\mathrm{M})$ and $\mathrm{CH}(\mathrm{M}\backslash i)$, respectively. However, the natural map $\mathrm{CH}(\mathrm{M}) \to \mathrm{CH}(\mathrm{M}\backslash i)$ sends $x_\varnothing$ to $x_\varnothing + x_{\{i\}}$, so $\underline{\beta}_{\mathrm{M}\backslash i}$ is sent to $\underline{\beta}_{\mathrm{M}} - x_{\{i\}}$. But $x_{\{i\}}$ does not act on $\underline{\mathrm{IH}}(\mathrm{M})$, so we must consider a larger space

$$\underline{\mathrm{IH}}_i(\mathrm{M}) = \underline{\mathrm{IH}}(\mathrm{M}) \oplus \underline{\psi}_{\mathrm{M}}^{\{i\}}\big(\underline{\mathrm{J}}(\mathrm{M}_{\{i\}}) \otimes \underline{\mathrm{CH}}(\mathrm{M}^{\{i\}})\big).$$

It is this space that we decompose into a sum of terms of the form ($**$) (Corollary 10.11).

The upshot is that we can show using the inductive assumptions for matroids $(\mathrm{M}\backslash i)^G$ that hard Lefschetz and Hodge–Riemann hold for the action of $\underline{\beta}_{\mathrm{M}} - x_{\{i\}}$ on $\underline{\mathrm{IH}}_i(\mathrm{M})$ (Corollaries 10.12 and 10.16). This statement, combined with $\underline{\mathsf{NS}}(\mathrm{M})$, implies hard Lefschetz for $\underline{\beta}_{\mathrm{M}}$ on $\underline{\mathrm{IH}}(\mathrm{M})$ (Proposition 12.1). By deforming $\underline{\beta}_{\mathrm{M}} - x_{\{i\}}$ to $\underline{\beta}_{\mathrm{M}}$, we get the Hodge–Riemann relations as well (Proposition 11.4). However, as noted in Section 4.2, in our first pass we only prove $\underline{\mathsf{NS}}(\mathrm{M})$ strictly below the critical degree $(d-2)/2$, so we only get hard Lefschetz and Hodge–Riemann in that range as well. When d is even, we need an additional chain of arguments to finish the proof in this degree.

4.6. **The middle degree.** Finally, we are faced with the problem of proving the Hodge–Riemann relations in the middle degree $\mathrm{IH}^{\frac{d}{2}}(\mathrm{M})$. Although the space of primitive classes depends on the choice of an ample class ℓ, if we already know the Hodge–Riemann relations in degrees below $d/2$, then showing them in middle degree is equivalent to showing that the signature of the Poincaré pairing on the whole space $\mathrm{IH}^{\frac{d}{2}}(\mathrm{M})$ is $\sum_{k\geqslant 0}(-1)^k \dim \mathrm{IH}^k(\mathrm{M})$ (Proposition 7.10).

We say that a graded vector space with non-degenerate pairing that satisfies this condition on the pairing in middle degree is **Hancock** (that is, "has a nice signature"). This condition is preserved by taking tensor products and orthogonal direct sums (Lemma 7.11). In [BHM+20], we showed that $\mathrm{CH}(\mathrm{M})$ satisfies Hodge–Riemann, so in particular it is Hancock. The fact that $\underline{\mathrm{IH}}(\mathrm{M})$ satisfies hard Lefschetz and Hodge–Riemann implies that $\underline{\mathrm{J}}(\mathrm{M})$ does too, so we can deduce that each summand $\psi_{\mathrm{M}}^F\big(\underline{\mathrm{J}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F)\big)$ in the decomposition $\mathsf{CD}(\mathrm{M})$ is Hancock (Corollary 7.14). If every term but one in an orthogonal direct sum decomposition is Hancock, and the whole space is as well, then the remaining summand is Hancock (Lemma 7.12). Thus, once we have the canonical decomposition $\mathsf{CD}(\mathrm{M})$, we can deduce that $\mathrm{IH}(\mathrm{M})$ is Hancock and thus satisfies Hodge–Riemann in middle degree (Proposition 7.16).

At this point, our induction still has a gap because we have not proved the decomposition $\mathsf{CD}(\mathrm{M})$ in the middle degree $d/2$. To fix this, we first work with $\mathrm{IH}_\circ(\mathrm{M})$, which we do know is a direct summand of $\mathrm{CH}(\mathrm{M})$. Following the argument of the previous paragraph shows that $\mathrm{IH}_\circ(\mathrm{M})$ satisfies the Hodge–Riemann relations in all degrees (Propositions 7.15 and 11.7), and this implies that $\mathrm{IH}_\circ(\mathrm{M})$ has no socle in degrees less than or equal to $d/2$ as an $\mathrm{H}_\circ(\mathrm{M})$-module (Proposition 12.4). Because $\underline{\mathrm{IH}}(\mathrm{M})$ is the quotient of $\mathrm{IH}_\circ(\mathrm{M})$ by the action of the generators of $\mathrm{H}(\mathrm{M})$, this implies the full condition $\underline{\mathsf{NS}}(\mathrm{M})$, including in the missing degree $(d-2)/2$ (Proposition 12.5). But the lack of socle in $\underline{\mathrm{IH}}^{\frac{d-2}{2}}(\mathrm{M})$ is equivalent to hard Lefschetz in that degree (Proposition 12.6), which gives the final ingredient needed to close the induction loop and prove the full canonical decomposition $\mathsf{CD}(\mathrm{M})$ (Proposition 12.2).

5. Modules over the graded Möbius algebra

We begin by defining and studying some basic constructions involving graded modules over the graded Möbius algebra $\mathrm{H}(\mathrm{M})$. This section is entirely independent of Section 3.

5.1. **Annihilators.**

We begin with a general lemma about annihilators of ideals in Poincaré duality algebras.

Lemma 5.1. Let R be a finite-dimensional commutative algebra equipped with a degree map with respect to which R satisfies Poincaré duality as in Theorems 2.19 (1) and 2.20 (1). Let $I, J \subseteq R$ be ideals. Let $\operatorname{Ann}(I)$ denote the annihilator of I in R. The following identities hold:

(1) If $J = \operatorname{Ann}(I)$, then $I = \operatorname{Ann}(J)$;

(2) $\operatorname{Ann}(I + J) = \operatorname{Ann}(I) \cap \operatorname{Ann}(J)$;

(3) $\operatorname{Ann}(I \cap J) = \operatorname{Ann}(I) + \operatorname{Ann}(J)$.

Proof. For the first item, notice that $\operatorname{Ann}(I) = I^\perp$, where the perp is taken with respect to the Poincaré duality pairing of R. Since $(I^\perp)^\perp = I$, the first assertion follows. The second item is obvious. For the third item, we use the first and second items to conclude

$$\begin{aligned}\operatorname{Ann}(I \cap J) &= \operatorname{Ann}\Big(\operatorname{Ann}\big(\operatorname{Ann}(I)\big) \cap \operatorname{Ann}\big(\operatorname{Ann}(J)\big)\Big)\\ &= \operatorname{Ann}\Big(\operatorname{Ann}\big(\operatorname{Ann}(I) + \operatorname{Ann}(J)\big)\Big)\\ &= \operatorname{Ann}(I) + \operatorname{Ann}(J).\end{aligned}$$

□

Lemma 5.2. The ideals $\langle x_\varnothing \rangle$ and $\langle y_i \mid i \in E\rangle$ are mutual annihilators inside of $\mathrm{CH}(\mathrm{M})$.

Proof. By Proposition 2.5 and Proposition 2.7, the annihilator of $x_\varnothing$ is equal to the kernel of $\varphi^\varnothing_{\mathrm{M}}$, which is equal to $\langle y_i \mid i \in E\rangle$. The opposite statement follows from Theorem 2.19 (1) and Lemma 5.1 (1). □

An upwardly closed subset $\Sigma \subseteq \mathcal{L}(\mathrm{M})$ is called an **order filter**. For any flat F of M, we will denote the order filters $\{G \mid G \geqslant F\}$ and $\{G \mid G > F\}$ by $\Sigma_{\geqslant F}$ and $\Sigma_{>F}$, respectively.

Definition 5.3. For any order filter Σ, we define an ideal of the graded Möbius algebra

$$\Upsilon_\Sigma \coloneqq \mathbb{Q}\{y_G \mid G \in \Sigma\} \subseteq \mathrm{H}(\mathrm{M}).$$

Note that $y_\varnothing = 1$ and $\Upsilon_{\mathcal{L}(\mathrm{M})} = \mathrm{H}(\mathrm{M})$. We will write

$$\Upsilon_{\geqslant F} \coloneqq \Upsilon_{\Sigma_{\geqslant F}} \ \text{ and } \ \Upsilon_{>F} \coloneqq \Upsilon_{\Sigma_{>F}}.$$

The following lemma generalizes Lemma 5.2.

Lemma 5.4. For any order filter Σ, the ideals $\mathrm{CH}(\mathrm{M}) \cdot \Upsilon_\Sigma$ and $\mathrm{CH}(\mathrm{M}) \cdot \{x_F \mid F \notin \Sigma\}$ are mutual annihilators in $\mathrm{CH}(\mathrm{M})$.

Proof. By Lemma 5.1 (1), it is sufficient to prove that $\mathrm{CH}(\mathrm{M}) \cdot \Upsilon_\Sigma$ is the annihilator of the set $\{x_F \mid F \notin \Sigma\}$. If $F \notin \Sigma$ and $G \in \Sigma$, then $G \nleqslant F$, and hence

$$y_G x_F = 0.$$

This proves that $\mathrm{CH}(\mathrm{M}) \cdot \Upsilon_\Sigma$ annihilates $\{x_F \mid F \notin \Sigma\}$. For the opposite inclusion, we use downward induction on the cardinality of Σ.

The base case $\Sigma = \mathcal{L}(\mathrm{M})$ is trivial. Now suppose that Σ is a proper order filter and that the statement is true for all order filters strictly containing Σ. Let η be an element of $\mathrm{CH}(\mathrm{M})$ satisfying $\eta x_F = 0$ for all $F \notin \Sigma$. We need to show that η is in the ideal $\Upsilon_\Sigma \cdot \mathrm{CH}(\mathrm{M})$.

Let H be a maximal flat not in Σ. Then $\eta x_H = 0$, and applying our inductive hypothesis to the order filter $\Sigma \cup \{H\}$, we find that

$$\eta \in \Upsilon_{\Sigma \cup \{H\}} \cdot \mathrm{CH}(\mathrm{M}) = y_H \,\mathrm{CH}(\mathrm{M}) + \Upsilon_\Sigma \cdot \mathrm{CH}(\mathrm{M}).$$

Now, for some $\xi, \xi_F \in \mathrm{CH}(\mathrm{M})$, we may write

$$\eta = y_H \xi + \sum_{F \in \Sigma} y_F \xi_F.$$

Since $H \notin \Sigma$, we have $x_H y_F = 0$ for all $F \in \Sigma$, and hence

$$0 = x_H \eta = x_H y_H \xi + \sum_{F \in \Sigma} x_H y_F \xi_F = x_H y_H \xi = x_H \psi^{\mathrm{M}}_H \varphi^{\mathrm{M}}_H(\xi) = \psi^{\mathrm{M}}_H\big(x_\varnothing \varphi^{\mathrm{M}}_H(\xi)\big).$$

Since ψ^{M}_H is injective, we have $x_\varnothing \varphi^{\mathrm{M}}_H(\xi) = 0 \in \mathrm{CH}(\mathrm{M}_H)$. By Lemma 5.2, it follows that $\varphi^{\mathrm{M}}_H(\xi)$ is in the ideal $\langle y_{K \setminus H} \mid K > H\rangle \subseteq \mathrm{CH}(\mathrm{M}_H)$. Applying ψ^{M}_H, we see that $y_H \xi = \psi^{\mathrm{M}}_H \varphi^{\mathrm{M}}_H(\xi)$ is in the ideal

$\langle y_K \mid K > H\rangle \subseteq \mathrm{CH}(\mathrm{M})$. By the maximality of H, any flat K strictly containing H is in Σ. Thus, y_H is in $\Upsilon_\Sigma \cdot \mathrm{CH}(\mathrm{M})$, and we conclude that η is in $\Upsilon_\Sigma \cdot \mathrm{CH}(\mathrm{M})$. $\square$

5.2. **Stalks and costalks.** For an order filter Σ and a graded $\mathrm{H}(\mathrm{M})$-module N, we define

$$\mathrm{N}_\Sigma \coloneqq \Upsilon_\Sigma \cdot \mathrm{N} \ \text{ and } \ \mathrm{N}^\Sigma \coloneqq \{n \in \mathrm{N} \mid \Upsilon_\Sigma \cdot n = 0\}.$$

We will write

$$\mathrm{N}_{\geqslant F} \coloneqq \mathrm{N}_{\Sigma_{\geqslant F}} \ \text{ and } \ \mathrm{N}^{\geqslant F} \coloneqq \mathrm{N}^{\Sigma_{\geqslant F}},$$

and similarly for the order filter $\Sigma_{>F}$. Clearly, if $\Sigma' \subseteq \Sigma$, then $\mathrm{N}_{\Sigma'} \subseteq \mathrm{N}_\Sigma$ and $\mathrm{N}^\Sigma \subseteq \mathrm{N}^{\Sigma'}$.

Definition 5.5. We define the **stalk** of N at F to be the quotient

$$\mathrm{N}_F \coloneqq \frac{\mathrm{N}_{\geqslant F}[\operatorname{rk} F]}{\mathrm{N}_{>F}[\operatorname{rk} F]}.$$

Dually, we define the **costalk** of N at F to be the quotient

$$\mathrm{N}_{[F]} \coloneqq \frac{\mathrm{N}^{>F}}{\mathrm{N}^{\geqslant F}}.$$

Both the stalk and costalk give endofunctors on the category of graded $\mathrm{H}(\mathrm{M})$-modules, and multiplication by y_F induces a natural transformation from the costalk functor to the stalk functor. The following lemma is immediate from the definitions.

Lemma 5.6. For any graded $\mathrm{H}(\mathrm{M})$-module N, we have $\mathrm{N}_{[\varnothing]} = \operatorname{soc}(\mathrm{N})$. If N is a direct summand of $\mathrm{CH}(\mathrm{M})$, then we also have a canonical isomorphism $\mathrm{N}_\varnothing \cong \varphi^\varnothing_{\mathrm{M}}\mathrm{N} \subseteq \underline{\mathrm{CH}}(\mathrm{M})$.

Lemma 5.7. For any graded $\mathrm{H}(\mathrm{M})$-module N, we have canonical isomorphisms

$$\mathrm{N}_F \cong (y_F\mathrm{N}[\operatorname{rk} F])_\varnothing \ \text{ and } \ \mathrm{N}_{[F]} \cong (y_F\mathrm{N}[\operatorname{rk} F])_{[\varnothing]}.$$

Proof. The first statement follows from

$$\mathrm{N}_F = \frac{\mathrm{N}_{\geqslant F}[\operatorname{rk} F]}{\mathrm{N}_{>F}[\operatorname{rk} F]} = \frac{(y_F\mathrm{N})_{\geqslant\varnothing}[\operatorname{rk} F]}{(y_F\mathrm{N})_{>\varnothing}[\operatorname{rk} F]} \cong (y_F\mathrm{N})_\varnothing[\operatorname{rk} F] \cong (y_F\mathrm{N}[\operatorname{rk} F])_\varnothing$$

The second statement follows from

$$\mathrm{N}_{[F]} = \frac{\mathrm{N}^{>F}}{\mathrm{N}^{\geqslant F}} \cong y_F\mathrm{N}^{>F}[\operatorname{rk} F] = (y_F\mathrm{N})^{>\varnothing}[\operatorname{rk} F] = (y_F\mathrm{N})_{[\varnothing]}[\operatorname{rk} F] \cong (y_F\mathrm{N}[\operatorname{rk} F])_{[\varnothing]}. \quad \square$$

For any graded $\mathrm{H}(\mathrm{M})$-module N, we write N^* for $\operatorname{Hom}_{\mathbb{Q}}(\mathrm{N}, \mathbb{Q})$. Note that N^* has a natural graded $\mathrm{H}(\mathrm{M})$-module structure.

Lemma 5.8. For any graded $\mathrm{H}(\mathrm{M})$-module N and any flat F, we have a canonical isomorphism of graded $\mathrm{H}(\mathrm{M})$-modules

$$(\mathrm{N}_F)^* \cong (\mathrm{N}^*)_{[F]}.$$

Proof. We first prove the lemma when $F = \varnothing$. The module $(\mathrm{N}_\varnothing)^*$ is equal to the submodule of N^* consisting of functions that vanish on $\mathrm{N}_{>\varnothing}$, which is the same as $(\mathrm{N}^*)_{[\varnothing]}$.

Now consider an arbitrary flat F. By Lemma 5.7 and the case that we just proved, we have

$$(\mathrm{N}_F)^* \cong \big((y_F\mathrm{N}[\operatorname{rk} F])_\varnothing\big)^* \cong \big((y_F\mathrm{N}[\operatorname{rk} F])^*\big)_{[\varnothing]} \cong (y_F\mathrm{N})^*[-\operatorname{rk} F]_{[\varnothing]}.$$

Since multiplication by y_F is an $\mathrm{H}(\mathrm{M})$-module homomorphism of degree $\operatorname{rk} F$, we have

$$(y_F\mathrm{N})^*[-\operatorname{rk} F] \cong y_F(\mathrm{N}^*)[\operatorname{rk} F].$$

Therefore, we have

$$(\mathrm{N}_F)^* \cong \big(y_F(\mathrm{N}^*)[\operatorname{rk} F]\big)_{[\varnothing]} \cong (\mathrm{N}^*)_{[F]},$$

where the second isomorphism follows from Lemma 5.7. □

5.3. **Pure modules.** We say that a graded $\mathrm{H}(\mathrm{M})$-module N is **pure** if it is isomorphic to a direct sum of direct summands of modules of the form $\mathrm{CH}(\mathrm{M}^F)[k]$, where $F \in \mathcal{L}(\mathrm{M})$ and $k \in \mathbb{Z}$. Fix an ordering $\{F_1, \ldots, F_r\}$ of $\mathcal{L}(\mathrm{M})$ refining the natural partial order, so that for any k, the set

$$\Sigma_k \coloneqq \{F_k, \ldots, F_r\}$$

is an order filter. Note that we have natural inclusions $\Upsilon_{\geqslant F_k} \subseteq \Upsilon_{\Sigma_k}$ and $\Upsilon_{>F_k} \subseteq \Upsilon_{\Sigma_{k+1}}$.

Proposition 5.9. Let N be a pure graded $\mathrm{H}(\mathrm{M})$-module.

(1) For all k, the above inclusions induce an isomorphism

$$\mathrm{N}_{F_k} = \frac{\mathrm{N}_{\geqslant F_k}[\operatorname{rk} F_k]}{\mathrm{N}_{>F_k}[\operatorname{rk} F_k]} \xrightarrow{\cong} \frac{\mathrm{N}_{\Sigma_k}[\operatorname{rk} F_k]}{\mathrm{N}_{\Sigma_{k+1}}[\operatorname{rk} F_k]}.$$

(2) For all k, the above inclusions induce an isomorphism

$$\frac{\mathrm{N}^{\Sigma_{k+1}}}{\mathrm{N}^{\Sigma_k}} \xrightarrow{\cong} \frac{\mathrm{N}^{>F_k}}{\mathrm{N}^{\geqslant F_k}} = \mathrm{N}_{[F_k]}.$$

Proof. The desired properties are preserved under taking direct sums, passing to direct summands, and shifting degree, so we may assume that $\mathrm{N} = \mathrm{CH}(\mathrm{M}^F)$ for some flat F. If $F \not\geqslant F_k$, then the source and target of both maps are zero, so both statements are trivial. Thus we may assume that $F \geqslant F_k$. Notice that if we replace M by M^F and each order filter of $\mathcal{L}(\mathrm{M})$ by its intersection with $\mathcal{L}(\mathrm{M}^F)$, none of the modules in the formulas change. So without loss of generality, we can also assume that $F = E$, that is, $\mathrm{M}^F = \mathrm{M}$.

Since $\mathrm{CH}(\mathrm{M})_{\Sigma_k} = \mathrm{CH}(\mathrm{M})_{\geqslant F_k} + \mathrm{CH}(\mathrm{M})_{\Sigma_{k+1}}$, the first map is surjective. To show that the first map is injective, notice that

$$\begin{aligned}
\mathrm{CH}(\mathrm{M})_{\geqslant F_k} \cap \mathrm{CH}(\mathrm{M})_{\Sigma_{k+1}} &= \mathrm{CH}(\mathrm{M}) \cdot \Upsilon_{\geqslant F_k} \cap \mathrm{CH}(\mathrm{M}) \cdot \Upsilon_{\Sigma_{k+1}} \\
&= \mathrm{Ann}\{x_G \mid G \ngeqslant F_k\} \cap \mathrm{Ann}\{x_G \mid G \notin \Sigma_{k+1}\} \\
&= \mathrm{Ann}\{x_G \mid G \ngtr F_k\} \\
&= \mathrm{CH}(\mathrm{M}) \cdot \Upsilon_{>F_k} \\
&= \mathrm{CH}(\mathrm{M})_{>F_k},
\end{aligned}$$

where the second and fourth equalities follow from Lemma 5.4 and the third equality follows from the fact that $\{G \mid G > F_k\} = \Sigma_{k+1} \cap \{G \mid G \geqslant F_k\}$. Thus, the first map is an isomorphism.

Since $\mathrm{CH}(\mathrm{M})^{\Sigma_{k+1}} \cap \mathrm{CH}(\mathrm{M})^{\geqslant F_k} = \mathrm{CH}(\mathrm{M})^{\Sigma_k}$, the second map is injective. To show that the second map is surjective, notice that

$$\begin{aligned}
\mathrm{CH}(\mathrm{M})^{\Sigma_{k+1}} + \mathrm{CH}(\mathrm{M})^{\geqslant F_k} &= \mathrm{Ann}\, \Upsilon_{\Sigma_{k+1}} + \mathrm{Ann}\, \Upsilon_{\geqslant F_k} \\
&= \mathrm{CH}(\mathrm{M}) \cdot \{x_G \mid G \notin \Sigma_{k+1}\} + \mathrm{CH}(\mathrm{M}) \cdot \{x_G \mid G \ngeqslant F_k\} \\
&= \mathrm{CH}(\mathrm{M}) \cdot \{x_G \mid G \ngtr F_k\} \\
&= \mathrm{Ann}\, \Upsilon_{>F_k} \\
&= \mathrm{CH}(\mathrm{M})^{>F_k},
\end{aligned}$$

where the second and fourth equalities follow from Lemma 5.4 and the third equality follows from the fact that $\Sigma_k = \Sigma_{k+1} \cup \{G \mid G \geqslant F_k\}$. Thus, the second map is an isomorphism. □

5.4. **Orlik–Solomon algebra.** For a matroid M with ground set E, we recall the definition and some basic facts about the Orlik–Solomon algebra of M. We refer to [OT92, Section 3.1] for more details.

Let $\mathcal{E}^1$ be the vector space over $\mathbb{Q}$ with basis $\{e_i\}_{i\in E}$, and let $\mathcal{E}$ be the exterior algebra generated by $\mathcal{E}^1$. Define a degree -1 linear map $\partial_{\mathcal{E}} \colon \mathcal{E} \to \mathcal{E}$ by setting $\partial_{\mathcal{E}} 1 = 0$, $\partial_{\mathcal{E}} e_i = 1$, and

$$\partial_{\mathcal{E}}(e_{i_1} \cdots e_{i_l}) = \sum_{k=1}^{l} (-1)^k e_{i_1} \cdots \widehat{e_{i_k}} \cdots e_{i_l} \ \text{ for any } i_1, \ldots, i_l \in E.$$

For any subset $S = \{i_1, \ldots, i_l\} \subseteq E$, we denote $e_{i_1} \cdots e_{i_l}$ by e_S. The **Orlik–Solomon** algebra of M, denoted by $\mathrm{OS}(\mathrm{M})$, is the quotient of $\mathcal{E}$ by the ideal generated by $\partial_{\mathcal{E}} e_S$ for all dependent sets S of M. The differential ∂ descends to a differential ∂ on $\mathrm{OS}(\mathrm{M})$, and the complex $(\mathrm{OS}(\mathrm{M}), \partial)$ is acyclic whenever the rank of M is positive.

For any flat F of M, we define a graded subspace $\mathcal{E}_F$ of $\mathcal{E}$ generated by those monomials e_S for all subsets $S \subseteq E$ with closure F. Then we have a direct sum decomposition

$$\mathcal{E} = \bigoplus_{F \in \mathcal{L}(\mathrm{M})} \mathcal{E}_F,$$

which induces a direct sum decomposition

$$\mathrm{OS}(\mathrm{M}) = \bigoplus_{F \in \mathcal{L}(\mathrm{M})} \mathrm{OS}_F(\mathrm{M}).$$

Moreover, the natural ring map $\mathrm{OS}(\mathrm{M}^F) \to \mathrm{OS}(\mathrm{M})$ induces an isomorphism of vector spaces

$$\mathrm{OS}^{\operatorname{rk} F}(\mathrm{M}^F) \cong \mathrm{OS}_F(\mathrm{M}).$$

5.5. **The costalk complex.** Let N be a graded $\mathrm{H}(\mathrm{M})$-module. For all $0 \leqslant k \leqslant d = \operatorname{rk} \mathrm{M}$, let

$$\mathrm{N}_!^k := \bigoplus_{F \in \mathcal{L}^k(\mathrm{M})} \mathrm{OS}_F(\mathrm{M})^* \otimes y_F \mathrm{N}.$$

Note that $\mathrm{OS}_F(\mathrm{M})$ sits entirely in degree $\operatorname{rk} F$ and $\mathrm{OS}_F(\mathrm{M})^*$ sits in degree $-\operatorname{rk} F$. In particular, tensoring with $\mathrm{OS}_F(\mathrm{M})^*$ and multiplying by y_F has no net effect on degrees.

We define a differential $\delta^k \colon \mathrm{N}_!^k \to \mathrm{N}_!^{k+1}$ as follows. If $F \in \mathcal{L}^k(\mathrm{M})$ and $G \in \mathcal{L}^{k+1}(\mathrm{M})$, then the (F, G)-component of δ^k is zero unless $F < G$. If $F < G$, choose $i \in G \backslash F$ so that $y_G = y_i y_F$. Then the (F, G)-component of δ^k is given on the first tensor factor by the (F, G)-component of $\partial^* \colon \mathrm{OS}_F(\mathrm{M})^* \to \mathrm{OS}_G(\mathrm{M})^*$ and on the second tensor factor by multiplication by y_i.

Proposition 5.10. If N is pure, then $\mathrm{H}^0(\mathrm{N}_!^\bullet) \cong \mathrm{N}_{[\varnothing]}$ and $\mathrm{H}^m(\mathrm{N}_!^\bullet) = 0$ for all $m > 0$.

Proof. Choose a total order on $\mathcal{L}(\mathrm{M})$ and define order filters Σ_k as in Section 5.3. Consider the filtration

$$0 = (\mathrm{N}^{\Sigma_1})_!^\bullet \subseteq \cdots \subseteq (\mathrm{N}^{\Sigma_r})_!^\bullet \subseteq (\mathrm{N}^{\Sigma_{r+1}})_!^\bullet = \mathrm{N}_!^\bullet$$

obtained by applying the functor $(\cdot)_!^\bullet$ to the filtration $0 = \mathrm{N}^{\Sigma_1} \subseteq \cdots \subseteq \mathrm{N}^{\Sigma_r} \subseteq \mathrm{N}^{\Sigma_{r+1}} = \mathrm{N}$.

We claim that the quotient complex

$$\frac{(\mathrm{N}^{\Sigma_{k+1}})_!^\bullet}{(\mathrm{N}^{\Sigma_k})_!^\bullet}$$

is acyclic when $k \neq 1$, and when $k = 1$, it is quasi-isomorphic to the module $\mathrm{N}_{[\varnothing]}$ concentrated in homological degree zero. Given the claim, the desired result then follows from the spectral sequence relating the cohomology of a filtered complex to the cohomology of its associated graded complexes.

To show the above claim, we consider the short exact sequence

$$0 \to \mathrm{N}^{\Sigma_k \cup \Sigma_{\geqslant F}} \to \mathrm{N}^{\Sigma_k} \xrightarrow{\cdot y_F} y_F \mathrm{N}^{\Sigma_k} \to 0,$$

for any k and any flat F. Taking the quotient of the sequence for $k+1$ by the sequence for k, we obtain a short exact sequence

$$0 \to \frac{\mathrm{N}^{\Sigma_{k+1} \cup \Sigma_{\geqslant F}}}{\mathrm{N}^{\Sigma_k \cup \Sigma_{\geqslant F}}} \to \frac{\mathrm{N}^{\Sigma_{k+1}}}{\mathrm{N}^{\Sigma_k}} \to \frac{y_F \mathrm{N}^{\Sigma_{k+1}}}{y_F \mathrm{N}^{\Sigma_k}} \to 0.$$

By Proposition 5.9 (2), the middle term of this sequence is isomorphic to $\mathrm{N}_{[F_k]}$. If $F \leqslant F_{k+1}$, then $\Sigma_{k+1} \cup \Sigma_{\geqslant F} = \Sigma_k \cup \Sigma_{\geqslant F}$, and the first term in our sequence is therefore zero. On the other hand, if $F \nleqslant F_{k+1}$, then Proposition 5.9 (2) implies that the first term of our sequence is $\mathrm{N}_{[F_k]}$, and therefore that the first map in our sequence is an isomorphism. Putting these two observations together, we conclude that

$$\frac{y_F \mathrm{N}^{\Sigma_{k+1}}}{y_F \mathrm{N}^{\Sigma_k}} \cong \begin{cases} \mathrm{N}_{[F_k]} & \text{if } F \leqslant F_{k+1}, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that there is an isomorphism of complexes

$$\frac{(\mathrm{N}^{\Sigma_{k+1}})_!^\bullet}{(\mathrm{N}^{\Sigma_k})_!^\bullet} \cong \mathrm{OS}(\mathrm{M}^{F_k})^* \otimes \mathrm{N}_{[F_k]},$$

where the right-hand side has the differential $\partial^* \otimes \mathrm{id}_{\mathrm{N}_{[F_k]}}$. Therefore, the complex is acyclic unless $\operatorname{rk} \mathrm{M}^{F_k} = 0$. This happens only when $k = 1$, in which case the quotient complex has only the module $\mathrm{N}_{[F_1]} = \mathrm{N}_{[\varnothing]}$ in homological degree zero. □

6. Intersection cohomology as a module over the graded Möbius algebra

The intersection cohomology $\mathrm{IH}(\mathrm{M}) \subseteq \mathrm{CH}(\mathrm{M})$ is a graded $\mathrm{H}(\mathrm{M})$-module. In addition, for any flat F, the ring homomorphism $\varphi_F^{\mathrm{M}}\colon \mathrm{CH}(\mathrm{M}) \to \mathrm{CH}(\mathrm{M}_F)$ induces a natural $\mathrm{H}(\mathrm{M})$-module structure on $\mathrm{CH}(\mathrm{M}_F)$. In this section, we apply some of the constructions from Section 5 to these modules.

For most of the remainder of the paper, we will prove very few absolute statements. Most of what we prove will be of the form "If X holds, then so does Y." At the end, we will use all of these results in a modular way to complete our inductive proof of Theorem 3.17.

Remark 6.1. The three main results of this section are Proposition 6.3, Corollary 6.4, and Proposition 6.5. Each of these statements has two parts, the first pertaining to the module $\mathrm{IH}(\mathrm{M})$ and the second pertaining to the module $\mathrm{IH}_\circ(\mathrm{M})$. We note that only the second parts of these three statements will be used in our large induction. The first parts require that we know $\mathrm{CD}(\mathrm{M})$, and will only be applied after the induction is complete. This was alluded to earlier in Remark 1.9.

6.1. Stalks and costalks of the intersection cohomology modules.

Lemma 6.2. Let F be a nonempty flat such that $\mathrm{CD}(\mathrm{M}_F)$ holds.

(1) If $\mathsf{CD}^k(\mathrm{M})$ holds, then $\varphi_F^{\mathrm{M}}\,\mathrm{IH}^k(\mathrm{M}) = \mathrm{IH}^k(\mathrm{M}_F)$. If this holds for all k, then we have a graded $\mathrm{H}(\mathrm{M})$-module isomorphism $y_F\,\mathrm{IH}(\mathrm{M}) \cong \mathrm{IH}(\mathrm{M}_F)[-\operatorname{rk} F]$.

(2) If $\mathsf{CD}_\circ^k(\mathrm{M})$ holds, then $\varphi_F^{\mathrm{M}}\,\mathrm{IH}_\circ^k(\mathrm{M}) = \mathrm{IH}^k(\mathrm{M}_F)$. If this holds for all k, then we have a graded $\mathrm{H}(\mathrm{M})$-module isomorphism $y_F\,\mathrm{IH}_\circ(\mathrm{M}) \cong \mathrm{IH}(\mathrm{M}_F)[-\operatorname{rk} F]$.

Proof. We prove statement (1); the proof of (2) is identical. For notational convenience, we will assume that $\mathsf{CD}^k(\mathrm{M})$ holds for all k, but in fact the argument makes sense one degree at a time.

For any nonempty proper flat G of M, we apply φ_F^{M} to the direct summand $\psi_{\mathrm{M}}^G \underline{\mathrm{J}}(\mathrm{M}_G)\otimes \mathrm{CH}(\mathrm{M}^G)$. By [BHM+20, Proposition 2.23], if $G \ngeqslant F$, then

$$\varphi_F^{\mathrm{M}}\psi_{\mathrm{M}}^G\underline{\mathrm{J}}(\mathrm{M}_G)\otimes \mathrm{CH}(\mathrm{M}^G) = 0.$$

By Lemma 3.6 (2), if $G \geqslant F$, then

$$\varphi_F^{\mathrm{M}}\psi_{\mathrm{M}}^G\underline{\mathrm{J}}(\mathrm{M}_G)\otimes \mathrm{CH}(\mathrm{M}^G) = \psi_{\mathrm{M}_F}^{G\setminus F}\,\underline{\mathrm{J}}(\mathrm{M}_G)\otimes \mathrm{CH}(\mathrm{M}_F^G).$$

Therefore, we have

$$\varphi_F^{\mathrm{M}}\Big(\bigoplus_{G<E}\psi_{\mathrm{M}}^G\underline{\mathrm{J}}(\mathrm{M}_G)\otimes \mathrm{CH}(\mathrm{M}^G)\Big) = \bigoplus_{F\leqslant G<E}\psi_{\mathrm{M}_F}^{G\setminus F}\,\underline{\mathrm{J}}(\mathrm{M}_G)\otimes \mathrm{CH}(\mathrm{M}_F^G).$$

By Lemma 3.6 (1), we also have $\varphi_F^{\mathrm{M}}\,\mathrm{IH}(\mathrm{M}) \subseteq \varphi_F^{\mathrm{M}}\,\mathrm{IH}_\circ(\mathrm{M}) \subseteq \mathrm{IH}(\mathrm{M}_F)$. Therefore, the map φ_F^{M} is compatible with the canonical decompositions in the sense that it maps $\mathrm{IH}(\mathrm{M})$ to $\mathrm{IH}(\mathrm{M}_F)$ and it maps the sum of the smaller summands to the sum of the smaller summands. Since φ_F^{M} is surjective, it must restrict to a surjective map from $\mathrm{IH}(\mathrm{M})$ to $\mathrm{IH}(\mathrm{M}_F)$, so $\varphi_F^{\mathrm{M}}\,\mathrm{IH}(\mathrm{M}) = \mathrm{IH}(\mathrm{M}_F)$. Applying the injective map ψ_F^{M} to this equality, we obtain the second part of statement (1). □

Proposition 6.3. Suppose that F is a proper flat for which $\mathsf{CD}(\mathrm{M}_F)$, $\mathsf{PD}(\mathrm{M}_F)$, and $\mathsf{NS}(\mathrm{M}_F)$ hold.

(1) If $\mathsf{CD}(\mathrm{M})$ holds, then the costalk $\mathrm{IH}(\mathrm{M})_{[F]}$ vanishes in degrees less than or equal to $(\operatorname{crk} F)/2$ and the stalk $\mathrm{IH}(\mathrm{M})_F$ vanishes in degrees greater than or equal to $(\operatorname{crk} F)/2$. In particular, $\mathrm{IH}(\mathrm{M})_{[F]} \to \mathrm{IH}(\mathrm{M})_F$ is the zero map.

(2) Suppose in addition that $F \neq \varnothing$. If $\mathsf{CD}_\circ(\mathrm{M})$ holds, then the costalk $\mathrm{IH}_\circ(\mathrm{M})_{[F]}$ vanishes in degrees less than or equal to $(\operatorname{crk} F)/2$ and the stalk $\mathrm{IH}_\circ(\mathrm{M})_F$ vanishes in degrees greater than or equal to $(\operatorname{crk} F)/2$. In particular, $\mathrm{IH}_\circ(\mathrm{M})_{[F]} \to \mathrm{IH}_\circ(\mathrm{M})_F$ is the zero map.

Proof. For any nonempty proper flat F, it follows from Lemmas 5.7 and 6.2 (2) that

$$\mathrm{IH}_\circ(\mathrm{M})_{[F]} \cong \big(y_F\,\mathrm{IH}_\circ(\mathrm{M})[\operatorname{rk} F]\big)_{[\varnothing]} \cong \mathrm{IH}(\mathrm{M}_F)_{[\varnothing]}.$$

Thus, $\mathsf{NS}(\mathrm{M}_F)$ implies that $\mathrm{IH}_\circ(\mathrm{M})_{[F]}$ vanishes in degrees less than or equal to $(\operatorname{crk} F)/2$. Similarly, we have

$$\mathrm{IH}_\circ(\mathrm{M})_F \cong \big(y_F\,\mathrm{IH}_\circ(\mathrm{M})[\operatorname{rk} F]\big)_{\varnothing} \cong \mathrm{IH}(\mathrm{M}_F)_{\varnothing}.$$

By $\mathsf{PD}(\mathrm{M}_F)$, there is a natural isomorphism $\mathrm{IH}(\mathrm{M}_F)^* \cong \mathrm{IH}(\mathrm{M}_F)[\operatorname{crk} F]$ of $\mathrm{H}(\mathrm{M})$-modules. Then by Lemma 5.8, we have

$$\mathrm{IH}(\mathrm{M}_F)_\varnothing \cong \big((\mathrm{IH}(\mathrm{M}_F)^*)_{[\varnothing]}\big)^* \cong \big(\mathrm{IH}(\mathrm{M}_F)_{[\varnothing]}[\operatorname{crk} F]\big)^*.$$

By $\mathsf{NS}(\mathrm{M}_F)$, it follows that $\mathrm{IH}(\mathrm{M}_F)_{[\varnothing]}$ vanishes in degrees less than or equal to $(\operatorname{crk} F)/2$, and hence $\mathrm{IH}(\mathrm{M}_F)_{[\varnothing]}[\operatorname{crk} F]$ vanishes in degrees less than or equal to $-(\operatorname{crk} F)/2$. Thus, $\mathrm{IH}_\circ(\mathrm{M})_F \cong \mathrm{IH}(\mathrm{M}_F)_{[\varnothing]}[\operatorname{crk} F]^*$ vanishes in degrees greater than or equal to $(\operatorname{crk} F)/2$.

This concludes the proof of statement (2). When F is a nonempty flat, the proof of (1) is identical. When $F = \varnothing$, $\mathsf{NS}(\mathrm{M})$ implies that $\mathrm{IH}(\mathrm{M})_{[\varnothing]}$ vanishes in degrees less than or equal to $d/2$. By $\mathsf{PD}(\mathrm{M})$, $\mathrm{IH}(\mathrm{M})_\varnothing$ vanishes in degrees greater than or equal to $d/2$. $\square$

Corollary 6.4. Let F and G be flats of M.

(1) Suppose that for any flats $F' < F$ (respectively $G' < G$), the conditions CD, PD, and NS hold for the matroid $\mathrm{M}^F_{F'}$ (respectively $\mathrm{M}^G_{G'}$). Let $\rho\colon\ \mathrm{IH}(\mathrm{M}^F) \to \mathrm{IH}(\mathrm{M}^G)[k]$ be a map of graded $\mathrm{H}(\mathrm{M})$-modules. If $F \neq G$, or if $F = G$ and $k \neq 0$, then the induced stalk/costalk maps

$$\rho_F\colon\ \mathrm{IH}(\mathrm{M}^F)_F \to \mathrm{IH}(\mathrm{M}^G)[k]_F \quad\text{and}\quad \rho_{[G]}\colon\ \mathrm{IH}(\mathrm{M}^F)_{[G]} \to \mathrm{IH}(\mathrm{M}^G)[k]_{[G]}$$

are both zero.

(2) Suppose that F and G are nonempty and that for any flats $F' < F$ (respectively $G' < G$), the conditions $\mathsf{CD}_\circ$, $\mathsf{PD}_\circ$, and $\mathsf{NS}_\circ$ hold for the matroid $\mathrm{M}^F_{F'}$ (respectively $\mathrm{M}^G_{G'}$). Let $\rho\colon\ \mathrm{IH}_\circ(\mathrm{M}^F) \to \mathrm{IH}_\circ(\mathrm{M}^G)[k]$ be a map of graded $\mathrm{H}_\circ(\mathrm{M})$-modules. If $F \neq G$, or if $F = G$ and $k \neq 0$, then the induced stalk/costalk maps

$$\rho_F\colon\ \mathrm{IH}_\circ(\mathrm{M}^F)_F \to \mathrm{IH}_\circ(\mathrm{M}^G)[k]_F \quad\text{and}\quad \rho_{[G]}\colon\ \mathrm{IH}_\circ(\mathrm{M}^F)_{[G]} \to \mathrm{IH}_\circ(\mathrm{M}^G)[k]_{[G]}$$

are both zero.

Proof. For statement (1), we first observe that $y_F\,\mathrm{IH}(\mathrm{M}^F) \cong \mathbb{Q}[-\operatorname{rk} F]$, and hence Lemma 5.7 implies that

$$\mathrm{IH}(\mathrm{M}^F)_F \cong \mathbb{Q}_\varnothing \cong \mathbb{Q} \quad\text{and}\quad \mathrm{IH}(\mathrm{M}^F)_{[F]} \cong \mathbb{Q}_{[\varnothing]} \cong \mathbb{Q},$$

with the induced map between them being the identity. When $F = G$ and $k \neq 0$, the statement follows immediately from this observation.

Now assume that $F \neq G$. To show the vanishing of ρ_F, we may further assume that $F < G$, as otherwise we would have $\mathrm{IH}(\mathrm{M}^G)_F = 0$. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{IH}(\mathrm{M}^F)_{[F]} \arrow[r] \arrow[d, "\rho_{[F]}"] & \mathrm{IH}(\mathrm{M}^F)_F \arrow[d, "\rho_F"] \\
\mathrm{IH}(\mathrm{M}^G)[k]_{[F]} \arrow[r] & \mathrm{IH}(\mathrm{M}^G)[k]_F.
\end{tikzcd}$$

The top map is an isomorphism by the observation in the previous paragraph and the bottom map is zero by Proposition 6.3 (1) applied to the matroid M^G, so $\rho_F = 0$.

To show the vanishing of $\rho_{[G]}$, consider the following commutative diagram:

$$\begin{tikzcd} \mathrm{IH}(\mathrm{M}^F)_{[G]} \arrow[r] \arrow[d, "\rho_{[G]}"] & \mathrm{IH}(\mathrm{M}^F)_G \arrow[d, "\rho_G"] \\ \mathrm{IH}(\mathrm{M}^G)[k]_{[G]} \arrow[r] & \mathrm{IH}(\mathrm{M}^G)[k]_G. \end{tikzcd}$$

Now the bottom map is an isomorphism and the top map is zero, so $\rho_{[G]} = 0$.

Statement (2) follows from the same arguments. □

6.2. **Indecomposability.**

Proposition 6.5. Let M be a matroid with ground set E.

(1) Suppose that $\mathsf{CD}(\mathrm{M}_F)$ and $\mathsf{NS}(\mathrm{M}_F)$ hold for all proper flats F. Any endomorphism of the graded $\mathrm{H}(\mathrm{M})$-module $\mathrm{IH}(\mathrm{M})$ that induces the zero map on the stalk $\mathrm{IH}(\mathrm{M})_E$ is in fact the zero endomorphism of $\mathrm{IH}(\mathrm{M})$. In particular, $\mathrm{IH}(\mathrm{M})$ has only scalar automorphisms, and is therefore indecomposable as an $\mathrm{H}(\mathrm{M})$-module.

(2) Suppose that E is nonempty, $\mathsf{CD}_\circ(\mathrm{M})$ holds, and $\mathsf{CD}(\mathrm{M}_F)$ and $\mathsf{NS}(\mathrm{M}_F)$ hold for all nonempty proper flats F. Any endomorphism of the graded $\mathrm{H}_\circ(\mathrm{M})$-module $\mathrm{IH}_\circ(\mathrm{M})$ that induces an automorphism of the stalk $\mathrm{IH}_\circ(\mathrm{M})_E$ is in fact an automorphism of $\mathrm{IH}_\circ(\mathrm{M})$. In particular, $\mathrm{IH}_\circ(\mathrm{M})$ is indecomposable as an $\mathrm{H}_\circ(\mathrm{M})$-module.

Proof. For statement (1), we proceed by induction on the cardinality of the ground set E. When E is empty or consists of a singleton, the proposition is trivial. Let f be an endomorphism of $\mathrm{IH}(\mathrm{M})$ that induces the zero map on $\mathrm{IH}(\mathrm{M})_E$. For each rank one flat G, Lemma 6.2 (1) implies that $y_G\,\mathrm{IH}(\mathrm{M}) \cong \mathrm{IH}(\mathrm{M}_G)[-1]$. Since f restricts to an endomorphism of the graded $\mathrm{H}(\mathrm{M}_G)$-module $\mathrm{IH}(\mathrm{M}_G)$ that induces the zero map on the stalk $\mathrm{IH}(\mathrm{M}_G)_{E\setminus G} \cong \mathrm{IH}(\mathrm{M})_E$, the inductive hypothesis implies that f restricts to zero on each submodule $y_G\,\mathrm{IH}(\mathrm{M})$. Thus, the map $f\colon \mathrm{IH}(\mathrm{M}) \to \mathrm{IH}(\mathrm{M})$ factors through the quotient module $\mathrm{IH}(\mathrm{M})_\varnothing$ of $\mathrm{IH}(\mathrm{M})$ and lands in the submodule $\mathrm{IH}(\mathrm{M})_{[\varnothing]}$ of $\mathrm{IH}(\mathrm{M})$. But then it must be the zero map by Proposition 6.3 (1). The conclusion that $\mathrm{IH}(\mathrm{M})$ has only scalar automorphisms follows from the fact that $\mathrm{IH}(\mathrm{M})_E \cong \mathbb{Q}$ is one-dimensional.

Next, we prove statement (2). Suppose that f is an endomorphism, but not an automorphism, of $\mathrm{IH}_\circ(\mathrm{M})$ that induces an automorphism of the stalk $\mathrm{IH}_\circ(\mathrm{M})_E$. Since $\mathrm{IH}_\circ(\mathrm{M})_E \cong \mathbb{Q}$ is one-dimensional, the induced automorphism of f on the stalk $\mathrm{IH}_\circ(\mathrm{M})_E$ must be a nonzero scalar multiple, which we denote by c.

By Lemma 6.2 (2), we have $y_F\,\mathrm{IH}_\circ(\mathrm{M}) \cong \mathrm{IH}(\mathrm{M}_F)[-\operatorname{rk} F]$ for any nonempty flat F. By statement (1), the restriction of f to $\mathrm{IH}_\circ(\mathrm{M})_{>\varnothing} = \sum_{F\neq\varnothing} y_F\,\mathrm{IH}_\circ(\mathrm{M})$ is equal to multiplication by c. Choose a nonzero homogeneous element η of minimal degree in the kernel of f. For any nonempty flat F, we have

$$cy_F\eta = f(y_F\eta) = y_F f(\eta) = y_F \cdot 0 = 0.$$

Thus, $y_F\eta = 0$ for any nonempty flat F. By Lemma 5.2, this implies that η is a multiple of $x_\varnothing$ in $\mathrm{CH}(\mathrm{M})$. By $\mathrm{CD}_\circ(\mathrm{M})$, $\mathrm{IH}_\circ(\mathrm{M})$ is a direct summand of $\mathrm{CH}(\mathrm{M})$ as an $\mathrm{H}_\circ(\mathrm{M})$-module. Hence, $\eta = x_\varnothing\xi$ for some $\xi \in \mathrm{IH}_\circ(\mathrm{M})$. We have

$$0 = f(\eta) = f(x_\varnothing\xi) = x_\varnothing f(\xi).$$

Thus $f(\xi)$ is in the intersection of the annihilator of $x_\varnothing$ and $\mathrm{IH}_\circ(\mathrm{M})$, which is equal to $\mathrm{IH}_\circ(\mathrm{M})_{>\varnothing}$.

Let $\xi' = f(\xi)/c$. Since $\xi' \in \mathrm{IH}_\circ(\mathrm{M})_{>\varnothing}$, we have $f(\xi') = c\xi' = f(\xi)$, and hence $f(\xi - \xi') = 0$. Since

$$0 \neq \eta = x_\varnothing\xi = x_\varnothing(\xi - \xi'),$$

we have $\xi - \xi' \neq 0$. This contradicts the minimality of the degree of η. □

7. The submodules indexed by flats

In order to define the modules $\mathrm{IH}(\mathrm{M}) \subseteq \mathrm{IH}_\circ(\mathrm{M}) \subseteq \mathrm{CH}(\mathrm{M})$ and $\underline{\mathrm{IH}}(\mathrm{M}) \subseteq \underline{\mathrm{CH}}(\mathrm{M})$, we made use of the submodules

$$\psi^F_{\mathrm{M}}\underline{\mathrm{J}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F) \subseteq \mathrm{CH}(\mathrm{M})$$

for all proper flats F, and the submodules

$$\underline{\psi}^F_{\mathrm{M}}\underline{\mathrm{J}}(\mathrm{M}_F) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F) \subseteq \underline{\mathrm{CH}}(\mathrm{M})$$

for all nonempty proper flats F. The purpose of this section is to understand the relationship between the intrinsic Poincaré pairings on these pieces and the pairings induced by the inclusions into the Chow ring and augmented Chow ring of M.

7.1. The Poincaré pairing on the F-submodule.

Suppose that

$$\mathrm{N} = \bigoplus_{0\leqslant i,j\leqslant d} \mathrm{N}^{i,j}$$

is a finite-dimensional bigraded $\mathbb{Q}$-vector space. Suppose that N is equipped with a bilinear pairing $\langle -,- \rangle$ such that, if $\mu \in \mathrm{N}^{i,j}$ and $\nu \in \mathrm{N}^{k,l}$, then $\langle \mu,\nu\rangle \neq 0$ only when $i+j+k+l = d$. Suppose that $r \in \mathbb{N}$. We say that the pairing is **adapted to r** if it satisfies the following properties:

(1) $\dim \mathrm{N}^{i,j} = \dim \mathrm{N}^{r-i,d-r-j}$ for any $0 \leqslant i \leqslant r$ and $0 \leqslant j \leqslant d-r$;

(2) if $\mu \in \mathrm{N}^{i,j}$, $\nu \in \mathrm{N}^{k,l}$, and $i+k < r$, then $\langle \mu,\nu\rangle = 0$.

We define the ***r*-reduction** of the original pairing to be the new pairing $\langle -,-\rangle_r$ defined by

$$\langle \mu,\nu\rangle_r := \sum_{\substack{i,j,k,l\\ i+k=r}} \langle \mu_{ij},\nu_{kl}\rangle,$$

where μ_{ij} is the projection of μ to $\mathrm{N}^{i,j}$, and similarly for ν_{kl}.

Lemma 7.1. Suppose that the bilinear form $\langle -,-\rangle$ is adapted to r. Then $\langle -,-\rangle_r$ is non-degenerate if and only if $\langle -,-\rangle$ is non-degenerate.

Proof. This translates to the statement that if a matrix is block upper triangular and its block diagonal part is nonsingular, then the original matrix is nonsingular. □

The following lemma is an immediate consequence of the definitions.

Lemma 7.2. Suppose that $\underline{\mathrm{PD}}(\mathrm{M})$, $\underline{\mathrm{HL}}(\mathrm{M})$, and $\underline{\mathrm{HR}}(\mathrm{M})$ all hold. Then $\underline{\mathrm{J}}(\mathrm{M})[-1]$ satisfies Poincaré duality, hard Lefschetz, and Hodge–Riemann, all of degree d, with respect to the hard Lefschetz operator

$$\mathrm{L}^{d-2k}\colon \underline{\mathrm{J}}(\mathrm{M})[-1]^k = \underline{\mathrm{J}}^{k-1}(\mathrm{M}) \to \underline{\mathrm{J}}^{d-k-1}(\mathrm{M}) = \underline{\mathrm{J}}(\mathrm{M})[-1]^{d-k}, \qquad \eta \mapsto \underline{\beta}^{d-2k}\eta$$

and the Poincaré pairing

$$\langle \eta,\xi\rangle_{\underline{\mathrm{J}}(\mathrm{M})[-1]} = -\underline{\deg}_{\mathrm{M}}(\underline{\beta}\,\eta\,\xi).$$

Let F be a proper flat, and consider the bigraded vector space $\underline{\mathrm{J}}(\mathrm{M}_F)[-1]\otimes \mathrm{CH}(\mathrm{M}^F)$. This vector space has two natural bilinear pairings. The first, which we denote $\langle\cdot\,,\cdot\rangle_{\underline{\mathrm{J}}(\mathrm{M}_F)[-1]\otimes\mathrm{CH}(\mathrm{M}^F)}$, is the tensor product of the Poincaré pairings on $\underline{\mathrm{J}}(\mathrm{M}_F)[-1]$ and $\mathrm{CH}(\mathrm{M}^F)$. The second, which we denote $\langle\cdot\,,\cdot\rangle_{\mathrm{CH}(\mathrm{M})}$, is the restriction of the Poincaré pairing on $\mathrm{CH}(\mathrm{M})$ via the inclusion

$$\underline{\mathrm{J}}(\mathrm{M}_F)[-1]\otimes \mathrm{CH}(\mathrm{M}^F) \to \mathrm{CH}(\mathrm{M})$$

induced by ψ^F_{M}, which matches the total grading on the source with the grading on the target. Similarly, the bigraded vector space $\underline{\mathrm{J}}(\mathrm{M}_F)[-1]\otimes \underline{\mathrm{CH}}(\mathrm{M}^F)$ has two natural bilinear pairings. The first, which we denote $\langle\cdot\,,\cdot\rangle_{\underline{\mathrm{J}}(\mathrm{M}_F)[-1]\otimes\underline{\mathrm{CH}}(\mathrm{M}^F)}$, is the tensor product of the Poincaré pairings on $\underline{\mathrm{J}}(\mathrm{M}_F)[-1]$ and $\underline{\mathrm{CH}}(\mathrm{M}^F)$. The second, which we denote $\langle\cdot\,,\cdot\rangle_{\underline{\mathrm{CH}}(\mathrm{M})}$, is the restriction of the Poincaré pairing on $\underline{\mathrm{CH}}(\mathrm{M})$ via the inclusion

$$\underline{\mathrm{J}}(\mathrm{M}_F)[-1]\otimes \underline{\mathrm{CH}}(\mathrm{M}^F) \to \underline{\mathrm{CH}}(\mathrm{M})$$

induced by $\underline{\psi}^F_{\mathrm{M}}$.

Proposition 7.3. Let $r = \operatorname{crk} F$.

(1) The pairing $\langle\cdot\,,\cdot\rangle_{\underline{\mathrm{J}}(\mathrm{M}_F)[-1]\otimes\mathrm{CH}(\mathrm{M}^F)}$ on $\underline{\mathrm{J}}(\mathrm{M}_F)[-1]\otimes\mathrm{CH}(\mathrm{M}^F)$ is adapted to r, and its r-reduction is equal to the pairing $\langle\cdot\,,\cdot\rangle_{\mathrm{CH}(\mathrm{M})}$.

(2) The pairing $\langle \cdot\,,\cdot \rangle_{\underline{\mathrm{J}}(\mathrm{M}_F)[-1]\otimes\underline{\mathrm{CH}}(\mathrm{M}^F)}$ on $\underline{\mathrm{J}}(\mathrm{M}_F)[-1]\otimes\underline{\mathrm{CH}}(\mathrm{M}^F)$ is adapted to r, and its r-reduction is equal to the pairing $\langle \cdot\,,\cdot \rangle_{\underline{\mathrm{CH}}(\mathrm{M})}$.

Proof. We prove only part (1); the proof of part (2) is identical. The first condition for adaptedness follows from the Poincaré duality statements of Lemma 7.2 and Theorem 2.19. For the second condition, let

$$\mu \in \underline{\mathrm{J}}(\mathrm{M}_F)[-1]^i \otimes \mathrm{CH}^j(\mathrm{M}^F) = \underline{\mathrm{J}}^{i-1}(\mathrm{M}_F)\otimes \mathrm{CH}^j(\mathrm{M}^F)$$

and

$$\nu \in \underline{\mathrm{J}}(\mathrm{M}_F)[-1]^k \otimes \mathrm{CH}^l(\mathrm{M}^F) = \underline{\mathrm{J}}^{k-1}(\mathrm{M}_F)\otimes \mathrm{CH}^l(\mathrm{M}^F).$$

By Lemma 2.17 (1), we have

$$\langle \mu,\nu\rangle_{\mathrm{CH}(\mathrm{M})} = \deg_{\mathrm{M}}\left(\psi^F_{\mathrm{M}}\mu\cdot\psi^F_{\mathrm{M}}\nu\right) = -\underline{\deg}_{\mathrm{M}_F}\otimes\deg_{\mathrm{M}^F}\left((\underline{\beta}_{\mathrm{M}_F}\otimes 1 + 1\otimes\alpha_{\mathrm{M}^F})\mu\nu\right).$$

If $i+k<r$, then

$$(\underline{\beta}_{\mathrm{M}_F}\otimes 1 + 1\otimes\alpha_{\mathrm{M}^F})\mu\nu \in \underline{\mathrm{CH}}^{<\mathrm{crk}(F)-1}(\mathrm{M}_F)\otimes\mathrm{CH}(\mathrm{M}^F)$$

and hence $\langle \mu,\nu\rangle_{\mathrm{CH}(\mathrm{M})} = 0$. This proves that the first pairing is adapted to r. If $i+k=r$, then

$$(1\otimes\alpha_{\mathrm{M}^F})\mu\nu \in \underline{\mathrm{CH}}^{r-2}(\mathrm{M}_F)\otimes\mathrm{CH}(\mathrm{M}^F),$$

hence we have

$$\langle \mu,\nu\rangle_{\mathrm{CH}(\mathrm{M})} = -\underline{\deg}_{\mathrm{M}_F}\otimes\deg_{\mathrm{M}^F}\left((\underline{\beta}_{\mathrm{M}_F}\otimes 1)\mu\nu\right) = \langle \mu,\nu\rangle_{\underline{\mathrm{J}}(\mathrm{M}_F)[-1]\otimes\mathrm{CH}(\mathrm{M}^F)}.$$

This completes the proof. □

7.2. **Things we get for free.** In this section, we use Proposition 7.3 to show that some statements follow immediately from the assumption that Theorem 3.17 holds for smaller matroids. Assume throughout the section that E is nonempty.

Corollary 7.4. Assume that all of the statements of Theorem 3.17 hold for M_F for every nonempty proper flat F. Then the statements $\mathrm{PD}_\circ(\mathrm{M})$, $\underline{\mathrm{PD}}(\mathrm{M})$, $\mathrm{CD}_\circ(\mathrm{M})$, and $\underline{\mathrm{CD}}(\mathrm{M})$ hold.

Proof. By Proposition 3.7, the subspaces $\psi^F_{\mathrm{M}}\underline{\mathrm{J}}(\mathrm{M}_F)\otimes\mathrm{CH}(\mathrm{M}^F)$ are mutually orthogonal as F varies through all nonempty proper flats of M. By Lemmas 7.1 and 7.2, Proposition 7.3, and Theorem 2.19 (1), the restriction of the Poincaré pairing on $\psi^F_{\mathrm{M}}\underline{\mathrm{J}}(\mathrm{M}_F)\otimes\mathrm{CH}(\mathrm{M}^F)\subseteq\mathrm{CH}(\mathrm{M})$ is non-degenerate. These statements imply that the sum of these subspaces of $\mathrm{CH}(\mathrm{M})$ is a direct sum and the restriction of the Poincaré pairing to this direct sum is non-degenerate. Since $\mathrm{IH}_\circ(\mathrm{M})$ is defined to be the orthogonal complement of the above direct sum, we have an orthogonal decomposition

$$\mathrm{CH}(\mathrm{M}) = \mathrm{IH}_\circ(\mathrm{M}) \oplus \bigoplus_{\varnothing<F<E}\psi^F_{\mathrm{M}}\underline{\mathrm{J}}(\mathrm{M}_F)\otimes\mathrm{CH}(\mathrm{M}^F)$$

and the restriction of the Poincaré pairing to $\mathrm{IH}_\circ(\mathrm{M})$ is also non-degenerate. Thus, $\mathsf{PD}_\circ(\mathrm{M})$ and $\mathsf{CD}_\circ(\mathrm{M})$ hold. The statements $\underline{\mathsf{PD}}(\mathrm{M})$ and $\underline{\mathsf{CD}}(\mathrm{M})$ follow from the same arguments. □

Proposition 7.5. If $\mathsf{CD}_\circ(\mathrm{M})$ holds, then $\langle x_\varnothing\rangle \cap \mathrm{IH}_\circ(\mathrm{M}) = x_\varnothing \cdot \mathrm{IH}_\circ(\mathrm{M})$.

Proof. By $\mathsf{CD}_\circ(\mathrm{M})$, we have

$$\begin{aligned}\langle x_\varnothing\rangle \cap \mathrm{IH}_\circ(\mathrm{M}) &= x_\varnothing\, \mathrm{CH}(\mathrm{M}) \cap \mathrm{IH}_\circ(\mathrm{M}) \\ &= \Big(x_\varnothing\, \mathrm{IH}_\circ(\mathrm{M}) \oplus \bigoplus_{\varnothing<F<E} x_\varnothing \psi^F_{\mathrm{M}} \underline{\mathrm{J}}(\mathrm{M}_F)\otimes \mathrm{CH}(\mathrm{M}^F)\Big) \cap \mathrm{IH}_\circ(\mathrm{M}) = x_\varnothing \cdot \mathrm{IH}_\circ(\mathrm{M}).\end{aligned}$$ □

Corollary 7.6. If $\mathsf{CD}_\circ(\mathrm{M})$ holds, then $\varphi^\varnothing_{\mathrm{M}}\, \mathrm{IH}_\circ(\mathrm{M}) = \underline{\mathrm{IH}}(\mathrm{M})$.

Proof. Let F be a nonempty proper flat of M. By the second commutative square of Lemma 2.18 (1),

$$\psi^\varnothing_{\mathrm{M}} \underline{\psi}^F_{\mathrm{M}} \underline{\mathrm{J}}(\mathrm{M}_F)\otimes \underline{\mathrm{CH}}(\mathrm{M}^F) = \psi^F_{\mathrm{M}}\Big(\underline{\mathrm{J}}(\mathrm{M}_F)\otimes \psi^\varnothing_{\mathrm{M}^F}\underline{\mathrm{CH}}(\mathrm{M}^F)\Big) \subseteq \psi^F_{\mathrm{M}}\underline{\mathrm{J}}(\mathrm{M}_F)\otimes \mathrm{CH}(\mathrm{M}^F).$$

Therefore, $\mathrm{IH}_\circ(\mathrm{M})$ is orthogonal to $\psi^\varnothing_{\mathrm{M}} \underline{\psi}^F_{\mathrm{M}} \underline{\mathrm{J}}(\mathrm{M}_F)\otimes \underline{\mathrm{CH}}(\mathrm{M}^F)$ with respect to the Poincaré pairing on $\mathrm{CH}(\mathrm{M})$. By Proposition 2.5 (5) and (6), $\varphi^\varnothing_{\mathrm{M}}\, \mathrm{IH}_\circ(\mathrm{M})$ is orthogonal to $\underline{\psi}^F_{\mathrm{M}} \underline{\mathrm{J}}(\mathrm{M}_F)\otimes \underline{\mathrm{CH}}(\mathrm{M}^F)$ with respect to the Poincaré pairing on $\underline{\mathrm{CH}}(\mathrm{M})$. Thus $\varphi^\varnothing_{\mathrm{M}}\, \mathrm{IH}_\circ(\mathrm{M}) \subseteq \underline{\mathrm{IH}}(\mathrm{M})$.

On the other hand, by the first commutative square of Lemma 2.18 (1), we have

$$\varphi^\varnothing_{\mathrm{M}} \psi^F_{\mathrm{M}} \underline{\mathrm{J}}(\mathrm{M}_F)\otimes \mathrm{CH}(\mathrm{M}^F) = \underline{\psi}^F_{\mathrm{M}}\Big(\underline{\mathrm{J}}(\mathrm{M}_F)\otimes \varphi^\varnothing_{\mathrm{M}^F}\, \mathrm{CH}(\mathrm{M}^F)\Big) = \underline{\psi}^F_{\mathrm{M}}\underline{\mathrm{J}}(\mathrm{M}_F)\otimes \underline{\mathrm{CH}}(\mathrm{M}^F).$$

Hence, $\underline{\mathrm{IH}}(\mathrm{M})$ is orthogonal to $\varphi^\varnothing_{\mathrm{M}} \psi^F_{\mathrm{M}} \underline{\mathrm{J}}(\mathrm{M}_F)\otimes \mathrm{CH}(\mathrm{M}^F)$ with respect to the Poincaré pairing on $\underline{\mathrm{CH}}(\mathrm{M})$, or equivalently (by Lemma 2.6) $\psi^\varnothing_{\mathrm{M}}\underline{\mathrm{IH}}(\mathrm{M})$ is orthogonal to $\psi^F_{\mathrm{M}} \underline{\mathrm{J}}(\mathrm{M}_F)\otimes \mathrm{CH}(\mathrm{M}^F)$ with respect to the Poincaré pairing on $\mathrm{CH}(\mathrm{M})$. Thus $\psi^\varnothing_{\mathrm{M}}\underline{\mathrm{IH}}(\mathrm{M}) \subseteq \mathrm{IH}_\circ(\mathrm{M})$.

By Proposition 2.7 and surjectivity of $\varphi^\varnothing_{\mathrm{M}}$, we have $\psi^\varnothing_{\mathrm{M}}\underline{\mathrm{IH}}(\mathrm{M}) \subseteq \langle x_\varnothing\rangle$. Then by Proposition 7.5, we have

$$\psi^\varnothing_{\mathrm{M}}\underline{\mathrm{IH}}(\mathrm{M}) \subseteq \mathrm{IH}_\circ(\mathrm{M}) \cap \langle x_\varnothing\rangle = x_\varnothing \cdot \mathrm{IH}_\circ(\mathrm{M}) = \psi^\varnothing_{\mathrm{M}}\varphi^\varnothing_{\mathrm{M}}\, \mathrm{IH}_\circ(\mathrm{M}).$$

By the injectivity of $\psi^\varnothing_{\mathrm{M}}$, it follows that $\underline{\mathrm{IH}}(\mathrm{M}) \subseteq \varphi^\varnothing_{\mathrm{M}}\, \mathrm{IH}_\circ(\mathrm{M})$. □

Corollary 7.7. If $\mathsf{CD}_\circ(\mathrm{M})$ holds, then $\langle x_\varnothing\rangle \cap \mathrm{IH}_\circ(\mathrm{M}) = \psi^\varnothing_{\mathrm{M}}\underline{\mathrm{IH}}(\mathrm{M})$.

Proof. By Corollary 7.6 and Proposition 2.7, we have

$$\psi^\varnothing_{\mathrm{M}}\underline{\mathrm{IH}}(\mathrm{M}) = \psi^\varnothing_{\mathrm{M}}\varphi^\varnothing_{\mathrm{M}}\, \mathrm{IH}_\circ(\mathrm{M}) = x_\varnothing \cdot \mathrm{IH}_\circ(\mathrm{M}).$$

The statement then follows from Proposition 7.5. □

Proposition 7.8. If $\mathsf{CD}_\circ(\mathrm{M})$ holds, then for any $k \leqslant d/2$ we have

$$\mathsf{CD}^k(\mathrm{M}) \implies \mathsf{NS}^k(\mathrm{M}).$$

Proof. Suppose that $\eta \in \mathrm{IH}^k(\mathrm{M})$ and $y_i\eta = 0$ for all $i \in E$. By Lemma 5.2, η is a multiple of $x_\varnothing$. Thus, Corollary 7.7 implies that

$$\eta \in \psi^\varnothing_\mathrm{M}\underline{\mathrm{IH}}^{k-1}(\mathrm{M}) = \psi^\varnothing_\mathrm{M}\underline{\mathrm{J}}^{k-1}(\mathrm{M}).$$

However, $\mathrm{CD}^k(\mathrm{M})$ implies that $\mathrm{IH}^k(\mathrm{M}) \cap \psi^\varnothing_\mathrm{M}\underline{\mathrm{J}}^{k-1}(\mathrm{M}) = 0$. Therefore, we have $\eta = 0$. □

7.3. **The Hancock condition.** Let $\mathrm{N} = \bigoplus_{k\geqslant 0} \mathrm{N}^k$ be a finite-dimensional graded $\mathbb{Q}$-vector space equipped with a symmetric bilinear form. Let

$$P_\mathrm{N}(t) \coloneqq \sum_{k\geqslant 0} t^k \dim \mathrm{N}^k$$

be the Poincaré polynomial of N. We say that N is **Hancock** if the signature of the bilinear form (the number of positive eigenvalues minus the number of negative eigenvalues) is equal to $P_\mathrm{N}(-1)$.

Remark 7.9. If the symmetric bilinear form on N satisfies Poincaré duality of degree d, then its signature is equal to the signature of its restriction to the degree $d/2$ piece. In particular, if d is odd, then the signature is necessarily zero, as is $P_\mathrm{N}(-1)$. Thus when d is odd, the Hancock condition follows automatically from Poincaré duality.

The motivation for the Hancock condition is the following proposition.

Proposition 7.10. Suppose that $\mathrm{L}\colon \mathrm{N} \to \mathrm{N}$ is a linear operator of degree 1 with respect to which N satisfies the hard Lefschetz theorem of degree d. Suppose that d is even and that N satisfies the Hodge–Riemann relations of degree d in all but the middle degree. Then N satisfies the Hodge–Riemann relations in middle degree if and only if N is Hancock.

Proof. The hard Lefschetz theorem implies that

$$\mathrm{N}^{d/2} = \bigoplus_{k=0}^{d/2} \mathrm{L}^{(d/2)-k} \ker(\mathrm{L}^{d-2k+1}).$$

For all $k \leqslant d/2$, the Hodge–Riemann relations in degree k are equivalent to the statement that the signature of the restriction of the bilinear form to $\mathrm{L}^{(d/2)-k}\ker(\mathrm{L}^{d-2k+1})$ is equal to $(-1)^k(\dim \mathrm{N}^k - \dim \mathrm{N}^{k-1})$. If we assume the Hodge–Riemann relations in all but one degree, this means that the Hodge–Riemann relations in the missing degree are equivalent to the statement that the signature of the bilinear form is equal to

$$\sum_{k=0}^{d/2} (-1)^k(\dim \mathrm{N}^k - \dim \mathrm{N}^{k-1}).$$

By hard Lefschetz and the fact that d is even,

$$-(-1)^k \dim \mathrm{N}^{k-1} = \dim \mathrm{N}^{d-k+1} = (-1)^{d-k+1} \dim \mathrm{N}^{d-k+1},$$

thus the expected signature is

$$\sum_{k=0}^{d/2}\Big((-1)^k \dim \mathrm{N}^k + (-1)^{d-k+1}\dim \mathrm{N}^{d-k+1}\Big) = P_{\mathrm{N}}(-1).$$

This completes the proof. □

Lemma 7.11. If N and N' are both Hancock, then so are $\mathrm{N}\oplus\mathrm{N}'$ and $\mathrm{N}\otimes\mathrm{N}'$.

Proof. This follows from the fact that signature and Poincaré polynomial are both additive with respect to direct sum and multiplicative with respect to tensor product. □

Lemma 7.12. Suppose that N is Hancock and $\mathrm{N} = \mathrm{N}_0\oplus\mathrm{N}_1\oplus\cdots\oplus\mathrm{N}_l$ is an orthogonal decomposition. If $\mathrm{N}_1,\ldots,\mathrm{N}_l$ are all Hancock, then so is N_0.

Proof. This follows from the fact that the signature and the Poincaré polynomial are both additive with respect to the orthogonal decomposition. □

Lemma 7.13. A graded bilinear form that is adapted to r is Hancock if and only if its r-reduction is Hancock.

Proof. This follows from the fact that the original matrix and its block diagonal part have the same multiset of eigenvalues. □

Corollary 7.14. Let F be a nonempty proper flat of M such that $\underline{\mathrm{PD}}(\mathrm{M}_F)$, $\underline{\mathrm{HL}}(\mathrm{M}_F)$, and $\underline{\mathrm{HR}}(\mathrm{M}_F)$ hold. The graded subspace $\psi^F_{\mathrm{M}}\underline{\mathrm{J}}(\mathrm{M}_F)\otimes\mathrm{CH}(\mathrm{M}^F)$ is Hancock with respect to the Poincaré pairing on $\mathrm{CH}(\mathrm{M})$, and the graded subspace $\underline{\psi}^F_{\mathrm{M}}\underline{\mathrm{J}}(\mathrm{M}_F)\otimes\underline{\mathrm{CH}}(\mathrm{M}^F)$ is Hancock with respect to the Poincaré pairing on $\underline{\mathrm{CH}}(\mathrm{M})$.

Proof. We prove the first statement; the proof of the second is the same. Let $r = \operatorname{crk} F$. By Proposition 7.3 and Lemma 7.13, this is equivalent to the statement that the graded vector space $\underline{\mathrm{J}}(\mathrm{M}_F)[-1]\otimes\mathrm{CH}(\mathrm{M}^F)$ is Hancock with respect to the pairing $\langle\cdot,\cdot\rangle_{\underline{\mathrm{J}}(\mathrm{M}_F)[-1]\otimes\mathrm{CH}(\mathrm{M}^F)}$. By Lemma 7.11, it is sufficient to prove that $\mathrm{CH}(\mathrm{M}^F)$ and $\underline{\mathrm{J}}(\mathrm{M}_F)[-1]$ are both Hancock. The first assertion follows from Theorem 2.19 and Proposition 7.10. The second assertion follows from Lemma 7.2 and Proposition 7.10. □

Proposition 7.15. Assume that $\underline{\mathrm{PD}}(\mathrm{M}_F)$, $\underline{\mathrm{HL}}(\mathrm{M}_F)$, and $\underline{\mathrm{HR}}(\mathrm{M}_F)$ hold for all nonempty proper flats F of M. Then

$$\mathsf{CD}_\circ(\mathrm{M}),\ \ \mathsf{HL}_\circ(\mathrm{M}),\ \text{ and }\ \mathsf{HR}_\circ^{<\frac{d}{2}}(\mathrm{M}) \implies \mathsf{HR}_\circ(\mathrm{M}).$$

Proof. Proposition 7.10 tells us that we need to show that $\mathrm{IH}_\circ(\mathrm{M})$ is Hancock. By Corollary 7.14, $\psi^F_{\mathrm{M}}\underline{\mathrm{J}}(\mathrm{M}_F)\otimes\mathrm{CH}(\mathrm{M}^F)$ is Hancock for all nonempty proper flats F of M. Theorem 2.19 and Proposition 7.10 tell us that $\mathrm{CH}(\mathrm{M})$ is Hancock, thus the subspace $\mathrm{IH}_\circ(\mathrm{M})$ is Hancock by $\mathrm{CD}_\circ(\mathrm{M})$, Proposition 3.7, and Lemma 7.12. □

Proposition 7.16. Suppose that E is nonempty and the following statements hold:

$$\mathsf{CD}(\mathrm{M}),\ \mathsf{HL}(\mathrm{M}),\ \mathsf{HR}^{<\frac{d}{2}}(\mathrm{M}),\ \mathsf{HL}_\circ(\mathrm{M}),\ \mathsf{HR}_\circ(\mathrm{M}),\ \underline{\mathsf{PD}}(\mathrm{M}),\ \underline{\mathsf{HL}}(\mathrm{M}),\ \text{and } \underline{\mathsf{HR}}(\mathrm{M}).$$

Then $\mathsf{HR}(\mathrm{M})$ also holds.

Proof. By Proposition 7.10, it suffices to show that $\mathrm{IH}(\mathrm{M})$ is Hancock. By $\mathsf{CD}(\mathrm{M})$, we have

$$\mathrm{IH}_\circ(\mathrm{M}) = \mathrm{IH}(\mathrm{M}) \oplus \psi^{\varnothing}_{\mathrm{M}}\underline{\mathrm{J}}(\mathrm{M}).$$

Since $\mathsf{PD}_\circ(\mathrm{M})$, $\mathsf{HL}_\circ(\mathrm{M})$, and $\mathsf{HR}_\circ(\mathrm{M})$ hold, Proposition 7.10 implies that $\mathrm{IH}_\circ(\mathrm{M})$ is Hancock. By $\underline{\mathsf{PD}}(\mathrm{M})$, $\underline{\mathsf{HL}}(\mathrm{M})$, and $\underline{\mathsf{HR}}(\mathrm{M})$, Lemma 7.2 and Proposition 7.10 combine to tell us that $\psi^{\varnothing}_{\mathrm{M}}\underline{\mathrm{J}}(\mathrm{M})$ is Hancock. Finally, $\mathrm{IH}(\mathrm{M})$ is Hancock by Lemma 7.12. □

8. Rouquier complexes

In this section, we define for any matroid M four complexes: the big Rouquier complex and the big reduced Rouquier complex, which are complexes of graded $\mathrm{CH}(\mathrm{M})$-modules; the small Rouquier complex, which is a complex of graded $\mathrm{H}(\mathrm{M})$-modules; and the small reduced Rouquier complex, which is a complex of graded $\mathrm{H}_\circ(\mathrm{M})$-modules.

Remark 8.1. We make an observation about the results of this section that is analogous to the observation in Remark 6.1. Our main result is Theorem 8.16, and only part (2) of this theorem will be part of our large induction. Part (1), which is in some sense the more natural statement, can only be established later, once we have proved $\mathsf{CD}(\mathrm{M})$.

8.1. The big complexes.

Consider the graded $\mathrm{CH}(\mathrm{M})$-module

$$\mathrm{C}^i(\mathrm{M}) \coloneqq \bigoplus_{\varnothing\leqslant F_1<\cdots<F_i<E} x_{F_1}\cdots x_{F_i}\,\mathrm{CH}(\mathrm{M})[i]$$

for $i>0$ and $\mathrm{C}^0(\mathrm{M}) \coloneqq \mathrm{CH}(\mathrm{M})$, along with the module homomorphism

$$\partial^i\colon \mathrm{C}^i(\mathrm{M}) \to \mathrm{C}^{i+1}(\mathrm{M})$$

defined component-wise by multiplication by a variable:

$$x_{F_1}\cdots\widehat{x_{F_j}}\cdots x_{F_{i+1}}\,\mathrm{CH}(\mathrm{M})[i] \xrightarrow{(-1)^j x_{F_j}} x_{F_1}\cdots x_{F_{i+1}}\,\mathrm{CH}(\mathrm{M})[i+1].$$

It is straightforward to check that $\partial^{i+1}\circ\partial^i = 0$, and hence $(\mathrm{C}^\bullet,\partial)$ is a complex of graded $\mathrm{CH}(\mathrm{M})$-modules. We call this complex the **big Rouquier complex**.

If E is nonempty, we define the **big reduced Rouquier complex** $\mathrm{C}^\bullet_\circ(\mathrm{M})$ to be the quotient of the big Rouquier complex by the subcomplex consisting of terms with $F_1 = \varnothing$. In other words, it is defined by

$$\mathrm{C}^i_\circ(\mathrm{M}) \coloneqq \bigoplus_{\varnothing < F_1 < \cdots < F_i < E} x_{F_1} \cdots x_{F_i} \,\mathrm{CH}(\mathrm{M})[i],$$

for $i > 0$ and $\mathrm{C}^0_\circ(\mathrm{M}) \coloneqq \mathrm{CH}(\mathrm{M})$. The differential of $\mathrm{C}^\bullet_\circ(\mathrm{M})$ is given by the same formula as in $\mathrm{C}^\bullet(\mathrm{M})$.

8.2. Basic properties.

Lemma 8.2. Let F be a flat of a loopless matroid M.

(1) We have an isomorphism

$$y_F \mathrm{C}^\bullet(\mathrm{M}) \cong \mathrm{C}^\bullet(\mathrm{M}_F)[-\operatorname{rk} F]$$

of complexes of graded $\mathrm{CH}(\mathrm{M})$-modules, where $\mathrm{CH}(\mathrm{M})$ acts on the right-hand side via the graded algebra homomorphism $\varphi^{\mathrm{M}}_F \colon \mathrm{CH}(\mathrm{M}) \to \mathrm{CH}(\mathrm{M}_F)$.

(2) If F is nonempty, we also have an isomorphism

$$y_F \mathrm{C}^\bullet_\circ(\mathrm{M}) \cong \mathrm{C}^\bullet(\mathrm{M}_F)[-\operatorname{rk} F]$$

of complexes of graded $\mathrm{CH}(\mathrm{M})$-modules.

Proof. The first statement follows from the fact that $\psi^{\mathrm{M}}_F \colon \mathrm{CH}(\mathrm{M}_F)[-\operatorname{rk} F] \to y_F \,\mathrm{CH}(\mathrm{M})$ is an isomorphism of graded $\mathrm{CH}(\mathrm{M})$-modules [BHM+20, Proposition 2.25]. Since $x_\varnothing y_F = 0$ for any nonempty flat F, the projection from $\mathrm{C}^\bullet(\mathrm{M})$ to $\mathrm{C}^\bullet_\circ(\mathrm{M})$ becomes an isomorphism after multiplying by y_F, and hence the second statement follows from the first one. □

Lemma 8.3. For all $i > 0$ and proper flats $F_1 < \cdots < F_i$, $x_{F_1} \cdots x_{F_i} \,\mathrm{CH}(\mathrm{M})[i]$ is isomorphic as an $\mathrm{H}(\mathrm{M})$-module to a direct sum of shifted copies of $\mathrm{CH}(\mathrm{M}^{F_1})$. In particular, both $\mathrm{C}^i(\mathrm{M})$ and $\mathrm{C}^i_\circ(\mathrm{M})$ are pure $\mathrm{H}(\mathrm{M})$-modules.

Proof. Using [BHM+20, Proposition 2.19] repeatedly, we have an isomorphism of $\mathrm{H}(\mathrm{M})$-modules

$$x_{F_1} \cdots x_{F_i} \,\mathrm{CH}(\mathrm{M})[i] \cong \underline{\mathrm{CH}}(\mathrm{M}_{F_i}) \otimes \underline{\mathrm{CH}}(\mathrm{M}^{F_i}_{F_{i-1}}) \otimes \cdots \otimes \underline{\mathrm{CH}}(\mathrm{M}^{F_2}_{F_1}) \otimes \mathrm{CH}(\mathrm{M}^{F_1}),$$

where the $\mathrm{H}(\mathrm{M})$-module structure on the right-hand side is induced by the $\mathrm{H}(\mathrm{M})$-module structure on $\mathrm{CH}(\mathrm{M}^{F_1})$ induced by the composition $\mathrm{H}(\mathrm{M}) \to \mathrm{CH}(\mathrm{M}) \to \mathrm{CH}(\mathrm{M}^{F_1})$, where the second map is the composition of $\varphi^{F_1}_{\mathrm{M}}$ with the map from $\underline{\mathrm{CH}}(\mathrm{M}_{F_1}) \otimes \mathrm{CH}(\mathrm{M}^{F_1})$ to $\mathrm{CH}(\mathrm{M}^{F_1})$ given by killing all classes of positive degree in the left-hand factor. Thus, the lemma follows. □

Corollary 8.4.

(1) If $\mathsf{CD}(\mathrm{M}^F)$ holds for all flats F of M, then $\mathrm{C}^i(\mathrm{M})$ is isomorphic to a direct sum of shifts of graded $\mathrm{H}(\mathrm{M})$-modules of the form $\mathrm{IH}(\mathrm{M}^G)$. The module $\mathrm{IH}(\mathrm{M})$ appears only in $\mathrm{C}^0(\mathrm{M})$, where it appears exactly once and without a shift.

(2) If E is nonempty and $\mathsf{CD}_\circ(\mathrm{M}^F)$ holds for all nonempty flats F, then $\mathrm{C}^i_\circ(\mathrm{M})$ is isomorphic to a direct sum of shifts of graded $\mathrm{H}_\circ(\mathrm{M})$-modules of the form $\mathrm{IH}_\circ(\mathrm{M}^G)$ for nonempty G. The module $\mathrm{IH}_\circ(\mathrm{M})$ appears only in $\mathrm{C}^0_\circ(\mathrm{M})$, where it appears exactly once and without a shift.

Proof. Given any flat F, we can apply the canonical decomposition CD repeatedly for various localizations of M to deduce that the $\mathrm{H}(\mathrm{M})$-module $\mathrm{CH}(\mathrm{M}^F)$ is isomorphic to a direct sum of modules of the form $\mathrm{IH}(\mathrm{M}^G)[k]$ for $G \leqslant F$ and $k \in \mathbb{Z}$. If we apply the coarser decomposition $\mathsf{CD}_\circ$ instead of CD, then the same argument proves statement (2). □

Lemma 8.5.

(1) If F is a proper flat, then the stalk complex $\mathrm{C}^\bullet(\mathrm{M})_F$ is acyclic. The stalk complex $\mathrm{C}^\bullet(\mathrm{M})_E$ is quasi-isomorphic to $\mathbb{Q}$ concentrated in degree zero.

(2) If F is a nonempty proper flat, the stalk complex $\mathrm{C}^\bullet_\circ(\mathrm{M})_F$ is acyclic. If E is nonempty, the stalk complex $\mathrm{C}^\bullet_\circ(\mathrm{M})_E$ is quasi-isomorphic to $\mathbb{Q}$ concentrated in degree zero.

Remark 8.6. Lemma 8.5 (2) does not tell us anything about the stalk of $\mathrm{C}^\bullet_\circ(\mathrm{M})$ at the empty flat. This will be the subject of Proposition 8.13.

Proof. We begin by proving statement (1) when F is the empty flat. We observe that multiplication by $x_\varnothing$ defines a map of complexes

$$\mathrm{C}^\bullet_\circ(\mathrm{M}) \to x_\varnothing \mathrm{C}^\bullet_\circ(\mathrm{M})[1],$$

and (after shifting by 1 in cohomological degree) the cone of this map is isomorphic to $\mathrm{C}^\bullet(\mathrm{M})$. To prove that $\mathrm{C}^\bullet(\mathrm{M})_\varnothing$ is acyclic, it is therefore sufficient to prove that for all i, the map from $\mathrm{C}^i_\circ(\mathrm{M})$ to $x_\varnothing \mathrm{C}^i_\circ(\mathrm{M})[1]$ induces an isomorphism on stalks at the empty flat. This follows from Lemmas 5.2 and 8.3.

Next we prove statement (1) for arbitrary proper flats. By Lemmas 5.7 and 8.2 (1),

$$\mathrm{C}^\bullet(\mathrm{M})_F \cong (y_F \mathrm{C}^\bullet(\mathrm{M})[\operatorname{rk} F])_\varnothing \cong \mathrm{C}^\bullet(\mathrm{M}_F)_\varnothing.$$

Since F is proper, M_F has positive rank, and the statement follows from the previous paragraph.

It follows from the definition of $\mathrm{C}^\bullet(\mathrm{M})$ that $\mathrm{C}^\bullet(\mathrm{M})_E = y_E \mathrm{C}^\bullet(\mathrm{M})[d]$ is quasi-isomorphic to a single copy of $\mathbb{Q}$ in both homological and grading degree zero, which implies the second sentence of (1).

For any nonempty flat F, we have $y_F x_\varnothing = 0$. Therefore, the natural quotient $\mathrm{C}^\bullet(\mathrm{M}) \to \mathrm{C}^\bullet_\circ(\mathrm{M})$ induces an isomorphism on the stalk at F. Thus, statement (2) follows from statement (1). □

Proposition 8.7. The complex $\mathrm{C}^\bullet(\mathrm{M})$ is acyclic except in degree zero, and $\mathrm{H}^0(\mathrm{C}^\bullet(\mathrm{M})) \cong \mathbb{Q}[-d]$.

Proof. Let Σ_k be a family of order filters defined as in Section 5.3. By Proposition 5.9, we have

$$\mathrm{C}^\bullet(\mathrm{M})_{\Sigma_k}/\mathrm{C}^\bullet(\mathrm{M})_{\Sigma_{k+1}} \cong \mathrm{C}^\bullet(\mathrm{M})_{F_k}[-\operatorname{rk} F_k],$$

which is acyclic for all $1 \leqslant k < r$ and quasi-isomorphic to $\mathbb{Q}[-d]$ in degree zero when $k = r$ by Lemma 8.5 (1). The result then follows from the spectral sequence relating the cohomology of a filtered complex to the cohomology of its associated graded. □

Proposition 8.8. For any j, we have $\mathrm{H}^j\left(\mathrm{C}^\bullet(\mathrm{M})_{[\varnothing]}\right) \cong \mathrm{OS}^j(\mathrm{M})^*[-d]$.

Proof. Let $\mathrm{C}^\bullet(\mathrm{M})^\bullet_!$ be the double complex obtained by applying the construction of Section 5.5 to each term in the big Rouquier complex. By Proposition 5.10, the i-th column of this double complex has no cohomology in positive degree, and its cohomology in degree zero is isomorphic to $\mathrm{C}^i(\mathrm{M})_{[\varnothing]}$. In particular, this implies that $\mathrm{C}^\bullet(\mathrm{M})_{[\varnothing]}$ is quasi-isomorphic to the total complex of $\mathrm{C}^\bullet(\mathrm{M})^\bullet_!$.

On the other hand, the j-th row of the double complex is equal to the direct sum over all flats F of rank j of

$$\mathrm{OS}_F(\mathrm{M})^* \otimes y_F \mathrm{C}^\bullet(\mathrm{M}) \cong \mathrm{OS}_F(\mathrm{M})^* \otimes \mathrm{C}^\bullet(\mathrm{M}_F)[-\operatorname{rk} F].$$

By Proposition 8.7, the j-th row has no cohomology in positive (cohomological) degree, and its cohomology in (cohomological) degree zero is isomorphic to

$$\bigoplus_{F \in \mathcal{L}^j(\mathrm{M})} \mathrm{OS}_F(\mathrm{M})^*[-d] \cong \mathrm{OS}^j(\mathrm{M})^*[-d].$$

Note that this graded vector space is concentrated in (grading) degree $d-j$, which means that the differential from the degree zero cohomology of the j-th column to the degree zero cohomology of the $(j+1)$-st column vanishes for degree reasons. In particular, this implies that the complex $\mathrm{OS}^\bullet(\mathrm{M})^*[-d]$ with zero differential is quasi-isomorphic to the total complex of $\mathrm{C}^\bullet(\mathrm{M})^\bullet_!$.

Putting together the two paragraphs above, we can conclude the proof. □

Corollary 8.9. Let F be a flat, and let j be a nonnegative integer.

(1) We have $\mathrm{H}^j\left(\mathrm{C}^\bullet(\mathrm{M})_{[F]}\right) \cong \mathrm{OS}^j(\mathrm{M}_F)^*[-\operatorname{crk} F]$.

(2) If F is nonempty, then $\mathrm{H}^j\left(\mathrm{C}^\bullet_\circ(\mathrm{M})_{[F]}\right) \cong \mathrm{OS}^j(\mathrm{M}_F)^*[-\operatorname{crk} F]$.

Proof. By Lemma 5.7 and Lemma 8.2 (1),

$$\mathrm{C}^\bullet(\mathrm{M})_{[F]} \cong (y_F \mathrm{C}^\bullet(\mathrm{M})[\operatorname{rk} F])_{[\varnothing]} \cong \mathrm{C}^\bullet(\mathrm{M}_F)_{[\varnothing]}.$$

Statement (1) then follows from Proposition 8.8. Similarly, we can deduce statement (2) using Lemma 8.2 (2), which says that $y_F \mathrm{C}^\bullet(\mathrm{M}) \cong y_F \mathrm{C}^\bullet_\circ(\mathrm{M})$ when F is nonempty. □

8.3. **The stalk of the big reduced Rouquier complex at the empty flat.** Throughout this section, we assume that E is nonempty. Our goal is to give a degree bound on the cohomology of the complex $\mathrm{C}^\bullet_\circ(\mathrm{M})_\varnothing$.

Given a complex $\mathrm{Q}^\bullet$ of graded $\mathrm{H}(\mathrm{M})$-modules, we denote by $\Delta(\mathrm{Q}^\bullet)$ the cone of the natural map $\mathrm{Q}^{\bullet-1}_{[\varnothing]} \to \mathrm{Q}^{\bullet-1}_\varnothing$. In particular, $\Delta(\mathrm{Q}^\bullet)^k = \mathrm{Q}^k_{[\varnothing]} \oplus \mathrm{Q}^{k-1}_\varnothing$, and we have a distinguished triangle

$$\mathrm{Q}^{\bullet-1}_{[\varnothing]} \to \mathrm{Q}^{\bullet-1}_\varnothing \to \Delta(\mathrm{Q}^\bullet) \to \mathrm{Q}^\bullet_{[\varnothing]}.$$

Lemma 8.10. The natural map $\Delta(\mathrm{C}^\bullet(\mathrm{M})) \to \mathrm{C}^\bullet(\mathrm{M})_{[\varnothing]}$ is a quasi-isomorphism.

Proof. This follows from the first part of Lemma 8.5, which says that $\mathrm{C}^\bullet(\mathrm{M})_\varnothing$ is acyclic. □

Lemma 8.11. The map $\mathrm{C}^\bullet(\mathrm{M}) \to \mathrm{C}^\bullet_\circ(\mathrm{M})$ induces a quasi-isomorphism $\Delta(\mathrm{C}^\bullet(\mathrm{M})) \to \Delta(\mathrm{C}^\bullet_\circ(\mathrm{M}))$.

Proof. Let $\mathrm{C}^\bullet_-$ be the kernel of $\mathrm{C}^\bullet(\mathrm{M}) \to \mathrm{C}^\bullet_\circ(\mathrm{M})$. In other words, the complex $\mathrm{C}^\bullet_-(\mathrm{M})$ is defined by

$$\mathrm{C}^i_-(\mathrm{M}) \coloneqq \bigoplus_{\varnothing = F_1 < \cdots < F_i < E} x_{F_1} \cdots x_{F_i} \, \mathrm{CH}(\mathrm{M})[i],$$

and with differential defined by the same component-wise formula as in the definition of $\mathrm{C}^\bullet(\mathrm{M})$. The big Rouquier complex $\mathrm{C}^\bullet(\mathrm{M})$ is isomorphic to the mapping cone of the map $\mathrm{C}^{\bullet-1}_\circ(\mathrm{M}) \to \mathrm{C}^\bullet_-(\mathrm{M})$, which is the direct sum of

$$x_{F_1} \cdots x_{F_i} \, \mathrm{CH}(\mathrm{M})[i] \xrightarrow{-x_\varnothing} x_\varnothing x_{F_1} \cdots x_{F_i} \, \mathrm{CH}(\mathrm{M})[i+1]$$

over all flags $\varnothing < F_1 < \cdots < F_i < E$. Thus, the mapping cone of $\mathrm{C}^\bullet(\mathrm{M}) \to \mathrm{C}^\bullet_\circ(\mathrm{M})$ is chain homotopy equivalent to $\mathrm{C}^{\bullet+1}_-(\mathrm{M})$, and hence the cone of $\Delta(\mathrm{C}^\bullet(\mathrm{M})) \to \Delta(\mathrm{C}^\bullet_\circ(\mathrm{M}))$ is chain homotopy equivalent to $\Delta(\mathrm{C}^{\bullet+1}_-(\mathrm{M}))$.

Since $\mathrm{C}^\bullet_-(\mathrm{M})$ is annihilated by $\Upsilon_{>\varnothing}$, we have $\mathrm{C}^\bullet_-(\mathrm{M})_{[\varnothing]} = \mathrm{C}^\bullet_-(\mathrm{M}) = \mathrm{C}^\bullet_-(\mathrm{M})_\varnothing$ and therefore the cohomology of $\Delta(\mathrm{C}^\bullet_-(\mathrm{M}))$ is zero in every degree. Thus, the cohomology of the cone of $\Delta(\mathrm{C}^\bullet(\mathrm{M})) \to \Delta(\mathrm{C}^\bullet_\circ(\mathrm{M}))$ is zero in every degree. Equivalently, the map $\Delta(\mathrm{C}^\bullet(\mathrm{M})) \to \Delta(\mathrm{C}^\bullet_\circ(\mathrm{M}))$ is a quasi-isomorphism. □

Lemma 8.12. The complex $\Delta(\mathrm{C}^\bullet_\circ(\mathrm{M}))$ is quasi-isomorphic to the cone of the map of complexes $\mathrm{C}^{\bullet-1}_\circ(\mathrm{M})_\varnothing[-1] \to \mathrm{C}^{\bullet-1}_\circ(\mathrm{M})_\varnothing$ given by multiplication by $x_\varnothing$.

Proof. By Lemma 5.2, the annihilator of $\Upsilon_{>\varnothing}$ in $\mathrm{CH}(\mathrm{M}^F)$ is equal to $x_\varnothing \, \mathrm{CH}(\mathrm{M}^F)$ for all nonempty flats F. Thus we have

$$\mathrm{CH}(\mathrm{M}^F)_{[\varnothing]} \cong x_\varnothing \, \mathrm{CH}(\mathrm{M}^F) \cong \mathrm{CH}(\mathrm{M}^F)_\varnothing[-1].$$

By Lemma 8.3, each $\mathrm{C}^i_\circ(\mathrm{M})$ is isomorphic to a direct sum of shifts of such modules, therefore

$$\mathrm{C}^{\bullet-1}_\circ(\mathrm{M})_{[\varnothing]} \cong \mathrm{C}^{\bullet-1}_\circ(\mathrm{M})_\varnothing[-1].$$

The lemma then follows from the definition of Δ. □

Proposition 8.13. Suppose that $d > 0$. Then the graded $\mathrm{H}_\circ(\mathrm{M})$-module $\mathrm{H}^i(\mathrm{C}^\bullet_\circ(\mathrm{M})_\varnothing)$ is concentrated in degree $d-1-i$.

Proof. Combining Lemmas 8.10, 8.11, and 8.12, it follows that $\mathrm{C}^\bullet(\mathrm{M})_{[\varnothing]}$ is quasi-isomorphic to the cone of the map $\mathrm{C}^{\bullet-1}_\circ(\mathrm{M})_\varnothing[-1] \to \mathrm{C}^{\bullet-1}_\circ(\mathrm{M})_\varnothing$ given by multiplication by $x_\varnothing$. This induces a long exact sequence

$$\cdots \to \mathrm{H}^i\big(\mathrm{C}^\bullet(\mathrm{M})_{[\varnothing]}\big) \to \mathrm{H}^i\big(\mathrm{C}^\bullet_\circ(\mathrm{M})_\varnothing\big)[-1] \xrightarrow{\cdot x_\varnothing} \mathrm{H}^i\big(\mathrm{C}^\bullet_\circ(\mathrm{M})_\varnothing\big) \to \mathrm{H}^{i+1}\big(\mathrm{C}^\bullet(\mathrm{M})_{[\varnothing]}\big) \to \cdots .$$

If $\mathrm{H}^i(\mathrm{C}^\bullet_\circ(\mathrm{M})_\varnothing) \neq 0$, let k be the smallest degree in which it does not vanish. That degree is not in the image of multiplication by $x_\varnothing$, so the long exact sequence implies that $\mathrm{H}^{i+1}(\mathrm{C}^\bullet(\mathrm{M})_{[\varnothing]})$ is nonzero in degree k. But that implies that $k = d-(i+1)$ by Proposition 8.8. Dually, if k is the largest nonvanishing degree, then it is killed by $x_\varnothing$, and our exact sequence implies that $\mathrm{H}^i(\mathrm{C}^\bullet(\mathrm{M})_{[\varnothing]})$ is nonzero in degree $k+1$, so we get $k+1 = d-i$ again by Proposition 8.8. Thus, the proposition follows. □

8.4. **The small complexes.** We begin with a standard lemma in homological algebra.

Lemma 8.14. Suppose that $(\mathrm{C}^\bullet, \partial)$ is a complex in some abelian category and we have direct sum decompositions of two consecutive objects

$$\mathrm{C}^k = \mathrm{P}^k \oplus \mathrm{Q}^k \quad \text{and} \quad \mathrm{C}^{k+1} = \mathrm{P}^{k+1} \oplus \mathrm{Q}^{k+1}$$

for some k with the property that the composition

$$\mathrm{P}^k \hookrightarrow \mathrm{C}^k \xrightarrow{\partial^k} \mathrm{C}^{k+1} \twoheadrightarrow \mathrm{P}^{k+1}$$

is an isomorphism. Then $(\mathrm{C}^\bullet, \partial)$ has as a direct summand a two-step acyclic complex whose k-th and $(k+1)$-st graded pieces are isomorphic to P^k.

Proof. First, we can replace P^{k+1} by the image of P^k in C^{k+1}. It is easy to check that the direct sum decomposition still holds, and now the differential sends P^k to P^{k+1} isomorphically. Next, replace Q^k by the kernel of the composition $\mathrm{C}^k \to \mathrm{C}^{k+1} \twoheadrightarrow \mathrm{P}^{k+1}$. It is again easy to check that our direct sum decomposition still holds and that the differential sends Q^k to Q^{k+1}. Now the differential $\partial^{k-1}\colon \mathrm{C}^{k-1} \to \mathrm{C}^k$ has image contained in $\ker \partial^k$, which is contained in Q^k, and $\partial^{k+1}(\mathrm{P}^{k+1}) = \partial^{k+1}\partial^k \mathrm{P}^k = 0$. So we obtain the desired direct sum of complexes. □

Regarding $\mathrm{C}^\bullet(\mathrm{M})$ as a complex of graded $\mathrm{H}(\mathrm{M})$-modules, we split off as many two-term acyclic complexes as possible until there do not exist k, $\mathrm{P}^k \neq 0$, P^{k+1}, Q^k, and Q^{k+1} such that the hypotheses of Lemma 8.14 hold. We call the resulting complex $\bar{\mathrm{C}}^\bullet(\mathrm{M}) \subseteq \mathrm{C}^\bullet(\mathrm{M})$ the **small Rouquier complex**. If E is nonempty, applying the same construction to $\mathrm{C}^\bullet_\circ(\mathrm{M})$ in the category of graded $\mathrm{H}_\circ(\mathrm{M})$-modules, we obtain the **small reduced Rouquier complex** $\bar{\mathrm{C}}^\bullet_\circ(\mathrm{M}) \subseteq \mathrm{C}^\bullet_\circ(\mathrm{M})$. The important features of these complexes are as follows:

(1) For any flat F, we have quasi-isomorphisms

$$\bar{\mathrm{C}}^\bullet(\mathrm{M})_F \simeq \mathrm{C}^\bullet(\mathrm{M})_F,\ \bar{\mathrm{C}}^\bullet(\mathrm{M})_{[F]} \simeq \mathrm{C}^\bullet(\mathrm{M})_{[F]},\ \bar{\mathrm{C}}^\bullet_\circ(\mathrm{M})_F \simeq \mathrm{C}^\bullet_\circ(\mathrm{M})_F,\ \text{and}\ \bar{\mathrm{C}}^\bullet_\circ(\mathrm{M})_{[F]} \simeq \mathrm{C}^\bullet_\circ(\mathrm{M})_{[F]}. \quad (1)$$

(2) If P is a nonzero direct summand of $\bar{\mathrm{C}}^k(\mathrm{M})$ (respectively $\bar{\mathrm{C}}^k_\circ(\mathrm{M})$), it is not possible to find an inclusion of P into $\bar{\mathrm{C}}^{k+1}(\mathrm{M})$ (respectively $\bar{\mathrm{C}}^{k+1}_\circ(\mathrm{M})$) as a direct summand with the property that the map from P to itself induced by ∂^k is an isomorphism.

Remark 8.15. Even though the subcomplex $\bar{\mathrm{C}}^\bullet(\mathrm{M})$ of $\mathrm{C}^\bullet(\mathrm{M})$ depends on the choices of splitting, its isomorphism class as a complex of $\mathrm{H}(\mathrm{M})$-modules is uniquely determined. In fact, the category of bounded complexes of finitely generated $\mathrm{H}(\mathrm{M})$-modules is an abelian category in which every element has finite length. By the Krull–Schmidt theorem, the complex $\mathrm{C}^\bullet(\mathrm{M})$ admits a decomposition into a direct sum of indecomposable complexes of $\mathrm{H}(\mathrm{M})$-modules, and the summands are uniquely determined up to isomorphisms. Removing all two-term acyclic summands, we obtain $\bar{\mathrm{C}}^\bullet(\mathrm{M})$. For the same reason, the isomorphism class of $\bar{\mathrm{C}}^\bullet_\circ(\mathrm{M})$ as a complex of $\mathrm{H}_\circ(\mathrm{M})$-modules is uniquely determined.

8.5. **Parity in the small Rouquier complexes.** The following theorem says that in a certain sense the small complexes are "perverse" objects.

Theorem 8.16.

(1) Suppose that $\mathsf{CD}(\mathrm{M}^F_G)$, $\mathsf{PD}(\mathrm{M}^F_G)$, and $\mathsf{NS}(\mathrm{M}^F_G)$ hold for all flats $G < F$. Then, for all i, $\bar{\mathrm{C}}^i(\mathrm{M})$ is isomorphic to a direct sum of modules of the form $\mathrm{IH}(\mathrm{M}^F)[k]$, where $k = (i - \operatorname{crk} F)/2$ is a nonpositive integer. Furthermore, $\bar{\mathrm{C}}^0(\mathrm{M}) \cong \mathrm{IH}(\mathrm{M})$.

(2) Suppose that $E \neq \varnothing$, that $\mathsf{CD}_\circ(\mathrm{M}^F_G)$ and $\mathsf{PD}_\circ(\mathrm{M}^F_G)$ hold for all flats $G < F$, and that $\mathsf{NS}_\circ(\mathrm{M}^F_G)$ holds for all flats $G < F < E$. Then, for all i, $\bar{\mathrm{C}}^i_\circ(\mathrm{M})$ is isomorphic to a direct sum of modules of the form $\mathrm{IH}_\circ(\mathrm{M}^F)[k]$, where F is nonempty and $k = (i - \operatorname{crk} F)/2$ is a nonpositive integer. Furthermore, $\bar{\mathrm{C}}^0_\circ(\mathrm{M}) \cong \mathrm{IH}_\circ(\mathrm{M})$.

Proof. We will give the proof of part (1); the proof of part (2) is identical. By Proposition 6.5 (1), the $\mathrm{H}(\mathrm{M})$-modules $\mathrm{IH}(\mathrm{M}^F)[k]$ are indecomposable for all flats F. By Corollary 8.4 (1) and the Krull–Schmidt theorem, the $\mathrm{H}(\mathrm{M})$-module $\bar{\mathrm{C}}^i(\mathrm{M})$ is isomorphic to a direct sum of modules of the form $\mathrm{IH}(\mathrm{M}^F)[k]$. Since $\mathrm{C}^i(\mathrm{M})$ vanishes in negative degree, we must have $k \leqslant 0$. We need to prove that $k = (i - \operatorname{crk} F)/2$.

Assume for the sake of contradiction that we have a summand of $\bar{\mathrm{C}}^i(\mathrm{M})$ of the form $\mathrm{IH}(\mathrm{M}^F)[k]$ with $k < (i - \operatorname{crk} F)/2$, and take i minimal with the property that such a summand exists. For such i and such summand $\mathrm{IH}(\mathrm{M}^F)[k]$, the flat F cannot be equal to E, because Corollary 8.4 (1) implies that $\mathrm{IH}(\mathrm{M})$ appears only in $\mathrm{C}^0(\mathrm{M})$ with multiplicity one and no shift. Thus F is a proper flat. By Lemma 8.5 (1) and Equation (1), the complex $\bar{\mathrm{C}}^\bullet(\mathrm{M})_F$ is acyclic. In particular, the summand

$\mathrm{IH}(\mathrm{M}^F)[k]_F \subseteq \bar{\mathrm{C}}^i(\mathrm{M})_F$ must either map nontrivially to some summand $\mathrm{IH}(\mathrm{M}^G)[l]_F \subseteq \bar{\mathrm{C}}^{i+1}(\mathrm{M})_F$ or receive a nontrivial map from some summand $\mathrm{IH}(\mathrm{M}^G)[l]_F \subseteq \bar{\mathrm{C}}^{i-1}(\mathrm{M})_F$.

We first assume that $\mathrm{IH}(\mathrm{M}^F)[k]_F$ maps nontrivially to some summand $\mathrm{IH}(\mathrm{M}^G)[l]_F \subseteq \bar{\mathrm{C}}^{i+1}(\mathrm{M})_F$. By Corollary 6.4 (1), we must have $G = F$ and $l = k$, and then Proposition 6.5 (1) implies that this map must come from an isomorphism between the summands $\mathrm{IH}(\mathrm{M}^F)[k] \subseteq \bar{\mathrm{C}}^i(\mathrm{M})$ and $\mathrm{IH}(\mathrm{M}^F)[k] \subseteq \bar{\mathrm{C}}^{i+1}(\mathrm{M})$. This contradicts the definition of the small Rouquier complex.

Next, we assume that the summand $\mathrm{IH}(\mathrm{M}^F)[k]_F \subseteq \bar{\mathrm{C}}^i(\mathrm{M})_F$ receives a nontrivial map from some summand $\mathrm{IH}(\mathrm{M}^G)[l]_F \subseteq \bar{\mathrm{C}}^{i-1}(\mathrm{M})_F$. We have $G \geqslant F$, otherwise $\mathrm{IH}(\mathrm{M}^G)[l]_F = 0$. If $G = F$, then Corollary 6.4 (1) implies that $l = k$, and again we obtain a contradiction from the definition of the small Rouquier complex. So we may assume that $G > F$. Since $\mathrm{IH}(\mathrm{M}^F)[k]_F$ is concentrated in degree $-k$, Proposition 6.3 (1) applied to the matroid M^G implies that

$$-k \leqslant \frac{\operatorname{rk} G - \operatorname{rk} F - 1}{2} - l = \frac{\operatorname{crk} F - \operatorname{crk} G - 1}{2} - l,$$

and therefore

$$l \leqslant k + \frac{\operatorname{crk} F - \operatorname{crk} G - 1}{2} < \frac{i - \operatorname{crk} F}{2} + \frac{\operatorname{crk} F - \operatorname{crk} G - 1}{2} = \frac{i - 1 - \operatorname{crk} G}{2}.$$

This contradicts the minimality of i and therefore completes the proof that $k \geqslant (i - \operatorname{crk} F)/2$.

Now assume for the sake of contradiction that we have a summand of $\bar{\mathrm{C}}^i(\mathrm{M})$ of the form $\mathrm{IH}(\mathrm{M}^F)[k]$ with $k > (i - \operatorname{crk} F)/2$, and take i maximal with the property that such a summand exists. We will make an argument similar to the one that we used above, but now using costalks instead of stalks. By Corollary 8.9 (1) and Equation (1), the i-th cohomology group of $\bar{\mathrm{C}}^\bullet(\mathrm{M})_{[F]}$ vanishes except in degree $\operatorname{crk} F - i$. On the other hand, the costalk $\mathrm{IH}(\mathrm{M}^F)[k]_{[F]}$ is nontrivial only in degree $-k$, which by assumption is strictly less than $(\operatorname{crk} F - i)/2$. Since $k \leqslant 0$, we have $-k \leqslant -2k < \operatorname{crk} F - i$. Therefore, the image of $\mathrm{IH}(\mathrm{M}^F)[k]_{[F]}$ in the cohomology of $\bar{\mathrm{C}}^\bullet(\mathrm{M})_F$ must be zero. In particular, the summand $\mathrm{IH}(\mathrm{M}^F)[k]_{[F]} \subseteq \bar{\mathrm{C}}^i(\mathrm{M})_{[F]}$ must either map nontrivially to some summand $\mathrm{IH}(\mathrm{M}^G)[l]_{[F]} \subseteq \bar{\mathrm{C}}^{i+1}(\mathrm{M})_{[F]}$ or receive a nontrivial map from some summand $\mathrm{IH}(\mathrm{M}^G)[l]_{[F]} \subseteq \bar{\mathrm{C}}^{i-1}(\mathrm{M})_{[F]}$.

We first assume that $\mathrm{IH}(\mathrm{M}^F)[k]_{[F]}$ receives a nontrivial map from some summand $\mathrm{IH}(\mathrm{M}^G)[l]_{[F]}$ of $\bar{\mathrm{C}}^{i-1}(\mathrm{M})_{[F]}$. By Corollary 6.4 (1), we have $G = F$ and $l = k$, and then Proposition 6.5 (1) implies that this map must come from an isomorphism between the summands $\mathrm{IH}(\mathrm{M}^F)[k] \subseteq \bar{\mathrm{C}}^{i-1}(\mathrm{M})$ and $\mathrm{IH}(\mathrm{M}^F)[k] \subseteq \bar{\mathrm{C}}^i(\mathrm{M})$. This contradicts the definition of the small Rouquier complex.

Next, we assume that the summand $\mathrm{IH}(\mathrm{M}^F)[k]_{[F]} \subseteq \bar{\mathrm{C}}^i(\mathrm{M})_{[F]}$ maps nontrivially to some summand $\mathrm{IH}(\mathrm{M}^G)[l]_{[F]} \subseteq \bar{\mathrm{C}}^{i+1}(\mathrm{M})_{[F]}$. We have $G \geqslant F$, otherwise $\mathrm{IH}(\mathrm{M}^G)[l]_{[F]} = 0$. If $G = F$, then Corollary 6.4 (1) implies that $l = k$, and again we obtain a contradiction from the definition of the small Rouquier complex. So we may assume that $G > F$. Since $\mathrm{IH}(\mathrm{M}^F)[k]_{[F]}$ is concentrated in degree $-k$, Proposition 6.3 (1) applied to the matroid M^G implies that

$$-k \geqslant \frac{\operatorname{rk} G - \operatorname{rk} F + 1}{2} - l = \frac{\operatorname{crk} F - \operatorname{crk} G + 1}{2} - l,$$

and therefore

$$l \geqslant k + \frac{\operatorname{crk} F - \operatorname{crk} G + 1}{2} > \frac{i - \operatorname{crk} F}{2} + \frac{\operatorname{crk} F - \operatorname{crk} G + 1}{2} = \frac{i + 1 - \operatorname{crk} G}{2}.$$

This contradicts the maximality of i and therefore completes the proof that $k \leqslant (i - \operatorname{crk} F)/2$. Together with the previous argument, we conclude that $k = (i - \operatorname{crk} F)/2$.

Finally, we prove that $\bar{\mathrm{C}}^0(\mathrm{M}) \cong \mathrm{IH}(\mathrm{M})$. By Proposition 6.5 (1), we know that $\mathrm{IH}(\mathrm{M})$ is an indecomposable $\mathrm{H}(\mathrm{M})$-module. By Corollary 8.4 (1), $\mathrm{C}^0(\mathrm{M})$ contains exactly one copy of $\mathrm{IH}(\mathrm{M})$ without shift, and the other $\mathrm{C}^k(\mathrm{M})$ do not contain any copy of $\mathrm{IH}(\mathrm{M})$ with or without shift. Thus, the summand $\mathrm{IH}(\mathrm{M})$ of $\mathrm{C}^0(\mathrm{M})$ is not canceled in the definition of $\bar{\mathrm{C}}^\bullet(\mathrm{M})$, and hence $\bar{\mathrm{C}}^0(\mathrm{M})$ contains one copy of $\mathrm{IH}(\mathrm{M})$ as a direct summand.

The fact that $\bar{\mathrm{C}}^0(\mathrm{M})$ does not contain $\mathrm{IH}(\mathrm{M}^F)[k]$ for any proper flat F follows from a similar argument to one that we used above. Indeed, suppose $\bar{\mathrm{C}}^0(\mathrm{M})$ does contain $\mathrm{IH}(\mathrm{M}^F)[k]$ for some proper flat F. Since $\bar{\mathrm{C}}^\bullet(\mathrm{M})_F$ is acyclic, $\mathrm{IH}(\mathrm{M}^F)[k]_F$ must map nontrivially to some summand $\mathrm{IH}(\mathrm{M}^G)[l]_F \subseteq \bar{\mathrm{C}}^1(\mathrm{M})_F$. By Corollary 6.4 (1), we have $G = F$ and $l = k$, and then Proposition 6.5 (1) implies that this map must come from an isomorphism between the summands $\mathrm{IH}(\mathrm{M}^F)[k] \subseteq \bar{\mathrm{C}}^0(\mathrm{M})$ and $\mathrm{IH}(\mathrm{M}^F)[k] \subseteq \bar{\mathrm{C}}^1(\mathrm{M})$. This contradicts the definition of the small Rouquier complex, thus concluding the proof. □

Corollary 8.17. Suppose that $\mathsf{CD}_\circ(\mathrm{M}^F_G)$ and $\mathsf{PD}_\circ(\mathrm{M}^F_G)$ hold for all flats $G < F$ and that $\mathsf{NS}_\circ(\mathrm{M}^F_G)$ and $\underline{\mathsf{NS}}(\mathrm{M}^F_G)$ hold for all flats $G < F < E$. Then $\underline{\mathsf{NS}}^{<\frac{d-2}{2}}(\mathrm{M})$ holds.

Proof. Consider the complex $\bar{\mathrm{C}}^\bullet_\circ(\mathrm{M})_\varnothing$. Lemma 5.6, Corollary 7.6, and Theorem 8.16 (2) together imply that

$$\bar{\mathrm{C}}^0_\circ(\mathrm{M})_\varnothing \cong \mathrm{IH}_\circ(\mathrm{M})_\varnothing \cong \underline{\mathrm{IH}}(\mathrm{M}),$$

and Theorem 8.16 (2) further implies that $\bar{\mathrm{C}}^1_\circ(\mathrm{M})_\varnothing$ is a direct sum of modules of the form

$$\mathrm{IH}_\circ(\mathrm{M}^F)[k]_\varnothing \cong \underline{\mathrm{IH}}(\mathrm{M}^F)[k],$$

where F is nonempty and $k = (1 - \operatorname{crk} F)/2 \leqslant 0$. Applying Proposition 8.13 with $i = 0$ and Equation (1), it follows that the kernel of the map

$$\partial^0_\varnothing \colon \bar{\mathrm{C}}^0_\circ(\mathrm{M})_\varnothing \to \bar{\mathrm{C}}^1_\circ(\mathrm{M})_\varnothing$$

is concentrated in degree $d - 1$, which is larger than $(d-2)/2$. Thus, it suffices to show that $\underline{\mathrm{IH}}(\mathrm{M}^F)[k]$ has no socle in degrees less than $(d-2)/2$.

The hypothesis $\underline{\mathrm{NS}}(\mathrm{M}^F)$ implies that the socle of $\underline{\mathrm{IH}}(\mathrm{M}^F)$ vanishes in degrees less than or equal to $(\operatorname{rk} F - 2)/2$, and therefore the socle of $\underline{\mathrm{IH}}(\mathrm{M}^F)[k]$ vanishes in degrees less than or equal to

$$\frac{\operatorname{rk} F - 2}{2} - \frac{1 - \operatorname{crk} F}{2} = \frac{d-3}{2} = \frac{d-2}{2} - \frac{1}{2}.$$

We can therefore conclude $\underline{\mathrm{NS}}^{<\frac{d-2}{2}}(\mathrm{M})$. □

8.6. **Multiplicities and inverse Kazhdan–Lusztig polynomials.** In this section we show that, under the assumption that Theorem 3.17 holds, the multiplicities of the modules $\mathrm{IH}(\mathrm{M}^F)$ in the complex $\bar{\mathrm{C}}(\mathrm{M})$ are given by coefficients of inverse Kazhdan–Lusztig polynomials of matroids. This implies that these coefficients are nonnegative, providing a proof of Theorem 1.4.

For a matroid M, define a polynomial $\tilde{Q}_{\mathrm{M}}(t) \in \mathbb{N}[t]$ whose coefficient of t^{-k} is the multiplicity of the module $\mathrm{IH}(\mathrm{M}^{\varnothing})[k]$ in $\bar{\mathrm{C}}^{d+2k}(\mathrm{M})$.

Proposition 8.18. The inverse Kazhdan–Lusztig polynomial $Q_{\mathrm{M}}(t)$ is equal to $\tilde{Q}_{\mathrm{M}}(t)$, so in particular it has nonnegative coefficients.

Lemma 8.19. Suppose that Theorem 3.17 holds. For any flat F and any integer k, the multiplicity of $\mathrm{IH}(\mathrm{M}^F)[k]$ in $\bar{\mathrm{C}}^{\operatorname{crk} F+2k}(\mathrm{M})$ is equal to the coefficient of t^{-k} in $\tilde{Q}_{\mathrm{M}_F}(t)$.

Proof. Lemma 6.2 (1) gives an isomorphism

$$y_F\, \mathrm{IH}(\mathrm{M}^G)[\ell] \cong \begin{cases} \mathrm{IH}(\mathrm{M}^G_F)[\ell - \operatorname{rk} F] & \text{if } F \leqslant G, \\ 0 & \text{otherwise.} \end{cases}$$

So to find the multiplicity of $\mathrm{IH}(\mathrm{M}^F)$ with any shift in $\mathrm{C}^\bullet(\mathrm{M})$, it is sufficient to find the multiplicity of $\mathrm{IH}(\mathrm{M}^F_F)$ in $y_F\mathrm{C}^\bullet(\mathrm{M})$. But $\mathrm{M}^F_F = (\mathrm{M}_F)^{\varnothing}$ has rank zero, so our result will follow if we can show that there is an isomorphism

$$y_F\bar{\mathrm{C}}^\bullet(\mathrm{M}) \cong \bar{\mathrm{C}}^\bullet(\mathrm{M}_F)[-\operatorname{rk} F].$$

By Lemma 8.2 (1), we have an isomorphism

$$y_F\mathrm{C}^\bullet(\mathrm{M}) \cong \mathrm{C}^\bullet(\mathrm{M}_F)[-\operatorname{rk} F]$$

for the big Rouquier complexes. Using Lemma 6.2 (1), the indecomposable summands of $y_F\mathrm{C}^\bullet(\mathrm{M})$ are in bijection with the summands of $\mathrm{C}^\bullet(\mathrm{M})$ of the form $\mathrm{IH}(\mathrm{M}^G)[\ell]$ with $G \geqslant F$. By Proposition 6.5 (1), the restriction map

$$\operatorname{End}_{\mathrm{H}(\mathrm{M})\text{-mod}}(\mathrm{IH}(\mathrm{M}^G)) \to \operatorname{End}_{\mathrm{H}(\mathrm{M}_F)\text{-mod}}(y_F\, \mathrm{IH}(\mathrm{M}^G))$$

is an isomorphism. Thus, the summands which get canceled from $\mathrm{C}^\bullet(\mathrm{M}_F)$ to form the minimal complex $\bar{\mathrm{C}}^\bullet(\mathrm{M}_F)$ are exactly the images under multiplication by y_F of canceling pairs from $\mathrm{C}^\bullet(\mathrm{M})$. The result follows. □

Proof of Proposition 8.18, assuming Theorem 3.17. If the rank d of M is equal to zero, then $\mathrm{C}^\bullet(\mathrm{M}) = \bar{\mathrm{C}}^\bullet(\mathrm{M})$ has only one component, which is $\mathrm{IH}(\mathrm{M}) = \mathrm{IH}(\mathrm{M}^\varnothing)$ in degree zero. So $Q_\mathrm{M}(t) = 1 = \tilde{Q}_\mathrm{M}(t)$ in this case.

When the rank of M is positive, by [GX20, Theorem 1.3], the inverse Kazhdan–Lusztig polynomial of M satisfies

$$\sum_{F \in \mathcal{L}(\mathrm{M})} (-1)^{\operatorname{rk} F} P_{\mathrm{M}^F}(t) Q_{\mathrm{M}_F}(t) = 0. \tag{2}$$

Thus, it suffices to show that $\tilde{Q}_\mathrm{M}(t)$ satisfies the displayed recurrence relation when d is positive.

By Lemma 8.5 (1), the complex $\mathrm{C}^\bullet(\mathrm{M})_\varnothing$ is acyclic, and since $\bar{\mathrm{C}}^\bullet(\mathrm{M})_\varnothing$ is a direct summand of this complex, it is acyclic as well. By Theorem 8.16 (1) and Lemma 8.19, $\bar{\mathrm{C}}^i(\mathrm{M})_\varnothing$ is the direct sum of $\mathrm{IH}(\mathrm{M}^F)_\varnothing[k]$ for various flats F, where $k = (i - \operatorname{crk} F)/2$ is a nonpositive integer. Moreover, the number of copies of $\mathrm{IH}(\mathrm{M}^F)_\varnothing[k]$ is equal to the coefficient of t^k in $\tilde{Q}_\mathrm{M}(t)$. Notice that when $k = (i - \operatorname{crk} F)/2$ is an integer, i and $\operatorname{crk} F$ have the same parity. Since the Poincaré polynomial of $\mathrm{IH}(\mathrm{M}^F)_\varnothing$ is equal to $P_{\mathrm{M}^F}(t)$, the alternating sum of the Poincaré polynomial of $\bar{\mathrm{C}}^i(\mathrm{M})_\varnothing$ for all i is equal to

$$\sum_{F \in \mathcal{L}(\mathrm{M})} (-1)^{\operatorname{crk} F} P_{\mathrm{M}^F}(t) \tilde{Q}_{\mathrm{M}_F}(t) = (-1)^{\operatorname{rk} \mathrm{M}} \sum_{F \in \mathcal{L}(\mathrm{M})} (-1)^{\operatorname{rk} F} P_{\mathrm{M}^F}(t) \tilde{Q}_{\mathrm{M}_F}(t).$$

Since $\bar{\mathrm{C}}^\bullet(\mathrm{M})_\varnothing$ is acyclic, the above sum is equal to zero.

All of the steps of this argument still hold when interpreted equivariantly with respect to any group of symmetries of M by Lemma A.1 and Definition A.6. □

When the matroid M has odd rank $2\ell + 1$, the coefficient of t^ℓ in (2) is nonzero only for $F = \varnothing$ and $F = E$, which implies that $P_\mathrm{M}(t)$ and $Q_\mathrm{M}(t)$ have the same coefficient of t^ℓ. Combining this with Proposition 8.18 gives the following result.

Corollary 8.20. For any matroid M, the second term $\bar{C}^1(\mathrm{M})$ of the small Rouquier complex is isomorphic to

$$\bigoplus_{j \geqslant 0} \bigoplus_{\operatorname{crk} F = 2k+1} \mathrm{IH}(\mathrm{M}^F)[-j]^{\oplus \tau(\mathrm{M}_F)},$$

where $\tau(\mathrm{M})$ denotes the coefficient of $t^{(\operatorname{rk} \mathrm{M} - 1)/2}$ in $P_\mathrm{M}(t)$ when $\operatorname{rk} \mathrm{M}$ is odd.

9. Deletion induction for $\mathrm{IH}(\mathrm{M})$

Let M be a matroid of rank $d > 0$ on the ground set E. The purpose of this section is to show that, if $\mathsf{CD}^{<\frac{d}{2}}(\mathrm{M})$ holds, and all of the statements of Theorem 3.17 hold for matroids whose ground sets are proper subsets of E, then $\mathsf{HL}_i(\mathrm{M})$ and $\mathsf{HR}_i^{<\frac{d}{2}}(\mathrm{M})$ also hold.

Throughout this section, we assume the following hypotheses:

(1) the element $i \in E$ is not a coloop and it does not have a parallel element;

(2) the statement $\mathsf{CD}^{<\frac{d}{2}}(\mathrm{M})$ holds;

(3) Theorem 3.17 holds for any matroid whose ground set is a proper subset of E.

In particular, $\mathsf{PD}_\circ(\mathrm{M})$ and $\mathsf{CD}_\circ(\mathrm{M})$ hold by Corollary 7.4, and $\mathsf{CD}^{>\frac{d}{2}}(\mathrm{M})$ holds by Remark 3.11. By Remark 3.12 and Proposition 7.8, the statement $\mathsf{CD}^{<\frac{d}{2}}(\mathrm{M})$ implies $\mathsf{PD}^{<\frac{d}{2}}(\mathrm{M})$ and $\mathsf{NS}^{<\frac{d}{2}}(\mathrm{M})$. Our goal is to show that these hypotheses imply $\mathsf{HL}_i(\mathrm{M})$ and $\mathsf{HR}_i^{<\frac{d}{2}}(\mathrm{M})$.

9.1. **The deletion map and the semi-small decomposition.** Fixing an element i of E, there is a graded algebra homomorphism

$$\theta_i^{\mathrm{M}}\colon\ \mathrm{CH}(\mathrm{M}\backslash i) \to \mathrm{CH}(\mathrm{M}), \quad x_F \mapsto x_F + x_{F\cup i},$$

where a variable in the target is set to zero if its label is not a flat of M. Let $\mathrm{CH}_{(i)}$ be the image of the homomorphism θ_i^{M}, and let

$$\mathcal{S}_i \coloneqq \big\{F \mid F \text{ is a proper subset of } E\backslash i \text{ such that } F \in \mathcal{L}(\mathrm{M}) \text{ and } F \cup i \in \mathcal{L}(\mathrm{M})\big\}.$$

We will use the following result [BHM+20, Theorem 1.2]. Even though we have assumed that i is not a coloop of M in this section, we include the coloop case of the theorem below because we will need it in the proof of Proposition 10.6.

Theorem 9.1. If i is not a coloop of M, there is a direct sum decomposition of $\mathrm{CH}(\mathrm{M})$ into indecomposable graded $\mathrm{CH}(\mathrm{M}\backslash i)$-modules

$$\mathrm{CH}(\mathrm{M}) = \mathrm{CH}_{(i)} \oplus \bigoplus_{F\in\mathcal{S}_i} x_{F\cup i}\mathrm{CH}_{(i)}. \tag{3}$$

If i is a coloop of M, there is a direct sum decomposition of $\mathrm{CH}(\mathrm{M})$ into indecomposable graded $\mathrm{CH}(\mathrm{M}\backslash i)$-modules

$$\mathrm{CH}(\mathrm{M}) = \mathrm{CH}_{(i)} \oplus x_{E\backslash i}\mathrm{CH}_{(i)} \oplus \bigoplus_{F\in\mathcal{S}_i} x_{F\cup i}\mathrm{CH}_{(i)}. \tag{4}$$

In the first case, all pairs of distinct summands are orthogonal for the Poincaré pairing of $\mathrm{CH}(\mathrm{M})$, while in the second case all pairs of summands except the first two are orthogonal. Moreover, the summands admit isomorphisms as $\mathrm{CH}(\mathrm{M}\backslash i)$-modules

$$\mathrm{CH}_{(i)} \cong \mathrm{CH}(\mathrm{M}\backslash i) \ \text{ and } \ x_{F\cup i}\,\mathrm{CH}_{(i)} \cong \underline{\mathrm{CH}}(\mathrm{M}_{F\cup i}) \otimes \mathrm{CH}(\mathrm{M}^F)[-1].$$

9.2. **The hard Lefschetz theorem.** Let $\delta\colon \mathcal{L}(\mathrm{M}) \to \mathcal{L}(\mathrm{M}\backslash i)$ be the map that takes F to $F\backslash i$ for all flats F. Let $\theta_i^{\mathrm{M}}\colon\ \mathrm{CH}(\mathrm{M}\backslash i) \to \mathrm{CH}(\mathrm{M})$ be the ring homomorphism defined in Section 9.1. It follows from the definition that $\theta_i^{\mathrm{M}}(y_j) = y_j$ for any $j \in E\backslash i$. More generally, for any flat $G \in \mathcal{L}(\mathrm{M})$, we have $\theta_i^{\mathrm{M}}(y_G) = y_{\bar{G}}$, where $\bar{G}$ is the closure of G in M.

Recall that, in Definition 5.3, we defined an ideal $\Upsilon_\Sigma \subseteq \mathrm{H}(\mathrm{M})$ for any order filter $\Sigma \subseteq \mathcal{L}(\mathrm{M})$. In this section, we will write $\Upsilon^{\mathrm{M}}_\Sigma$ for $\Sigma \subseteq \mathcal{L}(\mathrm{M})$ and $\Upsilon^{\mathrm{M}\setminus i}_\Sigma$ for $\Sigma \subseteq \mathcal{L}(\mathrm{M}\setminus i)$ to make it clear which matroid we are working with at any given time. The fact that $\theta^{\mathrm{M}}_i(y_G) = y_{\bar{G}}$ immediately implies the following lemma.

Lemma 9.2. For any order filter Σ in $\mathcal{L}(\mathrm{M}\setminus i)$, we have

$$\mathrm{H}(\mathrm{M}) \cdot \theta^{\mathrm{M}}_i(\Upsilon^{\mathrm{M}\setminus i}_\Sigma) = \Upsilon^{\mathrm{M}}_{\delta^{-1}(\Sigma)}.$$

We do not assume $\mathrm{CD}(\mathrm{M})$ in middle degree, so we do not yet know that $\mathrm{IH}(\mathrm{M})$ is a direct summand of $\mathrm{CH}(\mathrm{M})$. However, we can produce a direct summand artificially in the following manner. Let

$$\widetilde{\mathrm{IH}}^k(\mathrm{M}) \coloneqq \begin{cases} \mathrm{IH}^k(\mathrm{M}) & \text{if } k \neq d/2, \\ \mathrm{IH}^k_\circ(\mathrm{M}) & \text{if } k = d/2. \end{cases}$$

Equivalently, we can define

$$\widetilde{\underline{\mathrm{J}}}^k(\mathrm{M}) \coloneqq \begin{cases} \underline{\mathrm{J}}^k(\mathrm{M}) & \text{if } k \neq d/2, \\ 0 & \text{if } k = d/2, \end{cases}$$

and then define $\widetilde{\mathrm{IH}}(\mathrm{M})$ to be the orthogonal complement to $\psi^{\varnothing}_{\mathrm{M}}\widetilde{\underline{\mathrm{J}}}(\mathrm{M})$ inside of $\mathrm{IH}_\circ(\mathrm{M})$. In particular, when d is odd, $\widetilde{\mathrm{IH}}(\mathrm{M}) = \mathrm{IH}(\mathrm{M})$ and $\widetilde{\underline{\mathrm{J}}}(\mathrm{M}) = \underline{\mathrm{J}}(\mathrm{M})$.

Lemma 9.3. The subspace $\widetilde{\mathrm{IH}}(\mathrm{M}) \subseteq \mathrm{IH}_\circ(\mathrm{M})$ is an $\mathrm{H}(\mathrm{M})$-submodule. Moreover, $\widetilde{\mathrm{IH}}(\mathrm{M})$ satisfies Poincaré duality and it is a direct summand of $\mathrm{CH}(\mathrm{M})$.

Proof. The maximal ideal $\Upsilon_{>\varnothing}$ of $\mathrm{H}(\mathrm{M})$ annihilates $x_\varnothing$, and hence the image of $\psi^{\varnothing}_{\mathrm{M}}$. Therefore, $\psi^{\varnothing}_{\mathrm{M}}\widetilde{\underline{\mathrm{J}}}(\mathrm{M})$ is an $\mathrm{H}(\mathrm{M})$-submodule, and thus the same is true for its orthogonal complement. The statement $\mathrm{CD}^{<\frac{d}{2}}(\mathrm{M})$ implies that $\psi^{\varnothing}_{\mathrm{M}}\widetilde{\underline{\mathrm{J}}}(\mathrm{M})$ satisfies Poincaré duality, and the statement $\mathrm{CD}_\circ(\mathrm{M})$ implies that $\mathrm{IH}_\circ(\mathrm{M})$ satisfies Poincaré duality. Therefore, $\widetilde{\mathrm{IH}}(\mathrm{M})$ satisfies Poincaré duality and we have an orthogonal decomposition

$$\mathrm{IH}_\circ(\mathrm{M}) = \widetilde{\mathrm{IH}}(\mathrm{M}) \oplus \psi^{\varnothing}_{\mathrm{M}}\widetilde{\underline{\mathrm{J}}}(\mathrm{M}). \tag{5}$$

By $\mathrm{CD}_\circ(\mathrm{M})$, $\mathrm{IH}_\circ(\mathrm{M})$ is a direct summand of $\mathrm{CH}(\mathrm{M})$, and hence the lemma follows. □

Lemma 9.4. The inclusion $\widetilde{\mathrm{IH}}(\mathrm{M}) \subseteq \mathrm{IH}_\circ(\mathrm{M})$ induces an isomorphism

$$\widetilde{\mathrm{IH}}(\mathrm{M})_F \cong \mathrm{IH}_\circ(\mathrm{M})_F$$

for each nonempty flat F. If d is even, then the induced map $\mathrm{IH}^k(\mathrm{M})_\varnothing \to \widetilde{\mathrm{IH}}^k(\mathrm{M})_\varnothing$ is an isomorphism when $k \neq d/2, (d/2)+1$, and is surjective when $k = (d/2)+1$.

Proof. If F is nonempty, then the displayed isomorphism follows from multiplying Equation (5) by y_F, since the image of $\psi_{\mathrm{M}}^{\varnothing}$ is annihilated by y_F. The second statement follows from the fact that $\mathrm{IH}^k(\mathrm{M}) = \widetilde{\mathrm{IH}}^k(\mathrm{M})$ for $k \neq d/2$ and $\mathrm{IH}^{\frac{d}{2}}(\mathrm{M}) \subseteq \widetilde{\mathrm{IH}}^{\frac{d}{2}}(\mathrm{M})$. □

An $\mathrm{H}(\mathrm{M})$-module N can also be considered as an $\mathrm{H}(\mathrm{M}\backslash i)$-module. We will use notations $\mathrm{N}_{F\in\mathcal{L}(\mathrm{M})}$ and $\mathrm{N}_{F\in\mathcal{L}(\mathrm{M}\backslash i)}$ to emphasize the module structure under which the stalk is taken.

Lemma 9.5. Suppose that $F \in \mathcal{L}(\mathrm{M}\backslash i)$ is a proper flat. The stalk $\widetilde{\mathrm{IH}}(\mathrm{M})_{F\in\mathcal{L}(\mathrm{M}\backslash i)}$ vanishes in degrees strictly greater than $(\operatorname{crk} F)/2$.

Proof. For any order filter $\Sigma \subseteq \mathcal{L}(\mathrm{M}\backslash i)$, Lemma 9.2 says that $\Upsilon^{\mathrm{M}}_{\delta^{-1}\Sigma} = \mathrm{H}(\mathrm{M}) \cdot \theta_i^{\mathrm{M}}(\Upsilon^{\mathrm{M}\backslash i}_{\Sigma})$. Thus, we have

$$\widetilde{\mathrm{IH}}(\mathrm{M})_{F\in\mathcal{L}(\mathrm{M}\backslash i)} = \frac{\theta_i^{\mathrm{M}}\big(\Upsilon^{\mathrm{M}\backslash i}_{\geqslant F}\big)\widetilde{\mathrm{IH}}(\mathrm{M})[\operatorname{rk} F]}{\theta_i^{\mathrm{M}}\big(\Upsilon^{\mathrm{M}\backslash i}_{> F}\big)\widetilde{\mathrm{IH}}(\mathrm{M})[\operatorname{rk} F]} = \frac{\Upsilon^{\mathrm{M}}_{\delta^{-1}\Sigma_{\geqslant F}}\widetilde{\mathrm{IH}}(\mathrm{M})[\operatorname{rk} F]}{\Upsilon^{\mathrm{M}}_{\delta^{-1}\Sigma_{> F}}\widetilde{\mathrm{IH}}(\mathrm{M})[\operatorname{rk} F]}.$$

If $F \notin \mathcal{S}_i$ (defined in Section 9.1), then $\delta^{-1}(F)$ consists of a single element $\bar{F}$, the closure of F in M. In this case, we have

$$\delta^{-1}\Sigma_{\geqslant F} = \Sigma_{\geqslant \bar{F}} \quad \text{and} \quad \delta^{-1}\Sigma_{> F} = \Sigma_{> \bar{F}}.$$

Therefore, $\widetilde{\mathrm{IH}}(\mathrm{M})_{F\in\mathcal{L}(\mathrm{M}\backslash i)} \cong \widetilde{\mathrm{IH}}(\mathrm{M})_{\bar{F}\in\mathcal{L}(\mathrm{M})}$. By our assumption that i does not have any parallel element, it follows that $\varnothing \notin \mathcal{S}_i$, and hence $F \neq \varnothing$. Thus, by Proposition 6.3 (2), Lemma 9.4, and the fact that $\operatorname{rk} F = \operatorname{rk} \bar{F}$, the stalk $\widetilde{\mathrm{IH}}(\mathrm{M})_{F\in\mathcal{L}(\mathrm{M}\backslash i)}$ vanishes in degrees greater than or equal to $(\operatorname{crk} F)/2$.

If $F \in \mathcal{S}_i$, then both F and $F \cup i$ are flats of M. In this case, we have

$$\delta^{-1}\Sigma^{\mathrm{M}\backslash i}_{\geqslant F} = \Sigma^{\mathrm{M}}_{\geqslant F} \quad \text{and} \quad \delta^{-1}\Sigma^{\mathrm{M}\backslash i}_{> F} = \Sigma^{\mathrm{M}}_{> F}\backslash\{F \cup i\}.$$

Thus, we get an exact sequence of graded vector spaces

$$0 \to \widetilde{\mathrm{IH}}(\mathrm{M})_{F\cup i\in\mathcal{L}(\mathrm{M})}[-1] \to \widetilde{\mathrm{IH}}(\mathrm{M})_{F\in\mathcal{L}(\mathrm{M}\backslash i)} \to \widetilde{\mathrm{IH}}(\mathrm{M})_{F\in\mathcal{L}(\mathrm{M})} \to 0. \tag{6}$$

If F is nonempty, then by Proposition 6.3 (2) and Lemma 9.4, the stalk $\widetilde{\mathrm{IH}}(\mathrm{M})_{F\in\mathcal{L}(\mathrm{M})}$ vanishes in degrees greater than or equal to $(\operatorname{crk} F)/2$ and $\widetilde{\mathrm{IH}}(\mathrm{M})_{F\cup i\in\mathcal{L}(\mathrm{M})}[-1]$ vanishes in degrees greater than or equal to $1 + (\operatorname{crk} F \cup i)/2$, or equivalently in degrees greater than $(\operatorname{crk} F)/2$. Thus, $\widetilde{\mathrm{IH}}(\mathrm{M})_{F\in\mathcal{L}(\mathrm{M}\backslash i)}$ vanishes in degrees greater than $(\operatorname{crk} F)/2$.

If $F = \varnothing$, then $\widetilde{\mathrm{IH}}(\mathrm{M})_{i\in\mathcal{L}(\mathrm{M})}[-1]$ vanishes in degrees greater than $d/2$ for the same reason as above. For the remaining term, note that Lemmas 5.8 and 9.3 give an isomorphism $\widetilde{\mathrm{IH}}(\mathrm{M})_{\varnothing\in\mathcal{L}(\mathrm{M})} \cong (\widetilde{\mathrm{IH}}(\mathrm{M})_{[\varnothing]})^*[-d]$. On the other hand, we clearly have $\widetilde{\mathrm{IH}}(\mathrm{M})_{[\varnothing]} = \mathrm{IH}(\mathrm{M})_{[\varnothing]}$ in all degrees except $d/2$, and so $\mathsf{NS}^{\frac{d}{2}}(\mathrm{M})$ implies that $\widetilde{\mathrm{IH}}(\mathrm{M})_{[\varnothing]}$ vanishes in degrees below $d/2$. Thus $\widetilde{\mathrm{IH}}(\mathrm{M})_{\varnothing\in\mathcal{L}(\mathrm{M})}$ vanishes in degrees above $d/2$, and the sequence (6) gives the result. □

Proposition 9.6. The graded $\mathrm{H}(\mathrm{M})$-module $\widetilde{\mathrm{IH}}(\mathrm{M})$ is isomorphic as an $\mathrm{H}(\mathrm{M}\backslash i)$-module to a direct sum of modules of the form $\mathrm{IH}((\mathrm{M}\backslash i)^F)[-(\operatorname{crk} F)/2]$ for various flats $F \in \mathcal{L}(\mathrm{M}\backslash i)$ of even corank.

Proof. By Proposition 6.5 (1), the $\mathrm{H}(\mathrm{M}\backslash i)$-modules $\mathrm{IH}((\mathrm{M}\backslash i)^F)$ are indecomposable for all $F \in \mathcal{L}(\mathrm{M}\backslash i)$. Thus, the decomposition (3) and the isomorphism

$$x_{F\cup i}\,\mathrm{CH}_{(i)} \cong \underline{\mathrm{CH}}(\mathrm{M}_{F\cup i}) \otimes \mathrm{CH}(\mathrm{M}^F)[-1]$$

in Theorem 9.1 imply that as a graded $\mathrm{H}(\mathrm{M}\backslash i)$-module $\mathrm{CH}(\mathrm{M})$ is isomorphic to a direct sum of modules of the form $\mathrm{IH}((\mathrm{M}\backslash i)^F)[k]$ for various flats $F \in \mathcal{L}(\mathrm{M}\backslash i)$ and integers k. By Lemma 9.3, $\widetilde{\mathrm{IH}}(\mathrm{M})$ is also isomorphic to a sum of graded $\mathrm{H}(\mathrm{M}\backslash i)$-modules of this form. It suffices to show that if $\mathrm{IH}((\mathrm{M}\backslash i)^F)[k]$ appears as a summand of $\widetilde{\mathrm{IH}}(\mathrm{M})$, then $k = -(\operatorname{crk} F)/2$. Since only one copy of $\mathrm{IH}(\mathrm{M}\backslash i)$ appears in $\mathrm{CH}(\mathrm{M})$ without shift, the assertion holds for $F = E\backslash i$.

Suppose that $\mathrm{IH}((\mathrm{M}\backslash i)^F)[k]$ is a summand of $\widetilde{\mathrm{IH}}(\mathrm{M})$ with $F \in \mathcal{L}(\mathrm{M}\backslash i)$ a proper flat. From Lemma 5.7, we have $\mathrm{IH}((\mathrm{M}\backslash i)^F)[k]_F \cong \mathbb{Q}[k]$. Thus, Lemma 9.5 implies that $-k \leqslant (\operatorname{crk} F)/2$. Since both $\mathrm{IH}((\mathrm{M}\backslash i)^F)$ and $\widetilde{\mathrm{IH}}(\mathrm{M})$ satisfy Poincaré duality, we have isomophisms of $\mathrm{H}(\mathrm{M}\backslash i)$-modules

$$\mathrm{IH}((\mathrm{M}\backslash i)^F)^* \cong \mathrm{IH}((\mathrm{M}\backslash i)^F)[\operatorname{rk} F] \ \text{ and } \ \widetilde{\mathrm{IH}}(\mathrm{M})^* \cong \widetilde{\mathrm{IH}}(\mathrm{M})[d].$$

Therefore,

$$\widetilde{\mathrm{IH}}(\mathrm{M}) \cong \widetilde{\mathrm{IH}}(\mathrm{M})^*[-d]$$

must have a summand isomorphic to

$$\mathrm{IH}\big((\mathrm{M}\backslash i)^F\big)[k]^*[-d] \cong \mathrm{IH}\big((\mathrm{M}\backslash i)^F\big)[-k-d+\operatorname{rk} F] = \mathrm{IH}\big((\mathrm{M}\backslash i)^F\big)[-k-\operatorname{crk} F].$$

By the above arguments, we have $k + \operatorname{crk} F \leqslant (\operatorname{crk} F)/2$, or equivalently $-k \geqslant (\operatorname{crk} F)/2$. The above two inequalities between $-k$ and $(\operatorname{crk} F)/2$ imply that $-k = (\operatorname{crk} F)/2$. □

Corollary 9.7. The statement $\mathsf{HL}_i(\mathrm{M})$ holds.

Proof. Notice that the hard Lefschetz theorem for $\mathrm{IH}(\mathrm{M})$ as an $\mathrm{H}(\mathrm{M}\backslash i)$-module is equivalent to the hard Lefschetz theorem for $\widetilde{\mathrm{IH}}(\mathrm{M})$ as an $\mathrm{H}(\mathrm{M}\backslash i)$-module. By Proposition 9.6, the theorem follows from HL for matroids whose ground sets are subsets of $E\backslash i$, and hence proper subsets of E. □

9.3. **The Hodge–Riemann relations away from middle degree.** Let F be a nonempty flat of $\mathrm{M}\backslash i$ of even corank, and suppose we have an inclusion

$$f\colon\ \mathrm{IH}((\mathrm{M}\backslash i)^F)[-(\operatorname{crk} F)/2] \hookrightarrow \widetilde{\mathrm{IH}}(\mathrm{M})$$

as a direct summand. We have two pairings on $\mathrm{IH}((\mathrm{M}\backslash i)^F)$ that are *a priori* different: the one induced by the inclusion of $\mathrm{IH}((\mathrm{M}\backslash i)^F)$ into $\mathrm{CH}((\mathrm{M}\backslash i)^F)$, and the one induced by the inclusion f followed by the inclusion of $\widetilde{\mathrm{IH}}(\mathrm{M})$ into $\mathrm{CH}(\mathrm{M})$.

Lemma 9.8. The above two pairings are related by a constant factor $c \in \mathbb{Q}$ with $(-1)^{\frac{\operatorname{crk} F}{2}} c > 0$.

Proof. Both pairings are compatible with the $\mathrm{H}(\mathrm{M}\backslash i)$-module structure in the sense that $\langle \eta\xi, \sigma\rangle = \langle \xi, \eta\sigma\rangle$ for any $\eta \in \mathrm{H}(\mathrm{M}\backslash i)$ and $\xi, \sigma \in \mathrm{IH}((\mathrm{M}\backslash i)^F)$. Thus, both are given by isomorphisms

$$\mathrm{IH}((\mathrm{M}\backslash i)^F)^* \cong \mathrm{IH}((\mathrm{M}\backslash i)^F)[\operatorname{rk} F].$$

of graded $\mathrm{H}(\mathrm{M}\backslash i)$-modules. Proposition 6.5 (1) implies that $\mathrm{IH}((\mathrm{M}\backslash i)^F)$ has only scalar endomorphisms, and hence any two such isomorphisms must be related by a nonzero scalar factor $c \in \mathbb{Q}$.

To compute the sign of c, we pair the class $1 \in \mathrm{IH}((\mathrm{M}\backslash i)^F)$ with the class $y_F \in \mathrm{IH}((\mathrm{M}\backslash i)^F)$. Inside $\mathrm{CH}((\mathrm{M}\backslash i)^F)$, they pair to 1. By Proposition 2.13 and Proposition 2.15, the pairing of 1 and y_F inside $\widetilde{\mathrm{IH}}(\mathrm{M})$ is equal to the Poincaré pairing of $\varphi^{\mathrm{M}}_{\bar F} \circ f(1)$ with itself inside $\varphi^{\mathrm{M}}_{\bar F}\widetilde{\mathrm{IH}}^{\frac{\operatorname{crk} F}{2}}(\mathrm{M})$. Since F is nonempty, $(\operatorname{crk} F)/2$ is strictly less than $d/2$, so $\widetilde{\mathrm{IH}}^{\frac{\operatorname{crk} F}{2}}(\mathrm{M}) = \mathrm{IH}^{\frac{\operatorname{crk} F}{2}}(\mathrm{M})$. Applying Lemma 6.2 (1), we see that $\varphi^{\mathrm{M}}_{\bar F}\,\mathrm{IH}^{\frac{\operatorname{crk} F}{2}}(\mathrm{M})$ is equal to $\mathrm{IH}^{\frac{\operatorname{crk} F}{2}}(\mathrm{M}_{\bar F})$. We note that $\bar F$ is the closure of F in M, and $\theta^{\mathrm{M}}_i(y_F) = y_{\bar F}$. Since $\varphi^{\mathrm{M}}_{\bar F} \circ f(1)$ is annihilated by y_j for all $j \in E\backslash \bar F$, it is a primitive class in $\mathrm{IH}^{\frac{\operatorname{crk} F}{2}}(\mathrm{M}_{\bar F})$. Therefore, the sign of its pairing with itself is equal to $(-1)^{\frac{\operatorname{crk} F}{2}}$ by $\mathsf{HR}(\mathrm{M}_{\bar F})$. □

Corollary 9.9. The statement $\mathsf{HR}_i^{<\frac{d}{2}}(\mathrm{M})$ holds.

Proof. Since the statement does not involve the middle degree, we can replace $\mathrm{IH}(\mathrm{M})$ with $\widetilde{\mathrm{IH}}(\mathrm{M})$. By Proposition 9.6, it suffices to prove that each summand $\mathrm{IH}((\mathrm{M}\backslash i)^F)[-(\operatorname{crk} F)/2]$ of $\widetilde{\mathrm{IH}}(\mathrm{M})$ satisfies the Hodge–Riemann relations. Again, since the statement does not involve the middle degree, we can assume that F is nonempty. Then the statement follows from Lemma 9.8 and $\mathsf{HR}((\mathrm{M}\backslash i)^F)$. □

10. Deletion induction for $\underline{\mathrm{IH}}_i(\mathrm{M})$

Let M be a matroid on the ground set E. The purpose of this section is to show that, if we know everything for matroids with strictly smaller ground sets, then $\underline{\mathsf{HL}}_i(\mathrm{M})$ and $\underline{\mathsf{HR}}_i(\mathrm{M})$ hold.

Throughout this section, we assume the following:

(1) the element $i \in E$ is not a coloop and it does not have any parallel element;

(2) Theorem 3.17 holds for any matroid whose ground set is a proper subset of E.

The basic argument is similar to the one in the previous section. We show in Corollary 10.11 that $\underline{\mathrm{IH}}_i(\mathrm{M})$ is isomorphic to a direct sum of modules of the form $\underline{\mathrm{IH}}((\mathrm{M}\backslash i)^F)[-(\operatorname{crk} F)/2]$. Since the matroids $(\mathrm{M}\backslash i)^F$ have smaller ground sets than M, we know $\underline{\mathsf{HL}}((\mathrm{M}\backslash i)^F)$ and $\underline{\mathsf{HR}}((\mathrm{M}\backslash i)^F)$ by induction, and we use this to deduce $\underline{\mathsf{HL}}_i(\mathrm{M})$ and $\underline{\mathsf{HR}}_i(\mathrm{M})$. However, the $\mathbb{Q}[\underline{\beta}, x_{\{i\}}]$-module structure on $\underline{\mathrm{IH}}_i(\mathrm{M})$ is not rich enough to define the decomposition directly, so instead we work with a decomposition of $\mathrm{IH}_i(\mathrm{M})$ as an $\mathrm{H}(\mathrm{M}\backslash i)$-module, and then use it to produce the desired decomposition of $\underline{\mathrm{IH}}_i(\mathrm{M})$.

We have an identification $\underline{\mathrm{IH}}_i(\mathrm{M}) = \mathrm{IH}_i(\mathrm{M})_\varnothing$, where the stalk is taken at the empty flat in $\mathcal{L}(\mathrm{M})$. However, when $\mathrm{IH}_i(\mathrm{M})$ is considered as an $\mathrm{H}(\mathrm{M}\backslash i)$-module, we can only take the stalk of $\mathrm{IH}_i(\mathrm{M})$ at the empty flat of the matroid $\mathrm{M}\backslash i$. This stalk will be too large for what we want, because it includes a contribution from the stalk $\mathrm{IH}_i(\mathrm{M})_{\{i\}\in\mathcal{L}(\mathrm{M})}$, by the analog of the exact sequence (6). To get around this problem, we find a direct summand inside $\mathrm{IH}_i(\mathrm{M})$ that gives the correct stalk at $\varnothing \in \mathcal{L}(\mathrm{M}\backslash i)$. We find this summand by intersecting $\mathrm{IH}_i(\mathrm{M})$ with a direct summand of $\mathrm{CH}(\mathrm{M})$ obtained from the decomposition defined in [BHM+20].

Remark 10.1. Let us explain the geometry behind this decomposition when M is realizable. Following the notation of Section 1.3, we have the Schubert variety Y corresponding to M and its blow-up $Y_\circ$ at the point stratum corresponding to the flat $\varnothing$ of M. Recall from Remark 4.1 that the exceptional divisor $\underline{Y} \subseteq Y_\circ$ has intersection cohomology $\underline{\mathrm{IH}}(\mathrm{M})$. Let Y_i be the blow-up of $Y_\circ$ along the proper transform of $\overline{U^{\{i\}}}$, the closure of the stratum indexed by $\{i\}$, and let $\underline{Y}_i \subseteq Y_i$ be the inverse image of $\underline{Y}$. It is the blow-up of $\underline{Y}$ along $\underline{Y} \cap \overline{U^{\{i\}}}$, and its intersection cohomology is $\underline{\mathrm{IH}}_i(\mathrm{M})$.

As explained in Remark 4.3, the Schubert variety corresponding to $\mathrm{M}\backslash i$ is the image Y' of Y under the projection $(\mathbb{P}^1)^E \to (\mathbb{P}^1)^{E\backslash i}$. Let $Y'_\circ$ be the blow-up of Y' at the point stratum. The projection $Y \to Y'$ does not lift to a map $Y_\circ \to Y'_\circ$, but it does lift to a map $Y_i \to Y'_\circ$. The preimage of the exceptional divisor $\underline{Y}'$ of $Y'_\circ$ under this map has two components: $\underline{Y}_i$ and the exceptional divisor of $Y_i \to Y_\circ$. Taking the stalk of the $\mathrm{H}(\mathrm{M}\backslash i)$-module $\mathrm{IH}_i(\mathrm{M})$ at $\varnothing \in \mathcal{L}(\mathrm{M}\backslash i)$ gives the cohomology of the restriction of the IC sheaf of Y_i to the union of both components. The component $\underline{Y}_i$ gives $\underline{\mathrm{IH}}_i(\mathrm{M})$, but there is also a contribution from the other component. By finding the correct summand of $\mathrm{IH}_i(\mathrm{M})$, we are able to get only the part of this stalk that we want.

10.1. **A two-summand decomposition of CH(M).** Let $\mathcal{S}_i$ be the collection of subsets of $E\backslash i$ defined in Section 9.1. Let

$$R \coloneqq \mathrm{CH}_{(i)} \oplus \bigoplus_{F\in \mathcal{S}_i\backslash\{\varnothing\}} x_{F\cup i}\,\mathrm{CH}_{(i)} \quad \text{and} \quad P \coloneqq x_i\,\mathrm{CH}_{(i)} \subseteq \mathrm{CH}(\mathrm{M}).$$

By the decomposition (3), we have an orthogonal decomposition of $\mathrm{CH}(\mathrm{M}\backslash i)$-modules

$$\mathrm{CH}(\mathrm{M}) = R \oplus P.$$

Lemma 10.2. Let $F \in \mathcal{L}(\mathrm{M})$ be a flat different from both $\varnothing$ and $\{i\}$. Then $y_F R \subseteq R$ and $y_F P = 0$.

Proof. We have $y_F x_i = 0$ and hence $y_F P = 0$. Since $y_F P = 0$, R is orthogonal to $y_F P$, and equivalently, $y_F R$ is orthogonal to P. Since R is the orthogonal complement of P, the inclusion $y_F R \subseteq R$ follows. □

Consider the ideals

$$\Upsilon^{\mathrm{M}\backslash i}_{>\varnothing} \subseteq \mathrm{H}(\mathrm{M}\backslash i) \quad \text{and} \quad \Upsilon^{\mathrm{M}}_{>\varnothing} \subseteq \mathrm{H}(\mathrm{M}).$$

Recall that by Proposition 2.7 and Lemma 5.2, we have a natural isomorphism of $\mathrm{CH}(\mathrm{M})$-modules

$$\underline{\mathrm{CH}}(\mathrm{M}) \cong \frac{\mathrm{CH}(\mathrm{M})}{\Upsilon^{\mathrm{M}}_{>\varnothing} \cdot \mathrm{CH}(\mathrm{M})}.$$

Proposition 10.3. The inclusion of R into $\mathrm{CH}(\mathrm{M})$ induces an isomorphism of $\mathrm{CH}(\mathrm{M}\backslash i)$-modules

$$\frac{R}{\Upsilon^{\mathrm{M}\backslash i}_{>\varnothing} \cdot R} \cong \frac{\mathrm{CH}(\mathrm{M})}{\Upsilon^{\mathrm{M}}_{>\varnothing} \cdot \mathrm{CH}(\mathrm{M})}.$$

Proof. To prove surjectivity, it suffices to show that

$$R + \Upsilon^{\mathrm{M}}_{>\varnothing} \cdot \mathrm{CH}(\mathrm{M}) = \mathrm{CH}(\mathrm{M}),$$

or equivalently that $P \subseteq R + \Upsilon^{\mathrm{M}}_{>\varnothing} \cdot \mathrm{CH}(\mathrm{M})$. As a module over $\mathrm{CH}(\mathrm{M}\backslash i)$, P is generated by x_i, so it is enough to show that $x_i \in R + \Upsilon^{\mathrm{M}}_{>\varnothing} \cdot \mathrm{CH}(\mathrm{M})$. To see this, we observe that in $\mathrm{CH}(\mathrm{M})$,

$$y_i = \sum_{i \notin F} x_F = \sum_{F \in \mathcal{S}_i} \Big(\theta^{\mathrm{M}}_i(x_F) - x_{F \cup i}\Big) + \sum_{F \in \mathcal{L}(\mathrm{M}\backslash i), F \cup i \notin \mathcal{L}(\mathrm{M})} \theta^{\mathrm{M}}_i(x_F). \tag{7}$$

We have $y_i \in \Upsilon^{\mathrm{M}}_{>\varnothing} \mathrm{CH}(\mathrm{M})$ and all of the summands on the right-hand side of this expression are in R except for x_i, therefore $x_i \in R + \Upsilon^{\mathrm{M}}_{>\varnothing} \mathrm{CH}(\mathrm{M})$. This completes the proof of surjectivity.

We will prove injectivity by showing that the source and target have the same dimension. For this purpose, we factor the map as follows:

$$\frac{R}{\Upsilon^{\mathrm{M}\backslash i}_{>\varnothing} \cdot R} \to \frac{\mathrm{CH}(\mathrm{M})}{\Upsilon^{\mathrm{M}\backslash i}_{>\varnothing} \cdot \mathrm{CH}(\mathrm{M})} \to \frac{\mathrm{CH}(\mathrm{M})}{\Upsilon^{\mathrm{M}}_{>\varnothing} \cdot \mathrm{CH}(\mathrm{M})}.$$

Since R is a direct summand of $\mathrm{CH}(\mathrm{M})$ as a $\mathrm{CH}(\mathrm{M}\backslash i)$-module, it follows that

$$\Upsilon^{\mathrm{M}\backslash i}_{>\varnothing} \cdot R = R \cap \Upsilon^{\mathrm{M}\backslash i}_{>\varnothing} \cdot \mathrm{CH}(\mathrm{M}).$$

Thus, the first map is injective. We have shown that the composition (and therefore the second map) is surjective. It suffices to show that the cokernel of the first map has the same dimension as the kernel of the second map.

By Lemma 10.2, we have

$$\Upsilon^{\mathrm{M}\backslash i}_{>\varnothing} \cdot \mathrm{CH}(\mathrm{M}) = \Upsilon^{\mathrm{M}\backslash i}_{>\varnothing} \cdot R + \Upsilon^{\mathrm{M}\backslash i}_{>\varnothing} \cdot P = \Upsilon^{\mathrm{M}\backslash i}_{>\varnothing} \cdot R.$$

Therefore, the cokernel of the first map is isomorphic to $P = x_i \, \mathrm{CH}(\mathrm{M})_{(i)} \cong \underline{\mathrm{CH}}(\mathrm{M}_i)[-1]$.

On the other hand, the kernel of the second map is

$$\frac{\Upsilon^{\mathrm{M}}_{>\varnothing} \cdot \mathrm{CH}(\mathrm{M})}{\Upsilon^{\mathrm{M}\backslash i}_{>\varnothing} \cdot \mathrm{CH}(\mathrm{M})} = \frac{\Upsilon^{\mathrm{M}}_{>\varnothing} \cdot \mathrm{CH}(\mathrm{M})}{\Upsilon^{\mathrm{M}}_{\Sigma_{>\varnothing} \backslash \{i\}} \cdot \mathrm{CH}(\mathrm{M})}.$$

By Proposition 5.9 (1), the above quotient is isomorphic to

$$\frac{\Upsilon^{\mathrm{M}}_{\geqslant i} \cdot \mathrm{CH}(\mathrm{M})}{\Upsilon^{\mathrm{M}}_{>i} \cdot \mathrm{CH}(\mathrm{M})},$$

which by Lemma 5.7 is isomorphic to $(y_i \, \mathrm{CH}(\mathrm{M}))_\varnothing$. By Proposition 2.7 and Lemma 5.2, we have

$$(y_i \, \mathrm{CH}(\mathrm{M}))_\varnothing \cong \mathrm{CH}(\mathrm{M}_i)[-1]_\varnothing \cong \underline{\mathrm{CH}}(\mathrm{M}_i)[-1].$$

This completes the proof of injectivity. □

10.2. **The $(\boldsymbol{R}, \boldsymbol{P})$-decomposable modules.** We say that a graded subspace $V \subseteq \mathrm{CH}(\mathrm{M})$ is **$(\boldsymbol{R}, \boldsymbol{P})$-decomposable** if $V = (V \cap R) \oplus (V \cap P)$.

Lemma 10.4. If V is (R, P)-decomposable, so is the orthogonal complement $V^\perp$ with respect to the Poincaré pairing of $\mathrm{CH}(\mathrm{M})$.

Proof. Since V is (R, P)-decomposable, $V^\perp \cap R$ is equal to the orthogonal complement of $V \cap R$ inside of R and $V^\perp \cap P$ is equal to the orthogonal complement of $V \cap P$ inside of P. Since both R and P satisfy Poincaré duality, we have

$$\dim(V^\perp \cap R) = \dim R - \dim(V \cap R) \ \text{ and } \ \dim(V^\perp \cap P) = \dim P - \dim(V \cap P),$$

and hence

$$\begin{aligned} \dim(V^\perp \cap R) + \dim(V^\perp \cap P) &= \dim R + \dim P - \dim(V \cap R) - \dim(V \cap P) \\ &= \dim \mathrm{CH}(\mathrm{M}) - \dim V \\ &= \dim V^\perp. \end{aligned}$$

Since $(V^\perp \cap R) \oplus (V^\perp \cap P) \subseteq V^\perp$, the above equality implies that the inclusion is indeed an equality, and equivalently, $V^\perp$ is (R, P)-decomposable. □

Lemma 10.5. Suppose that $V \subseteq \mathrm{CH}(\mathrm{M})$ is a summand of $\mathrm{CH}(\mathrm{M})$ as graded $\mathrm{H}(\mathrm{M})$-modules and suppose that the restriction of the Poincaré pairing to V is non-degenerate. If $V \subseteq \mathrm{CH}(\mathrm{M})$ is (R, P)-decomposable, then the inclusion of V into $\mathrm{CH}(\mathrm{M})$ induces a graded vector space isomorphism

$$\frac{V \cap R}{\Upsilon^{\mathrm{M} \setminus i}_{>\varnothing} \cdot (V \cap R)} \cong \frac{V}{\Upsilon^{\mathrm{M}}_{>\varnothing} \cdot V}.$$

Proof. By the non-degeneracy assumption, we have a decomposition $\mathrm{CH}(\mathrm{M}) = V \oplus V^\perp$. By Lemma 10.4, $V^\perp$ is also (R, P)-decomposable. Thus, we have $R = (V \cap R) \oplus (V^\perp \cap R)$, and the isomorphism in Proposition 10.3 decomposes as a direct sum of isomorphisms

$$\frac{V \cap R}{V \cap \Upsilon^{\mathrm{M} \setminus i}_{>\varnothing} \cdot R} \cong \frac{V}{V \cap \Upsilon^{\mathrm{M}}_{>\varnothing} \cdot \mathrm{CH}(\mathrm{M})} \ \text{ and } \ \frac{V^\perp \cap R}{V^\perp \cap \Upsilon^{\mathrm{M} \setminus i}_{>\varnothing} \cdot R} \cong \frac{V^\perp}{V^\perp \cap \Upsilon^{\mathrm{M}}_{>\varnothing} \cdot \mathrm{CH}(\mathrm{M})}.$$

It therefore suffices to show that $V \cap \Upsilon^{\mathrm{M}\setminus i}_{>\varnothing} R = \Upsilon^{\mathrm{M}\setminus i}_{>\varnothing}(V \cap R)$ and $V \cap \Upsilon^{\mathrm{M}}_{>\varnothing} \mathrm{CH}(\mathrm{M}) = \Upsilon^{\mathrm{M}}_{>\varnothing} V$. They follow from decompositions $R = (V \cap R) \oplus (V^\perp \cap R)$ and $\mathrm{CH}(\mathrm{M}) = V \oplus V^\perp$ respectively. □

The main result of this section is the following.

Proposition 10.6. The subspace $\mathrm{IH}_i(\mathrm{M}) \subseteq \mathrm{CH}(\mathrm{M})$ is (R, P)-decomposable.

Proof. Clearly, a direct sum of (R, P)-decomposable modules is (R, P)-decomposable. By Lemma 10.4, it suffices to prove that every summand of $\mathrm{IH}_i(\mathrm{M})^\perp$ is (R, P)-decomposable. Thus, it suffices to show $\psi^F_{\mathrm{M}}(\underline{\mathrm{J}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F))$ is (R, P)-decomposable for all nonempty proper flats $F \neq \{i\}$. We divide the proof into three cases.

Case 1: $i \notin F$. In this case, $x_i x_F = 0$, so

$$\psi^F_{\mathrm{M}}(\underline{\mathrm{J}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F)) \subseteq x_F \, \mathrm{CH}(\mathrm{M}) \subseteq P^\perp = R.$$

Case 2: $i \in F$ and $F \setminus i \notin \mathcal{L}(\mathrm{M})$. In this case, i is not a coloop of M^F. By the decomposition (3), we have

$$\mathrm{CH}(\mathrm{M}^F) = \mathrm{CH}(\mathrm{M}^F)_{(i)} \oplus \bigoplus_{G \in \mathcal{S}_i(\mathrm{M}^F)} x_{G \cup i} \, \mathrm{CH}(\mathrm{M}^F)_{(i)}.$$

Let

$$Q \coloneqq \psi^F_{\mathrm{M}}\Big(\underline{\mathrm{J}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F)_{(i)}\Big).$$

Then,

$$\psi^F_{\mathrm{M}}\big(\underline{\mathrm{J}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F)\big) = Q \oplus \bigoplus_{G \in \mathcal{S}_i(\mathrm{M}^F)} x_{G \cup i} Q. \tag{8}$$

We will prove that $\psi^F_{\mathrm{M}}(\underline{\mathrm{J}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F))$ is (R, P)-decomposable, by proving every summand on the right-hand side of Equation (8) is (R, P)-decomposable.

Since $F \setminus i \notin \mathcal{L}(\mathrm{M})$, we have $\theta^{\mathrm{M}}_i(x_{F \setminus i}) = x_F$. Notice that

$$\underline{\mathrm{J}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F)_{(i)} \subseteq \underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F)_{(i)} = \varphi^F_{\mathrm{M}} \, \mathrm{CH}_{(i)}\,.$$

Therefore,

$$Q = \psi^F_{\mathrm{M}}\Big(\underline{\mathrm{J}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F)_{(i)}\Big) \subseteq \psi^F_{\mathrm{M}}\Big(\varphi^F_{\mathrm{M}} \, \mathrm{CH}_{(i)}\Big) = x_F \, \mathrm{CH}_{(i)} \subseteq \mathrm{CH}_{(i)} \subseteq R.$$

For any $G \in \mathcal{S}_i(\mathrm{M}^F) \subseteq \mathcal{S}_i(\mathrm{M})$, we have

$$x_{G \cup i} Q \subseteq x_{G \cup i} \, \mathrm{CH}_{(i)},$$

which is contained in R if G is nonempty and in P if G is empty. Therefore, each summand of the right-hand side of Equation (8) is (R, P)-decomposable.

Case 3: $i \in F$ and $F \setminus i \in \mathcal{L}(\mathrm{M})$. In this case, i is a coloop of M^F. By the decomposition (4), we have

$$\mathrm{CH}(\mathrm{M}^F) = \mathrm{CH}(\mathrm{M}^F)_{(i)} \oplus x_{F \setminus i} \, \mathrm{CH}(\mathrm{M}^F)_{(i)} \oplus \bigoplus_{G \in \mathcal{S}_i(\mathrm{M}^F)} x_{G \cup i} \, \mathrm{CH}(\mathrm{M}^F)_{(i)},$$

and hence

$$\psi^F_{\mathrm{M}}\big(\underline{\mathrm{J}}(\mathrm{M}_F)\otimes \mathrm{CH}(\mathrm{M}^F)\big) = Q \oplus x_{F\setminus i}Q \oplus \bigoplus_{G\in\mathcal{S}_i(\mathrm{M}^F)} x_{G\cup i}Q. \tag{9}$$

We still have $Q \subseteq x_F \mathrm{CH}_{(i)} \subseteq R$ (even though we no longer have $x_F \mathrm{CH}_{(i)} \subseteq \mathrm{CH}_{(i)}$). For any $G \in \mathcal{S}_i(\mathrm{M}^F) \subseteq \mathcal{S}_i(\mathrm{M})$, we have

$$x_{G\cup i}x_F = x_{G\cup i}(x_{F\setminus i} + x_F) = x_{G\cup i}\theta^{\mathrm{M}}_i(x_{F\setminus i}) \in x_{G\cup i}\mathrm{CH}_{(i)},$$

thus

$$x_{G\cup i}Q \subseteq x_{G\cup i}x_F \mathrm{CH}_{(i)} \subseteq x_{G\cup i}\mathrm{CH}_{(i)}\,.$$

If $G = \varnothing$, then $x_{G\cup i}\mathrm{CH}_{(i)}$ is contained in P, and otherwise it is contained in R. Since $\{i\}$ and $F\setminus i$ are incomparable, $x_{F\setminus i}Q$ is orthogonal to P, and hence contained in R. We have proved that every summand on the right-hand side of (9) is (R, P)-decomposable. Thus, $\psi^F_{\mathrm{M}}(\underline{\mathrm{J}}(\mathrm{M}_F)\otimes \mathrm{CH}(\mathrm{M}^F))$ is also (R, P)-decomposable. □

Combining Lemma 10.5 and Proposition 10.6, we obtain the following corollary.

Corollary 10.7. The inclusion of $\mathrm{IH}_i(\mathrm{M})$ into $\mathrm{CH}(\mathrm{M})$ induces a graded vector space isomorphism

$$\frac{\mathrm{IH}_i(\mathrm{M})\cap R}{\Upsilon^{\mathrm{M}\setminus i}_{>\varnothing}\cdot(\mathrm{IH}_i(\mathrm{M})\cap R)} \cong \frac{\mathrm{IH}_i(\mathrm{M})}{\Upsilon^{\mathrm{M}}_{>\varnothing}\cdot \mathrm{IH}_i(\mathrm{M})} \cong \underline{\mathrm{IH}}_i(\mathrm{M}).$$

10.3. **The hard Lefschetz theorem.**

Lemma 10.8. Suppose that $F \in \mathcal{L}(\mathrm{M})$ is different from $\varnothing$ and $\{i\}$. Then, $\varphi^{\mathrm{M}}_F \mathrm{IH}_i(\mathrm{M}) = \mathrm{IH}(\mathrm{M}_F)$.

Proof. By Lemma 6.2 (2), we have $\varphi^{\mathrm{M}}_F \mathrm{IH}_\circ(\mathrm{M}) = \mathrm{IH}(\mathrm{M}_F)$. By $\mathrm{CD}_\circ(\mathrm{M})$, we have

$$\mathrm{IH}_i(\mathrm{M}) = \mathrm{IH}_\circ(\mathrm{M}) \oplus \psi^i_{\mathrm{M}}\big(\underline{\mathrm{J}}(\mathrm{M}/i)\otimes \mathrm{CH}(\mathrm{M}^i)\big) \subseteq \mathrm{IH}_\circ(\mathrm{M}) \oplus x_i \mathrm{CH}(\mathrm{M}).$$

Since φ^{M}_F is a ring homomorphism and $\varphi^{\mathrm{M}}_F(x_i) = 0$, it follows that

$$\varphi^{\mathrm{M}}_F \mathrm{IH}_i(\mathrm{M}) = \varphi^{\mathrm{M}}_F \mathrm{IH}_\circ(\mathrm{M}) = \mathrm{IH}(\mathrm{M}_F).$$ □

We will now state and prove the analogues of Lemma 9.5 and Proposition 9.6.

Lemma 10.9. Suppose that $F \in \mathcal{L}(\mathrm{M}\setminus i)$ is a nonempty proper flat. The stalk $(\mathrm{IH}_i(\mathrm{M}) \cap R)_{F\in\mathcal{L}(\mathrm{M}\setminus i)}$ vanishes in degrees greater than $(\operatorname{crk} F)/2$.

Proof. First we note that $\mathrm{IH}_i(\mathrm{M}) \cap R$ and $\mathrm{IH}_i(\mathrm{M}) \cap P$ are not $\mathrm{H}(\mathrm{M})$-modules, because they are not closed under multiplication by y_i.[11] However, by Lemma 10.2, P is closed under multiplication

[11]For instance, one can easily check that $y_i \notin R$ using Equation (7). Since $1 \in \mathrm{IH}_i(\mathrm{M}) \cap R$, it follows that $\mathrm{IH}_i(\mathrm{M}) \cap R$ is not closed under multiplication by y_i.

by y_G for $G \neq \{i\}$. As the orthogonal complement, R is also closed under multiplication by y_G for $G \neq \{i\}$. Thus, the stalks $(\mathrm{IH}_i(\mathrm{M}) \cap R)_{G \in \mathcal{L}(\mathrm{M})}$ are well-defined for $G \neq \varnothing, \{i\}$.

We now follow the argument in the proof of Lemma 9.5. If $F \notin \mathcal{S}_i$, then

$$(\mathrm{IH}_i(\mathrm{M}) \cap R)_{F \in \mathcal{L}(\mathrm{M} \backslash i)} \cong (\mathrm{IH}_i(\mathrm{M}) \cap R)_{\bar{F} \in \mathcal{L}(\mathrm{M})},$$

where $\bar{F}$ is the closure of F in $\mathcal{L}(\mathrm{M})$. By Lemma 5.7 and Lemma 10.8, we have

$$\mathrm{IH}_i(\mathrm{M})_{\bar{F} \in \mathcal{L}(\mathrm{M})} \cong (y_{\bar{F}}\, \mathrm{IH}_i(\mathrm{M}))_{\varnothing} \cong \left(\varphi^{\mathrm{M}}_{\bar{F}}\, \mathrm{IH}_i(\mathrm{M})\right)_{\varnothing} = \mathrm{IH}(\mathrm{M}_{\bar{F}})_{\varnothing}.$$

By Proposition 6.3 (1) for the matroid $\mathrm{M}_{\bar{F}}$, this stalk vanishes in degrees greater than or equal to $(\operatorname{crk} F)/2$. Therefore, the stalk $(\mathrm{IH}_i(\mathrm{M}) \cap R)_{F \in \mathcal{L}(\mathrm{M} \backslash i)}$ also vanishes in degrees greater than or equal to $(\operatorname{crk} F)/2$.

If $F \in \mathcal{S}_i$, then

$$\delta^{-1} \Sigma^{\mathrm{M} \backslash i}_{\geqslant F} = \Sigma^{\mathrm{M}}_{\geqslant F} \text{ and } \delta^{-1} \Sigma^{\mathrm{M} \backslash i}_{> F} = \Sigma^{\mathrm{M}}_{> F} \backslash \{F \cup i\}.$$

Thus, we get a short exact sequence of graded vector spaces

$$0 \to (\mathrm{IH}_i(\mathrm{M}) \cap R)_{F \cup i \in \mathcal{L}(\mathrm{M})}[-1] \to (\mathrm{IH}_i(\mathrm{M}) \cap R)_{F \in \mathcal{L}(\mathrm{M} \backslash i)} \to (\mathrm{IH}_i(\mathrm{M}) \cap R)_{F \in \mathcal{L}(\mathrm{M})} \to 0.$$

By the same reasoning as above, we have $(\mathrm{IH}_i(\mathrm{M}) \cap R)_{F \in \mathcal{L}(\mathrm{M})}$ vanishes in degrees greater than or equal to $(\operatorname{crk} F)/2$ and $(\mathrm{IH}_i(\mathrm{M}) \cap R)_{F \cup i \in \mathcal{L}(\mathrm{M})}[-1]$ vanishes in degrees greater than or equal to $1 + (\operatorname{crk} F \cup i)/2$. Since $\operatorname{crk} F \cup i = \operatorname{crk} F - 1$, the short exact sequence implies that $(\mathrm{IH}_i(\mathrm{M}) \cap R)_{F \in \mathcal{L}(\mathrm{M} \backslash i)}$ vanishes in degrees greater than $(\operatorname{crk} F)/2$. □

Since $x_{F \cup i}\, \mathrm{CH}_{(i)} \cong \underline{\mathrm{CH}}(\mathrm{M}_{F \cup i}) \otimes \mathrm{CH}(\mathrm{M}^F)[-1]$ as $\mathrm{CH}_{(i)}$-modules, and hence as $\mathrm{H}_\circ(\mathrm{M} \backslash i)$-modules, R is isomorphic to the direct sum of $\mathrm{CH}(\mathrm{M} \backslash i)$ and shifted copies of $\mathrm{CH}(\mathrm{M}^F)$ for various $F \in \mathcal{S}_i$. Furthermore, iterating the decomposition $\mathrm{CD}_\circ$ to lower intervals of M, one sees that as $\mathrm{H}_\circ(\mathrm{M} \backslash i)$-modules, $\mathrm{CH}(\mathrm{M} \backslash i)$ and each $\mathrm{CH}(\mathrm{M}^F)$ are isomorphic to sums of shifted copies of $\mathrm{IH}_\circ((\mathrm{M} \backslash i)^G)$ for various nonempty flats $G \in \mathcal{L}(\mathrm{M} \backslash i)$. Thus, R is isomorphic to a direct sum of shifted copies of $\mathrm{IH}_\circ((\mathrm{M} \backslash i)^G)$ for various nonempty flats $G \in \mathcal{L}(\mathrm{M} \backslash i)$. Since $\mathrm{IH}_i(\mathrm{M}) \cap R$ is a direct summand of R as an $\mathrm{H}_\circ(\mathrm{M} \backslash i)$-module, by the indecomposability of $\mathrm{IH}_\circ((\mathrm{M} \backslash i)^G)$ (Proposition 6.5 (2)) and the Krull–Schmidt theorem, we know that $\mathrm{IH}_i(\mathrm{M}) \cap R$ is also isomorphic to a direct sum of shifted copies of $\mathrm{IH}_\circ((\mathrm{M} \backslash i)^G)$ for various nonempty flats $G \in \mathcal{L}(\mathrm{M} \backslash i)$.

Proposition 10.10. The graded $\mathrm{H}_\circ(\mathrm{M} \backslash i)$-module $\mathrm{IH}_i(\mathrm{M}) \cap R$ is isomorphic to a direct sum of modules of the form $\mathrm{IH}_\circ((\mathrm{M} \backslash i)^F)[-(\operatorname{crk} F)/2]$ for various nonempty flats $F \in \mathcal{L}(\mathrm{M} \backslash i)$.

Proof. The proof is very similar to that of Proposition 9.6. As we have argued before the proposition, each indecomposable summand of $\mathrm{IH}_i(\mathrm{M}) \cap R$ is isomorphic to $\mathrm{IH}_\circ((\mathrm{M} \backslash i)^F)[k]$ for some nonempty flat $F \in \mathcal{L}(\mathrm{M} \backslash i)$. What we need to show is that $k = -(\operatorname{crk} F)/2$. Since only one copy of $\mathrm{IH}_\circ(\mathrm{M}\backslash i)$ appears in $\mathrm{IH}_i(\mathrm{M}) \cap R$, and it appears without shift, the assertion holds if $F = E\backslash i$. Now, we assume that F is a proper flat of $\mathrm{M}\backslash i$.

Suppose that $\mathrm{IH}_\circ((\mathrm{M}\backslash i)^F)[k]$ is a summand of $\mathrm{IH}_i(\mathrm{M}) \cap R$ with $F \in \mathcal{L}(\mathrm{M}\backslash i)$ a nonempty proper flat. By Lemma 5.7, we have $\mathrm{IH}((\mathrm{M}\backslash i)^F)[k]_F \cong \mathbb{Q}[k]$. Thus Lemma 10.9 implies that $-k \leqslant (\operatorname{crk} F)/2$.

By Corollary 7.4 and its proof, we know that $\mathrm{IH}_i(\mathrm{M})$ satisfies Poincaré duality. Since $\mathrm{CH}(\mathrm{M}) = R \oplus Q$ is an orthogonal decomposition, and since $\mathrm{IH}_i(\mathrm{M})$ is (R, P)-decomposable, the Poincaré duality of $\mathrm{IH}_i(\mathrm{M})$ implies the Poincaré duality of $\mathrm{IH}_i(\mathrm{M}) \cap R$. Thus, we have

$$\big(\mathrm{IH}_i(\mathrm{M}) \cap R\big)^* \cong \big(\mathrm{IH}_i(\mathrm{M}) \cap R\big)[d].$$

as $\mathrm{H}_\circ(\mathrm{M}\backslash i)$-modules. By $\mathrm{PD}_\circ((\mathrm{M}\backslash i)^F)$, we also have

$$\mathrm{IH}_\circ\big((\mathrm{M}\backslash i)^F\big)^* \cong \mathrm{IH}_\circ\big((\mathrm{M}\backslash i)^F\big)[\operatorname{rk} F]$$

as $\mathrm{H}_\circ(\mathrm{M}\backslash i)$-modules. These two isomorphisms imply that

$$\mathrm{IH}_i(\mathrm{M}) \cap R \cong (\mathrm{IH}_i(\mathrm{M}) \cap R)^*[-d]$$

must have a summand isomorphic to

$$\mathrm{IH}_\circ\big((\mathrm{M}\backslash i)^F\big)[k]^*[-d] \cong \mathrm{IH}_\circ\big((\mathrm{M}\backslash i)^F\big)[-k-d+\operatorname{rk} F] = \mathrm{IH}_\circ\big((\mathrm{M}\backslash i)^F\big)[-k-\operatorname{crk} F].$$

Therefore, $k + \operatorname{crk} F \leqslant (\operatorname{crk} F)/2$, or equivalently $-k \geqslant (\operatorname{crk} F)/2$. Combining the two inequalities, we conclude that $k = -(\operatorname{crk} F)/2$, as desired. □

Corollary 10.11. The graded vector space $\underline{\mathrm{IH}}_i(\mathrm{M})$ is isomorphic to a direct sum of copies of modules of the form

$$\underline{\mathrm{IH}}\big((\mathrm{M}\backslash i)^F\big)[-(\operatorname{crk} F)/2]$$

for various nonempty flats $F \in \mathcal{L}(\mathrm{M}\backslash i)$ of even corank. Furthermore, multiplication on $\underline{\mathrm{IH}}_i(\mathrm{M})$ by $\underline{\beta}_{\mathrm{M}} - x_i$ corresponds to multiplication on these modules by $\underline{\beta}_{\mathrm{M}\backslash i}$.

Proof. By Lemma 5.2 and Corollary 7.6, we have

$$\frac{\mathrm{IH}_\circ\big((\mathrm{M}\backslash i)^F\big)}{\Upsilon^{\mathrm{M}\backslash i}_{>\varnothing} \cdot \mathrm{IH}_\circ\big((\mathrm{M}\backslash i)^F\big)} \cong \underline{\mathrm{IH}}\big((\mathrm{M}\backslash i)^F\big).$$

Thus, the first statement follows from Corollary 10.7 and Proposition 10.10. The second statement follows from that fact that $\varphi^{\varnothing}_{\mathrm{M}\backslash i}(x_\varnothing) = -\underline{\beta}_{\mathrm{M}\backslash i}$ and $\varphi^{\varnothing}_{\mathrm{M}}(\theta^{\mathrm{M}}_i(x_\varnothing)) = \varphi^{\varnothing}_{\mathrm{M}}(x_\varnothing + x_i) = -\underline{\beta}_{\mathrm{M}} + x_i$. □

Corollary 10.12. The statement $\underline{\mathrm{HL}}_i(\mathrm{M})$ holds.

Proof. This follows from Corollary 10.11 and $\underline{\mathrm{HL}}((\mathrm{M}\backslash i)^F)$ for all nonempty flats $F \in \mathcal{L}(\mathrm{M}\backslash i)$. □

10.4. **The Hodge–Riemann relations.** We start with an analysis analogous to what we had at the beginning of Section 9.3. Let F be a nonempty flat of $\mathrm{M}\backslash i$ of even corank, and suppose we have an inclusion

$$f\colon\ \mathrm{IH}_\circ\big((\mathrm{M}\backslash i)^F\big)[-(\operatorname{crk} F)/2] \hookrightarrow \mathrm{IH}_i(\mathrm{M}) \cap R$$

as a direct summand. We have two pairings on $\mathrm{IH}_\circ((\mathrm{M}\backslash i)^F)$ that are *a priori* different: the one induced by the inclusion of $\mathrm{IH}_\circ((\mathrm{M}\backslash i)^F)$ into $\mathrm{CH}((\mathrm{M}\backslash i)^F)$, and the one induced by the inclusion of $\mathrm{IH}_\circ((\mathrm{M}\backslash i)^F)[-(\operatorname{crk} F)/2]$ into $\mathrm{IH}_i(\mathrm{M}) \cap R$.

Lemma 10.13. These two pairings on $\mathrm{IH}_\circ((\mathrm{M}\backslash i)^F)$ are related by a constant factor $c \in \mathbb{Q}$ with $(-1)^{\frac{\operatorname{crk} F}{2}} c > 0$.

Proof. Both pairings are compatible with the $\mathrm{H}_\circ(\mathrm{M}\backslash i)$-module structure in the sense that $\langle \eta\xi, \sigma\rangle = \langle \xi, \eta\sigma\rangle$ for any $\eta \in \mathrm{H}_\circ(\mathrm{M}\backslash i)$ and $\xi, \sigma \in \mathrm{IH}_\circ((\mathrm{M}\backslash i)^F)$. Thus both pairings are given by isomorphisms $\mathrm{IH}_\circ((\mathrm{M}\backslash i)^F)^* \cong \mathrm{IH}_\circ((\mathrm{M}\backslash i)^F)[\operatorname{rk} F]$ of graded $\mathrm{H}_\circ(\mathrm{M}\backslash i)$-modules. By Proposition 6.5 (2), the $\mathrm{H}_\circ(\mathrm{M}\backslash i)$-module $\mathrm{IH}_\circ((\mathrm{M}\backslash i)^F)$ has only scalar endomorphisms, so any two such isomorphisms must be related by a scalar factor $c \in \mathbb{Q}$.

To compute the sign of c, we pair the class $1 \in \mathrm{IH}_\circ((\mathrm{M}\backslash i)^F)$ with the class $y_F \in \mathrm{IH}_\circ((\mathrm{M}\backslash i)^F)$. Inside of $\mathrm{CH}((\mathrm{M}\backslash i)^F)$, they pair to 1. Since $\theta_i^{\mathrm{M}}(y_F) = y_{\bar F}$, by Proposition 2.13, Proposition 2.15, and Lemma 10.8, their pairing inside of $\mathrm{IH}_i(\mathrm{M}) \cap R$, or equivalently inside of $\mathrm{CH}(\mathrm{M})$, is equal to the Poincaré pairing of $\varphi^{\mathrm{M}}_{\bar F}(f(1))$ with itself inside of $\mathrm{IH}^{\frac{\operatorname{crk} F}{2}}(\mathrm{M}_{\bar F})$. The class $\varphi^{\mathrm{M}}_{\bar F} \circ f(1)$ is annihilated by y_j for all $j \in E\backslash \bar F$, so it is primitive, and therefore the sign of its Poincaré pairing with itself is equal to $(-1)^{\frac{\operatorname{crk} F}{2}}$ by $\mathsf{HR}(\mathrm{M}_{\bar F})$. □

Since every summand $\underline{\mathrm{IH}}((\mathrm{M}\backslash i)^F)[-(\operatorname{crk} F)/2]$ in Corollary 10.11 is the image of a summand $\mathrm{IH}((\mathrm{M}\backslash i)^F)[-(\operatorname{crk} F)/2]$ of $\mathrm{IH}_i(\mathrm{M}) \cap R$. We denote the inclusion by

$$\underline{f}\colon \underline{\mathrm{IH}}((\mathrm{M}\backslash i)^F)[-(\operatorname{crk} F)/2] \hookrightarrow \underline{\mathrm{IH}}_i(\mathrm{M})$$

which is the quotient map of f. Now, we have two pairings on $\underline{\mathrm{IH}}((\mathrm{M}\backslash i)^F)$ that are *a priori* different: the one induced by the inclusion of $\underline{\mathrm{IH}}((\mathrm{M}\backslash i)^F)$ into $\underline{\mathrm{CH}}((\mathrm{M}\backslash i)^F)$, and the one induced by the above inclusion $\underline{f}$.

Lemma 10.14. These two pairings on $\underline{\mathrm{IH}}((\mathrm{M}\backslash i)^F)$ are related by the same constant factor $c \in \mathbb{Q}$ as in Lemma 10.13 with $(-1)^{\frac{\operatorname{crk} F}{2}} c > 0$.

Proof. We need to compare the Poincaré pairings in the Chow rings and the augmented Chow rings. Given two classes $\eta, \xi \in \mathrm{IH}_\circ((\mathrm{M}\backslash i)^F)$, we denote their images in $\underline{\mathrm{IH}}((\mathrm{M}\backslash i)^F)$ by $\underline{\eta}, \underline{\xi}$. By Propositions 2.5 and 2.7, we have

$$\langle \underline{\eta}, \underline{\xi}\rangle_{\underline{\mathrm{CH}}((\mathrm{M}\backslash i)^F)} = \langle \eta, x_\varnothing \xi\rangle_{\mathrm{CH}((\mathrm{M}\backslash i)^F)}.$$

On the other hand, we have

$$\langle \underline{f}(\eta), \underline{f}(\xi)\rangle_{\underline{\mathrm{CH}}(\mathrm{M})} = \langle f(\eta), x_\varnothing f(\xi)\rangle_{\mathrm{CH}(\mathrm{M})} = \langle f(\eta), (\theta_i^{\mathrm{M}}(x_\varnothing) - x_i) f(\xi)\rangle_{\mathrm{CH}(\mathrm{M})} = \langle f(\eta), f(x_\varnothing \xi)\rangle_{\mathrm{CH}(\mathrm{M})},$$

where the last equality follows from the next lemma and f being an $\mathrm{H}_\circ(\mathrm{M}\backslash i)$-module homomorphism. Thus, the two pairings are related by the same constant factor c as in Lemma 10.13. □

Lemma 10.15. For any $\mu, \nu \in R$, we have $\langle \mu, x_i \nu\rangle_{\mathrm{CH}(\mathrm{M})} = 0$.

Proof. By [BHM+20, Lemma 3.8], for any $F \in \mathcal{S}_i \backslash \{\varnothing\}$, we have

$$x_i x_{F \cup i} \, \mathrm{CH}_{(i)} \subseteq x_i \, \mathrm{CH}_{(i)} .$$

Since R is the direct sum of $\mathrm{CH}_{(i)}$ and $x_{F\cup i}\,\mathrm{CH}_{(i)}$ for all $F \in \mathcal{S}_i\backslash\{\varnothing\}$, it follows that $x_i R = x_i \, \mathrm{CH}_{(i)} = Q$, which is orthogonal to R with respect to the Poincaré pairing of $\mathrm{CH}(\mathrm{M})$. Thus, the lemma follows. □

Corollary 10.16. The statement $\underline{\mathsf{HR}}_i(\mathrm{M})$ holds.

Proof. This follows from Corollary 10.11, Lemma 10.14, and $\underline{\mathsf{HR}}((\mathrm{M}\backslash i)^F)$ for $\varnothing \neq F \in \mathcal{L}(\mathrm{M}\backslash i)$. □

11. Deformation arguments

This section is devoted to arguments that establish hard Lefschetz or Hodge–Riemann properties by considering families of Lefschetz arguments. We assume throughout that E is nonempty.

11.1. Establishing $\mathsf{HR}^{<\frac{d}{2}}(\mathrm{M})$.

Proposition 11.1. We have

$$\mathsf{HL}(\mathrm{M}),\ \mathsf{HL}_i(\mathrm{M}),\ \text{ and }\ \mathsf{HR}_i^{<\frac{d}{2}}(\mathrm{M}) \implies \mathsf{HR}^{<\frac{d}{2}}(\mathrm{M}).$$

Proof. Given $y = \sum_{j\in E} c_j y_j$ with every $c_j > 0$, to show that $\mathrm{IH}(\mathrm{M})$ satisfies the Hodge–Riemann relations with respect to multiplication by y in degrees less than $d/2$, we consider

$$y_t \coloneqq t \cdot c_i y_i + \sum_{j\in E, j\neq i} c_j y_j .$$

By $\mathsf{HL}(\mathrm{M})$ and $\mathsf{HL}_i(\mathrm{M})$, $\mathrm{IH}(\mathrm{M})$ satisfies the hard Lefschetz theorem with respect to multiplication by y_t for any $t \geqslant 0$. Therefore, for any $k < d/2$, the Hodge–Riemann form on $\mathrm{IH}^k(\mathrm{M})$ associated with any y_t with $t \geqslant 0$ has the same signature. Given the hard Lefschetz theorem, the Hodge–Riemann relations are conditions on the signature of the Hodge–Riemann forms [AHK18, Proposition 7.6], thus the fact that $\mathrm{IH}(\mathrm{M})$ satisfies the Hodge–Riemann relations with respect to multiplication by y_0 implies that it satisfies the Hodge–Riemann relations for any y_t with $t \geqslant 0$. □

11.2. **Establishing $\underline{\mathrm{HR}}(\mathrm{M})$.** The purpose of this section is to prove Proposition 11.4, which gives us a way to pass from $\underline{\mathrm{HR}}_i(\mathrm{M})$ to $\underline{\mathrm{HR}}(\mathrm{M})$. If i has a parallel element, then the statements $\underline{\mathrm{HR}}_i^{\leqslant k}(\mathrm{M})$ and $\underline{\mathrm{HR}}^{\leqslant k}(\mathrm{M})$ are the same. So, without loss of generality, we may assume that i has no parallel element, or equivalently that $\{i\}$ is a flat.

For any $t \geqslant 0$, consider the degree one linear operator $\underline{\mathrm{L}}_t$ on $\underline{\mathrm{IH}}_i(\mathrm{M})$ given by multiplication by $\underline{\beta} - tx_i$. We will assume $\underline{\mathrm{CD}}(\mathrm{M})$ throughout this section, so that we have

$$\underline{\mathrm{IH}}_i^k(\mathrm{M}) = \underline{\mathrm{IH}}^k(\mathrm{M}) \oplus \underline{\psi}^i_{\mathrm{M}}\underline{\mathrm{J}}^{k-1}(\mathrm{M}/i) \ \text{ and } \ \underline{\mathrm{IH}}_i^{d-k-1}(\mathrm{M}) = \underline{\mathrm{IH}}^{d-k-1}(\mathrm{M}) \oplus \underline{\psi}^i_{\mathrm{M}}\underline{\mathrm{J}}^{d-k-2}(\mathrm{M}/i).$$

Lemma 11.2. The map

$$\underline{\mathrm{L}}_t^{d-2k-1} \colon \underline{\mathrm{IH}}_i^k(\mathrm{M}) \to \underline{\mathrm{IH}}_i^{d-k-1}(\mathrm{M})$$

is block diagonal with respect to the above direct sum decompositions.

Proof. Since $\underline{\beta} = \sum_{i \notin G \neq \varnothing} x_G$, we have $\underline{\beta} x_{\{i\}} = 0$. Since the image of $\underline{\psi}^i_{\mathrm{M}}$ is equal to the ideal of $\underline{\mathrm{CH}}(\mathrm{M})$ generated by x_i, multiplication by $\underline{\beta}$ annihilates the image of $\underline{\psi}^i_{\mathrm{M}}$. Thus, we have

$$\begin{aligned}
\underline{\mathrm{L}}_t^{d-2k-1}\underline{\psi}^i_{\mathrm{M}}\underline{\mathrm{J}}^{k-1}(\mathrm{M}/i) &= \underline{\beta}^{d-2k-1}\underline{\psi}^i_{\mathrm{M}}\underline{\mathrm{J}}^{k-1}(\mathrm{M}/i) + (-t)^{d-2k-1}x_i^{d-2k-1}\underline{\psi}^i_{\mathrm{M}}\underline{\mathrm{J}}^{k-1}(\mathrm{M}/i)\\
&= x_i^{d-2k-1}\underline{\psi}^i_{\mathrm{M}}\underline{\mathrm{J}}^{k-1}(\mathrm{M}/i)\\
&= \underline{\psi}^i_{\mathrm{M}}\Big(\underline{\varphi}^i_{\mathrm{M}}(x_i^{d-2k-1})\underline{\mathrm{J}}^{k-1}(\mathrm{M}/i)\Big)\\
&= \underline{\psi}^i_{\mathrm{M}}\Big(\underline{\beta}_{\mathrm{M}/i}^{d-2k-1}\underline{\mathrm{J}}^{k-1}(\mathrm{M}/i)\Big)\\
&= \underline{\psi}^i_{\mathrm{M}}\Big(\underline{\beta}_{\mathrm{M}/i}^{d-2k-1}\underline{\mathrm{IH}}^{k-1}(\mathrm{M}/i)\Big)\\
&= \underline{\psi}^i_{\mathrm{M}}\big(\underline{\mathrm{J}}^{d-k-2}(\mathrm{M}/i)\big).
\end{aligned}$$

Thus $\underline{\mathrm{L}}_t^{d-2k-1}$ maps $\underline{\psi}^i_{\mathrm{M}}\underline{\mathrm{J}}^{k-1}(\mathrm{M}/i)$ to $\underline{\psi}^i_{\mathrm{M}}\underline{\mathrm{J}}^{d-k-2}(\mathrm{M}/i)$.

By Propositions 2.9, 2.11, and 3.7, we have

$$\begin{aligned}
x_i^{d-2k-1}\underline{\mathrm{IH}}^k(\mathrm{M}) \cdot \underline{\psi}^i_{\mathrm{M}}\underline{\mathrm{J}}^{k-1}(\mathrm{M}/i) &= \underline{\psi}^i_{\mathrm{M}}\Big(\underline{\varphi}^i_{\mathrm{M}}\big(x_i^{d-2k-1}\underline{\mathrm{IH}}^k(\mathrm{M})\big) \cdot \underline{\mathrm{J}}^{k-1}(\mathrm{M}/i)\Big)\\
&= \underline{\psi}^i_{\mathrm{M}}\Big(\underline{\varphi}^i_{\mathrm{M}}\big(\underline{\mathrm{IH}}^k(\mathrm{M})\big) \cdot \underline{\beta}^{d-2k-1}\underline{\mathrm{J}}^{k-1}(\mathrm{M}/i)\Big)\\
&= \underline{\psi}^i_{\mathrm{M}}\Big(\underline{\varphi}^i_{\mathrm{M}}\big(\underline{\mathrm{IH}}^k(\mathrm{M})\big) \cdot \underline{\mathrm{J}}^{d-k-2}(\mathrm{M}/i)\Big)\\
&= \underline{\mathrm{IH}}^k(\mathrm{M}) \cdot \underline{\psi}^i_{\mathrm{M}}\underline{\mathrm{J}}^{d-k-2}(\mathrm{M}/i)\\
&= 0.
\end{aligned}$$

Since $\underline{\mathrm{IH}}(\mathrm{M})$ is the orthogonal complement of $\underline{\psi}^i_{\mathrm{M}}\underline{\mathrm{J}}(\mathrm{M}/i)$ in $\underline{\mathrm{IH}}_i(\mathrm{M})$, it follows that

$$x_i^{d-2k-1}\underline{\mathrm{IH}}^k(\mathrm{M}) \subseteq \underline{\mathrm{IH}}^{d-k-1}(\mathrm{M}).$$

Hence $\underline{\mathrm{L}}_t^{d-2k-1}$ maps $\underline{\mathrm{IH}}^k(\mathrm{M})$ to $\underline{\mathrm{IH}}^{d-k-1}(\mathrm{M})$. □

Lemma 11.3. Let $k \leqslant (d-1)/2$ be given, and suppose that the statements $\underline{\mathrm{HR}}_i(\mathrm{M})$ and $\underline{\mathrm{HL}}^{\leqslant k}(\mathrm{M})$ hold. For any $0 < t \leqslant 1$, the map

$$\underline{\mathrm{L}}_t^{d-2k-1} \colon \underline{\mathrm{IH}}^k(\mathrm{M}) \to \underline{\mathrm{IH}}^{d-k-1}(\mathrm{M})$$

is an isomorphism.

Proof. First note that the statement for $t = 1$ holds by Lemma 11.2 and $\underline{\mathrm{HR}}_i(\mathrm{M})$. For $0 < t < 1$, assume for the sake of contradiction that $0 \neq \eta \in \underline{\mathrm{IH}}^k(\mathrm{M})$ and

$$\left(\underline{\beta}^{d-2k-1} + (-tx_i)^{d-2k-1}\right)\eta = 0. \tag{10}$$

Multiplying this equation by $\underline{\beta}$ and by x_i gives

$$\underline{\beta}^{d-2k}\eta = 0 \ \text{ and } \ x_i^{d-2k}\eta = 0.$$

Thus η is a primitive class in $\underline{\mathrm{IH}}^k(\mathrm{M})$ with respect to $\underline{\beta} - x_i$. By $\underline{\mathrm{HR}}_i(\mathrm{M})$,

$$(-1)^k \underline{\deg}_{\mathrm{M}}\left(\left(\underline{\beta}^{d-2k-1} + (-x_i)^{d-2k-1}\right)\eta^2\right) > 0.$$

But by an application of (10), this inequality is equivalent to

$$\begin{aligned}
0 &< (-1)^k \underline{\deg}_{\mathrm{M}}\left(\left(\underline{\beta}^{d-2k-1} + (-tx_i)^{d-2k-1} - (-tx_i)^{d-2k-1} + (-x_i)^{d-2k-1}\right)\eta^2\right)\\
&= (-1)^k \underline{\deg}_{\mathrm{M}}\left(\left(-(-tx_i)^{d-2k-1} + (-x_i)^{d-2k-1}\right)\eta^2\right)\\
&= (-1)^{d-k-1} \underline{\deg}_{\mathrm{M}}\left(x_i^{d-2k-1}(-t^{d-2k-1}+1)\eta^2\right).
\end{aligned}$$

Since $0 < t < 1$, this inequality reduces to

$$(-1)^{d-k-1}\underline{\deg}_{\mathrm{M}}(x_i^{d-2k-1}\eta^2) > 0.$$

On the other hand, by Lemma 3.5, we know that $\underline{\varphi}^i_{\mathrm{M}}\underline{\mathrm{IH}}(\mathrm{M}) \subseteq \underline{\mathrm{IH}}(\mathrm{M}/i)$. Since $\underline{\beta}_{\mathrm{M}}^{d-2k}\eta = 0$ and $\underline{\varphi}^i_{\mathrm{M}}(\underline{\beta}_{\mathrm{M}}) = \underline{\beta}_{\mathrm{M}/i}$, it follows that $\underline{\beta}_{\mathrm{M}/i}^{d-2k}\underline{\varphi}^i_{\mathrm{M}}(\eta) = 0$. In other words, $\underline{\varphi}^i_{\mathrm{M}}(\eta) \in \underline{\mathrm{IH}}^k(\mathrm{M}/i)$ is a primitive class with respect to $\underline{\beta}_{\mathrm{M}/i}$. Thus, by Proposition 2.9 and Proposition 2.11, we have

$$\begin{aligned}
0 &\leqslant (-1)^k \underline{\deg}_{\mathrm{M}/i}\left(\underline{\beta}^{d-2k-2}\underline{\varphi}^i_{\mathrm{M}}(\eta)^2\right)\\
&= (-1)^{d-k} \underline{\deg}_{\mathrm{M}/i}\left(\underline{\varphi}^i_{\mathrm{M}}(x_i^{d-2k-2}\eta^2)\right)\\
&= (-1)^{d-k} \underline{\deg}_{\mathrm{M}}\left(\underline{\psi}^i_{\mathrm{M}}\underline{\varphi}^i_{\mathrm{M}}(x_i^{d-2k-2}\eta^2)\right)\\
&= (-1)^{d-k} \underline{\deg}_{\mathrm{M}}(x_i^{d-2k-1}\eta^2).
\end{aligned}$$

Now, we have a contradiction between the above two sets of inequalities. $\square$

Proposition 11.4. For any $k \leqslant (d-1)/2$, we have

$$\underline{\mathrm{HR}}(\mathrm{M}/i),\ \underline{\mathrm{PD}}(\mathrm{M}),\ \underline{\mathrm{HR}}_i(\mathrm{M}),\ \text{ and }\ \underline{\mathrm{HL}}^{\leqslant k}(\mathrm{M}) \implies \underline{\mathrm{HR}}^{\leqslant k}(\mathrm{M}).$$

Proof. By induction on k, we may assume $\underline{\mathrm{HR}}^{<k}(\mathrm{M})$. To prove $\underline{\mathrm{HR}}^{k}(\mathrm{M})$, it suffices to prove that the Hodge–Riemann form on $\underline{\mathrm{IH}}^k(\mathrm{M})$ with respect to $\underline{\mathrm{L}}_0$ has the expected signature. More precisely, by the proof of [AHK18, Proposition 7.6], it suffices to show that

$$\mathrm{sig}_{\underline{\mathrm{L}}_0}\,\underline{\mathrm{IH}}^k(\mathrm{M}) - \mathrm{sig}_{\underline{\mathrm{L}}_0}\,\underline{\mathrm{IH}}^{k-1}(\mathrm{M}) = (-1)^k\left(\dim \underline{\mathrm{IH}}^k(\mathrm{M}) - \dim \underline{\mathrm{IH}}^{k-1}(\mathrm{M})\right),$$

where $\mathrm{sig}_{\underline{\mathrm{L}}_0}$ denotes the signature of the Hodge–Riemann form associated to $\underline{\mathrm{L}}_0$.

By Lemma 11.3 and $\underline{\mathrm{PD}}(\mathrm{M})$, the Hodge–Riemann form associated to $\underline{\mathrm{L}}_t$ is non-degenerate for all $0 < t \leqslant 1$, and by $\underline{\mathrm{HL}}^{\leqslant k}(\mathrm{M})$, the Hodge–Riemann form is also non-degenerate when $t = 0$. Thus, both $\mathrm{sig}_{\underline{\mathrm{L}}_t}\,\underline{\mathrm{IH}}^k(\mathrm{M})$ and $\mathrm{sig}_{\underline{\mathrm{L}}_t}\,\underline{\mathrm{IH}}^{k-1}(\mathrm{M})$ are constant as t varies in the closed interval $[0,1]$. Therefore, it suffices to show that

$$\mathrm{sig}_{\underline{\mathrm{L}}_1}\,\underline{\mathrm{IH}}^k(\mathrm{M}) - \mathrm{sig}_{\underline{\mathrm{L}}_1}\,\underline{\mathrm{IH}}^{k-1}(\mathrm{M}) = (-1)^k\left(\dim \underline{\mathrm{IH}}^k(\mathrm{M}) - \dim \underline{\mathrm{IH}}^{k-1}(\mathrm{M})\right). \tag{11}$$

By Lemma 11.2, we have

$$\mathrm{sig}_{\underline{\mathrm{L}}_1}\,\underline{\mathrm{IH}}_i^k(\mathrm{M}) = \mathrm{sig}_{\underline{\mathrm{L}}_1}\,\underline{\mathrm{IH}}^k(\mathrm{M}) + \mathrm{sig}_{\underline{\mathrm{L}}_1}\,\underline{\psi}^i_{\mathrm{M}}\underline{\mathrm{J}}^{k-1}(\mathrm{M}/i). \tag{12}$$

For any $\eta, \xi \in \underline{\mathrm{J}}^{k-1}(\mathrm{M}/i)$, since $\underline{\beta}$ annihilates the image of $\underline{\psi}^i_{\mathrm{M}}$, we have

$$\underline{\mathrm{L}}_1^{d-2k-1}(\underline{\psi}^i_{\mathrm{M}}\eta\cdot\underline{\psi}^i_{\mathrm{M}}\xi) = (-x_i)^{d-2k-1}(\underline{\psi}^i_{\mathrm{M}}\eta\cdot\underline{\psi}^i_{\mathrm{M}}\xi),$$

and hence

$$\underline{\deg}_{\mathrm{M}}\left(\underline{\mathrm{L}}_1^{d-2k-1}(\underline{\psi}^i_{\mathrm{M}}\eta\cdot\underline{\psi}^i_{\mathrm{M}}\xi)\right) = \underline{\deg}_{\mathrm{M}}\left((-x_i)^{d-2k-1}(\underline{\psi}^i_{\mathrm{M}}\eta\cdot\underline{\psi}^i_{\mathrm{M}}\xi)\right) = \underline{\deg}_{\mathrm{M}}\left(\underline{\psi}^i_{\mathrm{M}}(\underline{\beta}^{d-2k-1}_{\mathrm{M}/i}\eta)\cdot\underline{\psi}^i_{\mathrm{M}}\xi\right).$$

By Lemma 2.17 (2) with $F = \{i\}$ and the fact that $\underline{\alpha}_{\mathrm{M}^{\{i\}}} = 0$ for degree reasons, we have

$$\underline{\deg}_{\mathrm{M}}\left(\underline{\psi}^i_{\mathrm{M}}(\underline{\beta}^{d-2k-1}_{\mathrm{M}/i}\eta)\cdot\underline{\psi}^i_{\mathrm{M}}\xi\right) = -\underline{\deg}_{\mathrm{M}/i}\left(\underline{\beta}^{d-2k}_{\mathrm{M}/i}\eta\xi\right).$$

Combining the above two sets of equations, we have

$$\mathrm{sig}_{\underline{\mathrm{L}}_1}\,\underline{\psi}^i_{\mathrm{M}}\underline{\mathrm{J}}^{k-1}(\mathrm{M}/i) = -\,\mathrm{sig}_{\underline{\beta}_{\mathrm{M}/i}}\,\underline{\mathrm{J}}^{k-1}(\mathrm{M}/i) = -\,\mathrm{sig}_{\underline{\beta}_{\mathrm{M}/i}}\,\underline{\mathrm{IH}}^{k-1}(\mathrm{M}/i).$$

Therefore, by $\underline{\mathrm{HR}}(\mathrm{M}/i)$, we have

$$\mathrm{sig}_{\underline{\mathrm{L}}_1}\,\underline{\psi}^i_{\mathrm{M}}\underline{\mathrm{J}}^{k-1}(\mathrm{M}/i) - \mathrm{sig}_{\underline{\mathrm{L}}_1}\,\underline{\psi}^i_{\mathrm{M}}\underline{\mathrm{J}}^{k-2}(\mathrm{M}/i) = (-1)^k\left(\dim \underline{\psi}^i_{\mathrm{M}}\underline{\mathrm{J}}^{k-1}(\mathrm{M}/i) - \dim \underline{\psi}^i_{\mathrm{M}}\underline{\mathrm{J}}^{k-2}(\mathrm{M}/i)\right).$$

By $\underline{\mathrm{HR}}_i(\mathrm{M})$, we have

$$\mathrm{sig}_{\underline{\mathrm{L}}_1}\,\underline{\mathrm{IH}}_i^k(\mathrm{M}) - \mathrm{sig}_{\underline{\mathrm{L}}_1}\,\underline{\mathrm{IH}}_i^{k-1}(\mathrm{M}) = (-1)^k\left(\dim \underline{\mathrm{IH}}_i^k(\mathrm{M}) - \dim \underline{\mathrm{IH}}_i^{k-1}(\mathrm{M})\right).$$

The above two equations together with Equation (12) implies the desired Equation (11). $\square$

11.3. **Establishing $\mathsf{HL}_\circ(\mathrm{M})$ and $\mathsf{HR}_\circ^{<\frac{d}{2}}(\mathrm{M})$.** We now use similar arguments to those in the previous subsection in order to obtain the statements $\mathsf{HL}_\circ(\mathrm{M})$ and $\mathsf{HR}_\circ^{<\frac{d}{2}}(\mathrm{M})$. Fix a positive sum

$$y = \sum_{j \in E} c_j y_j.$$

For any $t \geqslant 0$, consider the degree one linear operator L_t on $\mathrm{IH}_\circ(\mathrm{M})$ given by multiplication by $y - tx_\varnothing$. We will assume $\mathsf{CD}^{<\frac{d}{2}}(\mathrm{M})$, so that for any $k < d/2$, we have a direct sum decomposition

$$\mathrm{IH}_\circ^k(\mathrm{M}) = \mathrm{IH}^k(\mathrm{M}) \oplus \psi_\mathrm{M}^\varnothing \underline{\mathrm{J}}^{k-1}(\mathrm{M}) \ \text{ and } \ \mathrm{IH}_\circ^{d-k}(\mathrm{M}) = \mathrm{IH}^{d-k}(\mathrm{M}) \oplus \psi_\mathrm{M}^\varnothing \underline{\mathrm{J}}^{d-k-1}(\mathrm{M}).$$

Lemma 11.5. For any $t \geqslant 0$, the linear map

$$\mathrm{L}_t^{d-2k} \colon \mathrm{IH}_\circ^k(\mathrm{M}) \to \mathrm{IH}_\circ^{d-k}(\mathrm{M})$$

is block diagonal with respect to the above decompositions.

Proof. Since $yx_\varnothing = 0$ and y annihilates the image of $\psi_\mathrm{M}^\varnothing$, we have

$$\begin{aligned} \mathrm{L}_t^{d-2k}\, \psi_\mathrm{M}^\varnothing \underline{\mathrm{J}}^{k-1}(\mathrm{M}) &= y^{d-2k}\psi_\mathrm{M}^\varnothing \underline{\mathrm{J}}^{k-1}(\mathrm{M}) + (-t)^{d-2k} x_\varnothing^{d-2k} \psi_\mathrm{M}^\varnothing \underline{\mathrm{J}}^{k-1}(\mathrm{M}) \\ &= (-t)^{d-2k} x_\varnothing^{d-2k} \psi_\mathrm{M}^\varnothing \underline{\mathrm{J}}^{k-1}(\mathrm{M}) \\ &= t^{d-2k} \psi_\mathrm{M}^\varnothing \left(\underline{\beta}^{d-2k} \underline{\mathrm{J}}^{k-1}(\mathrm{M}) \right) \\ &= t^{d-2k} \psi_\mathrm{M}^\varnothing \underline{\mathrm{J}}^{d-k-1}(\mathrm{M}), \end{aligned}$$

which is equal to $\psi_\mathrm{M}^\varnothing \underline{\mathrm{J}}^{d-k-1}(\mathrm{M})$ if $t > 0$ and 0 if $t = 0$. In either case, we have

$$\mathrm{L}_t^{d-2k}\, \psi_\mathrm{M}^\varnothing \underline{\mathrm{J}}^{k-1}(\mathrm{M}) \subseteq \psi_\mathrm{M}^\varnothing \underline{\mathrm{J}}^{d-k-1}(\mathrm{M}).$$

By the above inclusion, for any $\eta \in \mathrm{IH}^k(\mathrm{M})$ and $\xi \in \psi_\mathrm{M}^\varnothing \underline{\mathrm{J}}^{k-1}(\mathrm{M})$, we have

$$\deg_\mathrm{M} \left(\mathrm{L}_t^{d-2k}(\eta) \cdot \xi \right) = \deg_\mathrm{M} \left(\eta \cdot \mathrm{L}_t^{d-2k}(\xi) \right) = 0.$$

Notice that the graded subspace $\mathrm{IH}(\mathrm{M}) \subseteq \mathrm{IH}_\circ(\mathrm{M})$ is the orthogonal complement of $\psi_\mathrm{M}^\varnothing \underline{\mathrm{J}}(\mathrm{M})$. Thus, we also have

$$\mathrm{L}_t^{d-2k}\, \mathrm{IH}^k(\mathrm{M}) \subseteq \mathrm{IH}^{d-k}(\mathrm{M}).$$

□

Proposition 11.6. We have

$$\mathsf{CD}^{<\frac{d}{2}}(\mathrm{M}), \ \ \underline{\mathsf{HL}}^{<\frac{d-2}{2}}(\mathrm{M}), \ \text{ and } \ \mathsf{HL}(\mathrm{M}) \implies \mathsf{HL}_\circ(\mathrm{M}).$$

Proof. By Lemma 11.5, we need to show that L_t^{d-2k} induces isomorphisms $\mathrm{IH}^k(\mathrm{M}) \cong \mathrm{IH}^{d-k}(\mathrm{M})$ and $\psi_\mathrm{M}^\varnothing \underline{\mathrm{J}}^{k-1}(\mathrm{M}) \cong \psi_\mathrm{M}^\varnothing \underline{\mathrm{J}}^{d-k-1}(\mathrm{M})$ for some $t > 0$. In the proof of Lemma 11.5, we have shown that when $t > 0$ the induced isomorphism $\psi_\mathrm{M}^\varnothing \underline{\mathrm{J}}^{k-1}(\mathrm{M}) \cong \psi_\mathrm{M}^\varnothing \underline{\mathrm{J}}^{d-k-1}(\mathrm{M})$ follows from $\underline{\mathsf{HL}}^{<\frac{d-2}{2}}(\mathrm{M})$.

The statement $\mathsf{HL}(\mathrm{M})$ implies that $\mathrm{L}_0^{d-2k} \colon \mathrm{IH}^k(\mathrm{M}) \to \mathrm{IH}^{d-k}(\mathrm{M})$ is an isomorphism. Therefore, for sufficiently small t, the map $\mathrm{L}_t^{d-2k} \colon \mathrm{IH}^k(\mathrm{M}) \to \mathrm{IH}^{d-k}(\mathrm{M})$ is also an isomorphism. □

Proposition 11.7. We have

$$\mathsf{CD}^{<\frac{d}{2}}(\mathrm{M}),\ \mathsf{HL}(\mathrm{M}),\ \mathsf{HR}^{<\frac{d}{2}}(\mathrm{M}),\ \underline{\mathsf{HL}}^{<\frac{d-2}{2}}(\mathrm{M}),\ \text{ and }\ \underline{\mathsf{HR}}^{<\frac{d-2}{2}}(\mathrm{M}) \implies \mathsf{HR}_\circ^{<\frac{d}{2}}(\mathrm{M}).$$

Proof. For $k < d/2$, we prove $\mathsf{HR}_\circ^k(\mathrm{M})$ by induction on k. It is clear that $\mathrm{IH}_\circ(\mathrm{M})$ satisfies the Hodge–Riemann relations in degree zero with respect to L_t for t sufficiently small. Now fix $0 < k < d/2$ and suppose that $\mathsf{HR}_\circ^{<k}(\mathrm{M})$ holds. We need to show that, for t sufficiently small,

$$\mathrm{sig}_{\mathrm{L}_t}\,\mathrm{IH}_\circ^k(\mathrm{M}) - \mathrm{sig}_{\mathrm{L}_t}\,\mathrm{IH}_\circ^{k-1}(\mathrm{M}) = (-1)^k\left(\dim \mathrm{IH}_\circ^k(\mathrm{M}) - \dim \mathrm{IH}_\circ^{k-1}(\mathrm{M})\right).$$

By Lemma 11.5, we have

$$\mathrm{sig}_{\mathrm{L}_t}\,\mathrm{IH}_\circ^k(\mathrm{M}) = \mathrm{sig}_{\mathrm{L}_t}\,\mathrm{IH}^k(\mathrm{M}) + \mathrm{sig}_{\mathrm{L}_t}\,\psi_{\mathrm{M}}^{\varnothing}\underline{\mathrm{J}}^{k-1}(\mathrm{M}).$$

For $\eta, \xi \in \underline{\mathrm{J}}^{k-1}(\mathrm{M}) = \underline{\mathrm{IH}}^{k-1}(\mathrm{M})$, since each y_i annihilates the image of $\psi_{\mathrm{M}}^{\varnothing}$, we have

$$\mathrm{L}_t^{d-2k}\left(\psi_{\mathrm{M}}^{\varnothing}\eta\cdot\psi_{\mathrm{M}}^{\varnothing}\xi\right) = (-tx_{\varnothing})^{d-2k}\left(\psi_{\mathrm{M}}^{\varnothing}\eta\cdot\psi_{\mathrm{M}}^{\varnothing}\xi\right),$$

and hence

$$\begin{aligned}\deg_{\mathrm{M}}\left(\mathrm{L}_t^{d-2k}\left(\psi_{\mathrm{M}}^{\varnothing}\eta\cdot\psi_{\mathrm{M}}^{\varnothing}\xi\right)\right) &= \deg_{\mathrm{M}}\left((-tx_{\varnothing})^{d-2k}\,\psi_{\mathrm{M}}^{\varnothing}\eta\cdot\psi_{\mathrm{M}}^{\varnothing}\xi\right)\\ &= t^{d-2k}\deg_{\mathrm{M}}\left(\psi_{\mathrm{M}}^{\varnothing}\left(\underline{\beta}^{d-2k}\eta\right)\cdot\psi_{\mathrm{M}}^{\varnothing}\xi\right).\end{aligned}$$

By Lemma 2.17 (1) with $F = \varnothing$ and the fact that $\alpha_{\mathrm{M}^{\varnothing}} = 0$ for degree reasons, we have

$$\deg_{\mathrm{M}}\left(\psi_{\mathrm{M}}^{\varnothing}\left(\underline{\beta}^{d-2k}\eta\right)\cdot\psi_{\mathrm{M}}^{\varnothing}\xi\right) = -\underline{\deg}_{\mathrm{M}}\left(\underline{\beta}^{d-2k+1}\eta\xi\right).$$

When t is positive, by the above two sets of equations, we have

$$\mathrm{sig}_{\mathrm{L}_t}\,\psi_{\mathrm{M}}^{\varnothing}\underline{\mathrm{J}}^{k-1}(\mathrm{M}) = -\,\mathrm{sig}_{\underline{\beta}}\,\underline{\mathrm{J}}^{k-1}(\mathrm{M}) = -\,\mathrm{sig}_{\underline{\beta}}\,\underline{\mathrm{IH}}^{k-1}(\mathrm{M}),$$

and therefore

$$\mathrm{sig}_{\mathrm{L}_t}\,\mathrm{IH}_\circ^k(\mathrm{M}) = \mathrm{sig}_{\mathrm{L}_t}\,\mathrm{IH}^k(\mathrm{M}) - \mathrm{sig}_{\underline{\beta}}\,\underline{\mathrm{IH}}^{k-1}(\mathrm{M}).$$

By $\mathsf{HL}(\mathrm{M})$ and $\mathsf{HR}^{<\frac{d}{2}}(\mathrm{M})$, the Hodge–Riemann forms on $\mathrm{IH}^k(\mathrm{M})$ and $\mathrm{IH}^{k-1}(\mathrm{M})$ associated to L_0 are non-degenerate. Thus, for t sufficiently small, we have

$$\begin{aligned}\mathrm{sig}_{\mathrm{L}_t}\,\mathrm{IH}^k(\mathrm{M}) - \mathrm{sig}_{\mathrm{L}_t}\,\mathrm{IH}^{k-1}(\mathrm{M}) &= \mathrm{sig}_{\mathrm{L}_0}\,\mathrm{IH}^k(\mathrm{M}) - \mathrm{sig}_{\mathrm{L}_0}\,\mathrm{IH}^{k-1}(\mathrm{M})\\ &= (-1)^k\left(\dim \mathrm{IH}^k(\mathrm{M}) - \dim \mathrm{IH}^{k-1}(\mathrm{M})\right).\end{aligned}$$

We also have

$$\mathrm{sig}_{\underline{\beta}}\,\underline{\mathrm{IH}}^{k-1}(\mathrm{M}) - \mathrm{sig}_{\underline{\beta}}\,\underline{\mathrm{IH}}^{k-2}(\mathrm{M}) = (-1)^{-1}\left(\dim \underline{\mathrm{IH}}^{k-1}(\mathrm{M}) - \dim \underline{\mathrm{IH}}^{k-2}(\mathrm{M})\right)$$

by $\underline{\mathsf{HL}}^{<\frac{d-2}{2}}(\mathrm{M})$ and $\underline{\mathsf{HR}}^{<\frac{d-2}{2}}(\mathrm{M})$. Combining the above three sets of equations, we have

$$\begin{aligned}&\operatorname{sig}_{\mathrm{L}_t} \mathrm{IH}^k_\circ(\mathrm{M}) - \operatorname{sig}_{\mathrm{L}_t} \mathrm{IH}^{k-1}_\circ(\mathrm{M}) \\ &= (-1)^k \left(\dim \mathrm{IH}^k(\mathrm{M}) - \dim \mathrm{IH}^{k-1}(\mathrm{M})\right) - (-1)^{k-1} \left(\dim \underline{\mathrm{IH}}^{k-1}(\mathrm{M}) - \dim \underline{\mathrm{IH}}^{k-2}(\mathrm{M})\right) \\ &= (-1)^k \left(\dim \mathrm{IH}^k_\circ(\mathrm{M}) - \dim \mathrm{IH}^{k-1}_\circ(\mathrm{M})\right).\end{aligned}$$

□

12. Proof of the main theorem

Sections 12.1 and 12.2 are devoted to combining the results that we have obtained in the previous sections in order to complete the proof of Theorem 3.17. In Section 12.3 we prove Propositions 1.6 and 1.7, thus concluding the proof of Theorem 1.2.

12.1. Proof of Theorem 3.17 for non-Boolean matroids.

We now complete the inductive proof of Theorem 3.17 when M is not the Boolean matroid; the Boolean case will be addressed in Section 12.2. Let M be a matroid that is not Boolean, and assume that Theorem 3.17 holds for any matroid whose ground set is a proper subset of E. Since M is not Boolean, we may fix an element $i \in E$ which is not a coloop. If i has a parallel element, then all of our statements about M are equivalent to the corresponding statements about $\mathrm{M}\backslash i$, so we may assume that it does not.

We recall the main results in the previous five sections. By Corollary 7.4, we have $\mathsf{PD}_\circ(\mathrm{M})$, $\underline{\mathsf{PD}}(\mathrm{M})$, $\mathsf{CD}_\circ(\mathrm{M})$, and $\underline{\mathsf{CD}}(\mathrm{M})$. By Corollary 8.17, we have

$$\underline{\mathsf{NS}}^{<\frac{d-2}{2}}(\mathrm{M}) \text{ holds.}$$

By Corollaries 10.12 and 10.16, we have

$$\text{both } \underline{\mathsf{HL}}_i(\mathrm{M}) \text{ and } \underline{\mathsf{HR}}_i(\mathrm{M}) \text{ hold,}$$

and by Corollaries 9.7 and 9.9, we have

$$\mathsf{CD}^{<\frac{d}{2}}(\mathrm{M}) \implies \mathsf{HL}_i^{<\frac{d}{2}}(\mathrm{M}) \text{ and } \mathsf{HR}_i^{<\frac{d}{2}}(\mathrm{M}).$$

Proposition 12.1. We have

$$\underline{\mathsf{NS}}^{<\frac{d-2}{2}}(\mathrm{M}) \text{ and } \underline{\mathsf{HR}}_i(\mathrm{M}) \implies \underline{\mathsf{HL}}^{<\frac{d-2}{2}}(\mathrm{M}).$$

Proof. Given $1 \leqslant k < d/2$, let $\eta \in \underline{\mathrm{IH}}^{k-1}(\mathrm{M})$ be a nonzero class such that

$$\underline{\beta}^{d-2k+1}\eta = 0.$$

Recall from the proof of Lemma 11.2 that $\underline{\beta} x_{\{i\}} = 0$, and therefore

$$(\underline{\beta} - x_{\{i\}})^{d-2k} \cdot (\underline{\beta}\eta) = 0.$$

In other words, the class $\underline{\beta}\eta$ is primitive in $\underline{\mathrm{IH}}_i^k(\mathrm{M})$ with respect to multiplication by $\underline{\beta} - x_{\{i\}}$. By $\underline{\mathsf{NS}}^{<\frac{d-2}{2}}(\mathrm{M})$, we have $\underline{\beta}\eta \neq 0$. Now, $\underline{\mathsf{HR}}_i(\mathrm{M})$ implies that

$$0 < (-1)^k \underline{\deg}_{\mathrm{M}}\Big((\underline{\beta} - x_{\{i\}})^{d-2k-1} \cdot (\underline{\beta}\eta)^2\Big) = (-1)^k \underline{\deg}_{\mathrm{M}}\big(\underline{\beta}^{d-2k+1} \cdot \eta^2\big).$$

This contradicts the assumption that $\underline{\beta}^{d-2k+1}\eta = 0$. □

Proposition 12.2. For any $k \leqslant d/2$, we have

$$\underline{\mathsf{PD}}^{\leqslant k-1}(\mathrm{M}) \ \text{ and } \ \underline{\mathsf{HL}}^{\leqslant k-1}(\mathrm{M}) \implies \mathsf{CD}^{\leqslant k}(\mathrm{M}).$$

Proof. By $\mathsf{CD}_\circ(\mathrm{M})$, the statement $\mathsf{CD}^{\leqslant k}(\mathrm{M})$ is equivalent to the direct sum decomposition

$$\mathrm{IH}_\circ^{\leqslant k}(\mathrm{M}) = \mathrm{IH}^{\leqslant k}(\mathrm{M}) \oplus \psi_{\mathrm{M}}^{\varnothing}\underline{\mathrm{J}}^{\leqslant k-1}(\mathrm{M}).$$

By definition, $\mathrm{IH}(\mathrm{M})$ is the orthogonal complement of $\psi_{\mathrm{M}}^{\varnothing}\underline{\mathrm{J}}(\mathrm{M})$ in $\mathrm{IH}_\circ(\mathrm{M})$. Thus, the above direct sum decomposition is equivalent to the statement that the Poincaré pairing of $\mathrm{CH}(\mathrm{M})$ restricts to a non-degenerate pairing between $\psi_{\mathrm{M}}^{\varnothing}\underline{\mathrm{J}}^{\leqslant k-1}(\mathrm{M})$ and $\psi_{\mathrm{M}}^{\varnothing}\underline{\mathrm{J}}^{\geqslant d-k-1}(\mathrm{M})$.

By Lemma 2.17 (1) with $F = \varnothing$ and the fact that $\alpha_{\mathrm{M}^{\varnothing}} = 0$ for degree reasons, we have

$$\deg_{\mathrm{M}}\left(\psi_{\mathrm{M}}^{\varnothing}\mu \cdot \psi_{\mathrm{M}}^{\varnothing}\nu\right) = -\underline{\deg}_{\mathrm{M}}\big(\underline{\beta}_{\mathrm{M}} \cdot \mu\nu\big)$$

for $\mu, \nu \in \underline{\mathrm{CH}}(\mathrm{M})$. Thus, by $\underline{\mathsf{PD}}^{\leqslant k-1}(\mathrm{M})$ and $\underline{\mathsf{HL}}^{\leqslant k-1}(\mathrm{M})$, the Poincaré pairing of $\mathrm{CH}(\mathrm{M})$ restricts to a non-degenerate pairing between $\psi_{\mathrm{M}}^{\varnothing}\underline{\mathrm{J}}^{\leqslant k-1}(\mathrm{M})$ and $\psi_{\mathrm{M}}^{\varnothing}\underline{\mathrm{J}}^{\geqslant d-k-1}(\mathrm{M})$. □

By Proposition 7.8, we have

$$\mathsf{CD}_\circ(\mathrm{M}) \ \text{ and } \ \mathsf{CD}^{<\frac{d}{2}}(\mathrm{M}) \implies \mathsf{NS}^{<\frac{d}{2}}(\mathrm{M}).$$

Proposition 12.3. We have

$$\underline{\mathsf{HL}}^{<\frac{d-2}{2}}(\mathrm{M}) \ \text{ and } \ \mathsf{NS}^{<\frac{d}{2}}(\mathrm{M}) \implies \mathsf{HL}(\mathrm{M}).$$

Proof. Given positive numbers c_j for $j \in E$, we let $y = \sum_{j \in E} c_j y_j$. Suppose $\eta \in \mathrm{IH}^k(\mathrm{M})$ satisfies $y^{d-2k}\eta = 0$. For any rank one flat G, we have $\varphi_G^{\mathrm{M}}(y) = \sum_{j \notin G} c_j y_j \in \mathrm{CH}^1(\mathrm{M}_G)$. Since $y^{d-2k}\eta = 0$, we have

$$\varphi_G^{\mathrm{M}}(y)^{d-2k} \cdot \varphi_G^{\mathrm{M}}(\eta) = 0.$$

By Lemma 3.6 (1), we know that $\varphi_G^{\mathrm{M}}(\eta) \in \mathrm{IH}^k(\mathrm{M}_G)$. Thus, the class $\varphi_G^{\mathrm{M}}(\eta) \in \mathrm{IH}^k(\mathrm{M}_G)$ is primitive with respect to $\varphi_G^{\mathrm{M}}(y)$. By $\mathsf{HR}(\mathrm{M}_G)$, Proposition 2.13, and Proposition 2.15, for every rank one flat G we have

$$0 \leqslant (-1)^k \deg_{\mathrm{M}_G}\left(\varphi_G^{\mathrm{M}}(y)^{d-2k-1} \cdot \varphi_G^{\mathrm{M}}(\eta)^2\right) = (-1)^k \deg_{\mathrm{M}}\left(y_G \cdot y^{d-2k-1}\eta^2\right),$$

and the equality holds if and only if $\varphi_G^{\mathrm{M}}(\eta) = 0$.

On the other hand, since $y^{d-2k}\eta = 0$, we have

$$\begin{aligned} 0 &= (-1)^k \deg_{\mathrm{M}}\left(y^{d-2k}\eta^2\right) \\ &= (-1)^k \deg_{\mathrm{M}}\left(\left(\sum_{j\in E} c_j y_j\right)\cdot y^{d-2k-1}\eta^2\right) \\ &= (-1)^k \sum_{j\in E} c_j \deg_{\mathrm{M}}\left(y_{\overline{\{j\}}}\cdot y^{d-2k-1}\eta^2\right), \end{aligned}$$

where $\overline{\{j\}}$ denotes the closure of $\{j\}$ in M. Since each $c_j > 0$, the above two sets of equations imply that $\varphi^{\mathrm{M}}_G(\eta) = 0$ for every rank one flat G. Thus,

$$y_G\eta = \psi_G\big(\varphi^{\mathrm{M}}_G(\eta)\big) = \psi_G(0) = 0$$

for every rank one flat G. By $\mathsf{NS}^{<\frac{d}{2}}(\mathrm{M})$, it follows that $\eta = 0$.

We have proved that multiplication by y^{d-2k} is an injective map from $\mathrm{IH}^k(\mathrm{M})$ to $\mathrm{IH}^{d-k}(\mathrm{M})$. To conclude it is an isomorphism, it is enough to know that these spaces have the same dimension. We know that $\mathsf{PD}_\circ(\mathrm{M})$ holds, and since $\mathrm{IH}(\mathrm{M})$ is the perpendicular space to $\psi^{\varnothing}_{\mathrm{M}}(\underline{\mathrm{J}}(\mathrm{M}))$ in $\mathrm{IH}_\circ(\mathrm{M})$, it is enough to know that $\dim \underline{\mathrm{J}}^{k-1}(\mathrm{M}) = \dim \underline{\mathrm{J}}^{d-k-1}(\mathrm{M})$. This follows from $\underline{\mathsf{HL}}^{<\frac{d-2}{2}}(\mathrm{M})$. □

Proposition 12.4. We have

$$\mathsf{HR}_\circ(\mathrm{M}) \implies \mathsf{NS}_\circ(\mathrm{M}).$$

Proof. Let $y = \sum_{j\in E} y_j$. By $\mathsf{HR}_\circ(\mathrm{M})$, we can choose $\epsilon > 0$ such that $\mathrm{IH}_\circ(\mathrm{M})$ satisfies the Hodge–Riemann relations with respect to multiplication by $y - \epsilon x_\varnothing$. Suppose that η is a nonzero element of the socle of $\mathrm{IH}^k_\circ(\mathrm{M})$ for some $k \leqslant d/2$. By $\mathsf{HR}_\circ(\mathrm{M})$, we have

$$(-1)^k \deg_{\mathrm{M}}\left((y-\epsilon x_\varnothing)^{d-2k}\eta^2\right) > 0. \tag{13}$$

Since η is annihilated by every y_j, Lemma 5.2 implies that η is a multiple of $x_\varnothing$. On the other hand, since η is annihilated by $x_\varnothing$, Lemma 5.2 implies that η is in the ideal spanned by the y_j. Thus we have $\eta^2 = 0$, which contradicts Equation (13). □

Proposition 12.5. We have

$$\mathsf{NS}_\circ(\mathrm{M}) \implies \underline{\mathsf{NS}}(\mathrm{M}).$$

Proof. Suppose that $k \leqslant d/2$ and $\eta \in \underline{\mathrm{IH}}^{k-1}(\mathrm{M})$ is an element of the socle, that is, $\underline{\beta}\eta = 0$. By Corollary 7.7, it follows that $\psi^{\varnothing}_{\mathrm{M}}(\eta)$ is a multiple of $x_\varnothing$, and hence annihilated by each y_j by Lemma 5.2. Furthermore, by Proposition 2.5, we have

$$x_\varnothing\psi^{\varnothing}_{\mathrm{M}}(\eta) = \psi^{\varnothing}_{\mathrm{M}}\left(\varphi^{\varnothing}_{\mathrm{M}}(x_\varnothing)\eta\right) = \psi^{\varnothing}_{\mathrm{M}}(-\underline{\beta}\eta) = 0.$$

Thus, $\psi^{\varnothing}_{\mathrm{M}}(\eta) \in \mathrm{IH}^k_\circ(\mathrm{M})$ is annihilated by each y_j and $x_\varnothing$. Then $\mathsf{NS}_\circ(\mathrm{M})$ implies that $\psi^{\varnothing}_{\mathrm{M}}(\eta) = 0$, and the injectivity of $\psi^{\varnothing}_{\mathrm{M}}$ implies that $\eta = 0$. □

Proposition 12.6. We have

$$\underline{\mathsf{HL}}^{<\frac{d-2}{2}}(\mathrm{M}) \text{ and } \underline{\mathsf{NS}}(\mathrm{M}) \implies \underline{\mathsf{HL}}(\mathrm{M}).$$

Proof. When d is odd, the statements $\underline{\mathsf{HL}}^{<\frac{d-2}{2}}(\mathrm{M})$ and $\underline{\mathsf{HL}}(\mathrm{M})$ are the same. When d is even, the only missing case is $\underline{\mathsf{HL}}^{\frac{d-2}{2}}(\mathrm{M})$, which is exactly the same as $\underline{\mathsf{NS}}^{\frac{d-2}{2}}(\mathrm{M})$. $\square$

From Corollary 7.4, we have $\mathsf{PD}_\circ$, $\underline{\mathsf{PD}}$, $\mathsf{CD}_\circ$, and $\underline{\mathsf{CD}}$ of M. Following Figure 1, we have obtained CD, NS, $\mathsf{NS}_\circ$, HL, $\mathsf{HL}_\circ$, $\underline{\mathsf{HL}}$, HR, $\mathsf{HR}_\circ$, and $\underline{\mathsf{HR}}$. The statement PD follows from HL and HR. The statement $\underline{\mathsf{NS}}$ is proved in Proposition 12.5. So we have completed the proof of Theorem 3.17 assuming M is not the Boolean matroid.

12.2. **Proof of Theorem 3.17: Boolean case.** Suppose M is the Boolean matroid on $E = \{1, 2, \ldots, d\}$ with $d > 0$.

Proposition 12.7. The canonical decomposition $\underline{\mathsf{CD}}(\mathrm{M})$ of $\underline{\mathrm{CH}}(\mathrm{M})$ holds. We have $\underline{\mathrm{IH}}(\mathrm{M}) = \underline{\mathrm{H}}(\mathrm{M})$, and the space $\underline{\mathrm{J}}(\mathrm{M})$ is spanned by $1, \underline{\beta}, \ldots, \underline{\beta}^{d-2}$.

Proof. Let $\underline{\mathrm{J}}'(\mathrm{M})$ be the subspace of $\underline{\mathrm{H}}(\mathrm{M})$ spanned by $1, \underline{\beta}, \ldots, \underline{\beta}^{d-2}$. We have $\underline{\mathrm{H}}(\mathrm{M}) \subseteq \underline{\mathrm{IH}}(\mathrm{M})$, since $\underline{\mathrm{IH}}(\mathrm{M})$ is an $\mathrm{H}(\mathrm{M})$-module that contains 1. Since $\underline{\beta}^{d-2}$ is not zero, we have $\underline{\mathrm{J}}'(\mathrm{M}) \subseteq \underline{\mathrm{J}}(\mathrm{M})$.

Thus if we can show there is a direct sum decomposition

$$\underline{\mathrm{CH}}(\mathrm{M}) = \underline{\mathrm{H}}(\mathrm{M}) \oplus \bigoplus_{\varnothing < F < E} \underline{\psi}^F_{\mathrm{M}} \left(\underline{\mathrm{J}}'(\mathrm{M}_F) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F) \right), \tag{14}$$

the proposition will follow.

For a Boolean matroid M, $\underline{\mathrm{CH}}(\mathrm{M})$ admits an automorphism

$$\tau \colon \underline{\mathrm{CH}}(\mathrm{M}) \to \underline{\mathrm{CH}}(\mathrm{M}), \quad x_F \mapsto x_{E \setminus F}.$$

The automorphism τ exchanges $\underline{\alpha}$ and $\underline{\beta}$. It is then easy to see that the decomposition (14) is the result of applying τ to the decomposition ($\underline{\mathrm{D}}_3$) of [BHM+20].

Alternatively, one can use the basis of $\underline{\mathrm{CH}}(\mathrm{M})$ given by Feichtner and Yuzvinsky [FY04, Corollary 1]. Their basis is given by all products

$$x_{G_1}^{m_1} x_{G_2}^{m_2} \cdots x_{G_k}^{m_k} \underline{\alpha}^{m_{k+1}},$$

where $G_1 < G_2 < \cdots < G_k$ is a (possibly empty) flag of nonempty proper flats and we have $m_1 < \operatorname{rk} G_1$, $m_i < \operatorname{rk} G_i - \operatorname{rk} G_{i-1}$ for $1 < i \leqslant k$, and $m_{k+1} < \operatorname{crk} G_k$. Applying τ gives

$$\underline{\beta}^{m_{k+1}} (x_{F_k})^{m_k} \cdots (x_{F_1})^{m_1},$$

where $F_i = E \setminus G_i$. If $k \neq 0$ this is in $\underline{\psi}^{F_k}_{\mathrm{M}} \left((\underline{\beta}_{\mathrm{M}_{F_k}})^{m_k} \otimes \underline{\mathrm{CH}}(\mathrm{M}^{F_k}) \right)$, while if $k = 0$ it is in $\underline{\mathrm{H}}(\mathrm{M})$. The direct sum decomposition (14) follows. $\square$

Since $\underline{\mathrm{IH}}(\mathrm{M})$ is isomorphic to $\underline{\mathrm{H}}(\mathrm{M})$, which is spanned by $1, \underline{\beta}, \underline{\beta}^2, \ldots, \underline{\beta}^{d-1}$, we immediately deduce $\underline{\mathsf{NS}}(\mathrm{M})$ and $\underline{\mathsf{HL}}(\mathrm{M})$. Notice that the involution τ induces the identity map on $\underline{\mathrm{CH}}^{d-1}(\mathrm{M})$. Therefore, $\underline{\deg}_{\mathrm{M}}(\underline{\beta}^{d-1}) = \underline{\deg}_{\mathrm{M}}(\underline{\alpha}^{d-1}) = 1$, and we have $\underline{\mathsf{PD}}(\mathrm{M})$ and $\underline{\mathsf{HR}}(\mathrm{M})$.

The proof of Proposition 12.2 also works for the Boolean matroid, so from $\underline{\mathsf{HL}}(\mathrm{M})$ and $\underline{\mathsf{HR}}(\mathrm{M})$ we get $\mathsf{CD}(\mathrm{M})$. By Lemma 5.2 and Corollary 7.6, we have an isomorphism of graded vector spaces

$$\mathrm{IH}_\circ(\mathrm{M})_\varnothing \cong \varphi^\varnothing_{\mathrm{M}}(\mathrm{IH}_\circ(\mathrm{M})) = \underline{\mathrm{IH}}(\mathrm{M}).$$

Since $\psi^\varnothing_{\mathrm{M}}(\underline{\beta}^i) = x_\varnothing^{i+1}$, it follows that $\psi^\varnothing_{\mathrm{M}}\underline{\mathrm{J}}(\mathrm{M})$ is spanned by $x_\varnothing, \ldots, x_\varnothing^{d-1}$. Since $x_\varnothing y_j = 0$ for any $j \in E$, we have an isomorphism of vector spaces

$$\left(\psi^\varnothing_{\mathrm{M}}\underline{\mathrm{J}}(\mathrm{M})\right)_\varnothing \cong \psi^\varnothing_{\mathrm{M}}\underline{\mathrm{J}}(\mathrm{M}).$$

Since $\underline{\mathrm{IH}}(\mathrm{M})$ has total dimension d and $\underline{\mathrm{J}}(\mathrm{M})$ has total dimension $d-1$, the stalk $\mathrm{IH}(\mathrm{M})_\varnothing$ is one-dimensional, and hence $\mathrm{IH}(\mathrm{M})_\varnothing \cong \mathrm{IH}^0(\mathrm{M}) \cong \mathbb{Q}$. Therefore, $\mathrm{IH}(\mathrm{M})$ is generated in degree zero as a module over $\mathrm{H}(\mathrm{M})$. Equivalently, $\mathrm{IH}(\mathrm{M})$ is isomorphic to a quotient of $\mathrm{H}(\mathrm{M})$.

On the other hand, since M is the Boolean matroid, $\mathrm{H}(\mathrm{M}) = \mathbb{Q}[y_1, \ldots, y_d]/(y_1^2, \ldots, y_d^2)$ is a Poincaré duality algebra. Since $\mathrm{IH}^d(\mathrm{M})$ is one-dimensional, the quotient map $\mathrm{H}(\mathrm{M}) \to \mathrm{IH}(\mathrm{M})$ is an isomorphism in degree d. Therefore, the quotient map must be an isomorphism, that is,

$$\mathrm{IH}(\mathrm{M}) \cong \mathrm{H}(\mathrm{M}) = \mathbb{Q}[y_1, \ldots, y_d]/(y_1^2, \ldots, y_d^2).$$

Now, it is a well-known fact that $\mathrm{H}(\mathrm{M})$ satisfies Poincaré duality, the hard Lefschetz theorem, and the Hodge–Riemann relations. The statement $\mathsf{PD}_\circ(\mathrm{M})$ follows from $\mathsf{PD}(\mathrm{M})$, $\underline{\mathsf{PD}}(\mathrm{M})$, and $\underline{\mathsf{HL}}(\mathrm{M})$. By Lemma 11.2, the statement $\mathsf{HL}_\circ(\mathrm{M})$ follows from $\mathsf{HL}(\mathrm{M})$ and $\underline{\mathsf{HL}}(\mathrm{M})$, and the statement $\mathsf{HR}_\circ(\mathrm{M})$ follows from $\mathsf{HR}(\mathrm{M})$ and $\underline{\mathsf{HR}}(\mathrm{M})$.

12.3. **Proofs of Propositions 1.6 and 1.7.** Recall from Section 1.2 that the proof of Theorem 1.2 relies on Theorem 1.5, which we have already proved as part of Theorem 3.17, as well as on Propositions 1.6 and 1.7. In this subsection, we will prove these remaining two propositions.

Proof of Proposition 1.6. As parts of Theorem 3.17, we have already obtained $\mathsf{PD}(\mathrm{M})$ and $\mathsf{NS}(\mathrm{M})$. By $\mathsf{PD}(\mathrm{M})$, the socle of $\mathrm{IH}(\mathrm{M})$ is equal to the orthogonal complement $(\mathfrak{m}\,\mathrm{IH}(\mathrm{M}))^\perp$ in $\mathrm{IH}(\mathrm{M})$. By $\mathsf{NS}(\mathrm{M})$, we know that $(\mathfrak{m}\,\mathrm{IH}(\mathrm{M}))^\perp = 0$ in degrees less than or equal to $d/2$. Thus, $\mathfrak{m}\,\mathrm{IH}(\mathrm{M}) = \mathrm{IH}(\mathrm{M})$ in degrees greater than or equal to $d/2$, or equivalently, $\mathrm{IH}(\mathrm{M})_\varnothing = 0$ in degrees greater than or equal to $d/2$. □

Proof of Proposition 1.7. Choose an ordering $F_1, \ldots, F_r$ of $\mathcal{L}(\mathrm{M})$ refining the natural partial order as in Section 5.3 with the further property that $\Sigma_\mu = \{F_\mu, \ldots, F_r\} = \mathcal{L}^{\geqslant p}(\mathrm{M})$ and $\Sigma_\nu = \{F_\nu, \ldots, F_r\} =$

$\mathcal{L}^{\geqslant p+1}(\mathrm{M})$. By definition, we have

$$\mathfrak{m}^p \operatorname{IH}(\mathrm{M})/\mathfrak{m}^{p+1} \operatorname{IH}(\mathrm{M}) \cong \frac{\operatorname{IH}(\mathrm{M})_{\Sigma_\mu}}{\operatorname{IH}(\mathrm{M})_{\Sigma_\nu}}. \tag{15}$$

We claim that there exists a canonical isomorphism

$$\bigoplus_{F \in \mathcal{L}^p(\mathrm{M})} \frac{\operatorname{IH}(\mathrm{M})_{\geqslant F}}{\operatorname{IH}(\mathrm{M})_{>F}} \cong \frac{\operatorname{IH}(\mathrm{M})_{\Sigma_\mu}}{\operatorname{IH}(\mathrm{M})_{\Sigma_\nu}}. \tag{16}$$

In fact, the natural maps

$$\frac{\operatorname{IH}(\mathrm{M})_{\geqslant F}}{\operatorname{IH}(\mathrm{M})_{>F}} \to \frac{\operatorname{IH}(\mathrm{M})_{\Sigma_\mu}}{\operatorname{IH}(\mathrm{M})_{\Sigma_\nu}}$$

induce a surjective map

$$\bigoplus_{F \in \mathcal{L}^p(\mathrm{M})} \frac{\operatorname{IH}(\mathrm{M})_{\geqslant F}}{\operatorname{IH}(\mathrm{M})_{>F}} \to \frac{\operatorname{IH}(\mathrm{M})_{\Sigma_\mu}}{\operatorname{IH}(\mathrm{M})_{\Sigma_\nu}}.$$

To show that the above map is an isomorphism, it suffices to show that both sides have the same dimension. By Proposition 5.9 (1), we have

$$\dim\left(\frac{\operatorname{IH}(\mathrm{M})_{\Sigma_\mu}}{\operatorname{IH}(\mathrm{M})_{\Sigma_\nu}}\right) = \sum_{\mu \leqslant k \leqslant \nu-1} \dim\left(\frac{\operatorname{IH}(\mathrm{M})_{\Sigma_k}}{\operatorname{IH}(\mathrm{M})_{\Sigma_{k+1}}}\right) = \sum_{\mu \leqslant k \leqslant \nu-1} \dim \operatorname{IH}(\mathrm{M})_{F_k} = \sum_{F \in \mathcal{L}^p(\mathrm{M})} \dim \operatorname{IH}(\mathrm{M})_F.$$

Thus, the isomorphism in Equation (16) follows.

By Lemma 5.7 and Lemma 6.2 (1), for any flat F, we have canonical isomorphisms

$$\frac{\operatorname{IH}(\mathrm{M})_{\geqslant F}}{\operatorname{IH}(\mathrm{M})_{>F}} \cong \big(\operatorname{IH}(\mathrm{M})[-\operatorname{rk} F]\big)_F \cong \big(y_F \operatorname{IH}(\mathrm{M})\big)_\varnothing \cong \big(\operatorname{IH}(\mathrm{M}_F)[-\operatorname{rk} F]\big)_\varnothing. \tag{17}$$

Now, the proposition follows from Equations (15), (16), and (17). □

Appendix A. Equivariant polynomials

The purpose of this appendix is to give precise definitions of equivariant Kazhdan–Lusztig polynomials, equivariant Z-polynomials, and equivariant inverse Kazhdan–Lusztig polynomials. We also prove an equivariant analogue of the characterization of Kazhdan–Lusztig polynomials and Z-polynomials that appears in [BV20, Theorem 2.2].

Let Γ be a finite group, and let $\operatorname{VRep}(\Gamma)$ be the ring of virtual representations of Γ over $\mathbb{Q}$ with coefficients in $\mathbb{Q}$. For any finite-dimensional representation V of Γ, let $[V]$ be its class in $\operatorname{VRep}(\Gamma)$. If Γ acts on a set S and $x \in S$, we write $\Gamma_x \subseteq \Gamma$ for the stabilizer of x. We use the following standard lemma [Pro20, Lemma 2.7].

Lemma A.1. Let $V = \bigoplus_{x\in S} V_x$ be a vector space that decomposes as a direct sum of pieces indexed by a finite set S, and suppose that Γ acts linearly on V and acts by permutations on S. If $\gamma{\cdot}V_x = V_{\gamma\cdot x}$ for all $x \in S$ and $\gamma \in \Gamma$, then

$$[V] = \bigoplus_{x\in S} \frac{|\Gamma_x|}{|\Gamma|} \operatorname{Ind}_{\Gamma_x}^{\Gamma}[V_x] \in \operatorname{VRep}(\Gamma).$$

Let M be a matroid on the ground set E, and let Γ be a finite group acting on M. In other words, the set E is equipped with an action of Γ by permutations that take flats of M to flats of M. We define the **equivariant characteristic polynomial**

$$\chi^{\Gamma}_{\mathrm{M}}(t) \coloneqq \sum_{k=0}^{\operatorname{rk}\mathrm{M}} (-1)^k [\mathrm{OS}^k(\mathrm{M})]\, t^{\operatorname{rk}\mathrm{M}-k} \in \operatorname{VRep}(\Gamma)[t],$$

where $\mathrm{OS}^k(\mathrm{M})$ is the degree k part of the Orlik–Solomon algebra of M. The dimension homomorphism from $\operatorname{VRep}(\Gamma)[t]$ to $\mathbb{Z}[t]$ takes the equivariant characteristic polynomial $\chi^{\Gamma}_{\mathrm{M}}(t)$ to the ordinary characteristic polynomial $\chi_{\mathrm{M}}(t)$; see [OT92, Chapter 3]. The following statement appears in [GPY17, Theorem 2.8].

Theorem A.2. To each matroid M and symmetry group Γ, there is a unique way to assign a polynomial $P^{\Gamma}_{\mathrm{M}}(t)$ with coefficients in $\operatorname{VRep}(\Gamma)$ with the following properties:

(a) If the rank of M is zero, then $P^{\Gamma}_{\mathrm{M}}(t) = 1$.

(b) For every matroid M of positive rank, the degree of $P^{\Gamma}_{\mathrm{M}}(t)$ is strictly less than $\operatorname{rk}\mathrm{M}/2$.

(c) For every matroid M, we have $t^{\operatorname{rk}\mathrm{M}} P_{\mathrm{M}}(t^{-1}) \;=\; \sum_{F\in\mathcal{L}(\mathrm{M})} \frac{|\Gamma_F|}{|\Gamma|} \operatorname{Ind}_{\Gamma_F}^{\Gamma} \Big(\chi^{\Gamma_F}_{\mathrm{M}^F}(t) P^{\Gamma_F}_{\mathrm{M}_F}(t)\Big)$.

The polynomial $P^{\Gamma}_{\mathrm{M}}(t)$ is called the **equivariant Kazhdan–Lusztig polynomial** of M with respect to the action of Γ.

The following definition appears in [PXY18, Section 6].

Definition A.3. The **equivariant Z-polynomial** of M with respect to the action of Γ is

$$Z^{\Gamma}_{\mathrm{M}}(t) \;\coloneqq \sum_{F\in\mathcal{L}(\mathrm{M})} \frac{|\Gamma_F|}{|\Gamma|} \operatorname{Ind}_{\Gamma_F}^{\Gamma} \Big(P^{\Gamma_F}_{\mathrm{M}_F}(t)\Big)\, t^{\operatorname{rk}F} \in \operatorname{VRep}(\Gamma)[t].$$

A polynomial $f(t) \in \operatorname{VRep}(\Gamma)[t]$ is called **palindromic** if $t^{\deg f(t)} f(t^{-1}) = f(t)$. The fact that the equivariant Z-polynomial is palindromic is asserted without proof in [PXY18, Section 6]; a full proof appears in [Pro20, Corollary 4.5].

Lemma A.4. For any polynomial $f(t)$ of degree d, there is a unique polynomial $g(t)$ of degree strictly less than $d/2$ such that $f(t) + g(t)$ is palindromic.

Proof. We must take $g(t)$ to be the truncation of $t^d f(t^{-1}) - f(t)$ to degree $\lfloor (d-1)/2 \rfloor$. $\square$

The following proposition is an equivariant analogue of [BV20, Theorem 2.2].

Corollary A.5. Let M be a matroid of positive rank, let $\tilde{P}_{\mathrm{M}}^{\Gamma}(t)$ be a polynomial of degree strictly less than $\operatorname{rk} \mathrm{M}/2$ in $\mathrm{VRep}(\Gamma)[t]$, and let

$$\tilde{Z}_{\mathrm{M}}^{\Gamma}(t) \coloneqq \tilde{P}_{\mathrm{M}}^{\Gamma}(t) \ + \sum_{\varnothing \neq F \in \mathcal{L}(\mathrm{M})} \frac{|\Gamma_F|}{|\Gamma|} \operatorname{Ind}_{\Gamma_F}^{\Gamma} \left(P_{\mathrm{M}_F}^{\Gamma_F}(t) \right) t^{\operatorname{rk} F}.$$

If $\tilde{Z}_{\mathrm{M}}^{\Gamma}(t)$ is a palindromic polynomial, then $\tilde{P}_{\mathrm{M}}^{\Gamma}(t) = P_{\mathrm{M}}^{\Gamma}(t)$ and $\tilde{Z}_{\mathrm{M}}^{\Gamma}(t) = Z_{\mathrm{M}}^{\Gamma}(t)$.

Proof. By definition of $Z_{\mathrm{M}}^{\Gamma}(t)$, we have

$$Z_{\mathrm{M}}^{\Gamma}(t) = P_{\mathrm{M}}^{\Gamma}(t) \ + \sum_{\varnothing \neq F \in \mathcal{L}(\mathrm{M})} \frac{|\Gamma_F|}{|\Gamma|} \operatorname{Ind}_{\Gamma_F}^{\Gamma} \left(P_{\mathrm{M}_F}^{\Gamma_F}(t) \right) t^{\operatorname{rk} F}.$$

The corollary then follows from Lemma A.4 and the palindromicity of $\tilde{Z}_{\mathrm{M}}^{\Gamma}(t)$. $\square$

When the rank of M is positive, by [GX20, Theorem 1.3], the inverse Kazhdan–Lusztig polynomial of M satisfies

$$\sum_{F \in \mathcal{L}(\mathrm{M})} (-1)^{\operatorname{rk} F} P_{\mathrm{M}^F}(t) Q_{\mathrm{M}_F}(t) = 0.$$

We use the recurrence relation to define an equivariant analogue of $Q_{\mathrm{M}}(t)$.

Definition A.6. The **equivariant inverse Kazhdan–Lusztig polynomial** of M with respect to the action of Γ is defined by the condition that $Q_{\mathrm{M}}^{\Gamma}(t)$ is equal to the trivial representation if the rank of M is zero, and otherwise

$$\sum_{F \in \mathcal{L}(\mathrm{M})} (-1)^{\operatorname{rk} F} \frac{|\Gamma_F|}{|\Gamma|} \operatorname{Ind}_{\Gamma_F}^{\Gamma} \left(P_{\mathrm{M}^F}^{\Gamma_F}(t) Q_{\mathrm{M}_F}^{\Gamma_F}(t) \right) = 0.$$

Equivalently, we recursively put

$$Q_{\mathrm{M}}^{\Gamma}(t) = - \sum_{\varnothing \neq F \in \mathcal{L}(\mathrm{M})} (-1)^{\operatorname{rk} F} \frac{|\Gamma_F|}{|\Gamma|} \operatorname{Ind}_{\Gamma_F}^{\Gamma} \left(P_{\mathrm{M}^F}^{\Gamma_F}(t) Q_{\mathrm{M}_F}^{\Gamma_F}(t) \right) \in \mathrm{VRep}(\Gamma)[t].$$

For equivalent definitions of $P_{\mathrm{M}}^{\Gamma}(t)$, $Z_{\mathrm{M}}^{\Gamma}(t)$, and $Q_{\mathrm{M}}^{\Gamma}(t)$ in the framework of equivariant incidence algebras and equivariant Kazhdan–Lusztig–Stanley theory, we refer to [Pro20, Section 4].

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