The combinatorics behind the leading Kazhdan–Lusztig coefficients of braid matroids

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Abstract. Ferroni and Larson gave a combinatorial interpretation of the braid Kazhdan–Lusztig polynomials in terms of series-parallel matroids. As a consequence, they confirmed an explicit formula for the leading Kazhdan–Lusztig coefficients of braid matroids with odd rank, as conjectured by Elias, Proudfoot, and Wakefield. Based on Ferroni and Larson's work, we further explore the combinatorics behind the leading Kazhdan–Lusztig coefficients of braid matroids. The main results of this paper include an explicit formula for the leading Kazhdan–Lusztig coefficients of braid matroids with even rank, a simple expression for the number of simple series-parallel matroids of rank $k+1$ on $2k$ elements, and explicit formulas for the leading coefficients of inverse Kazhdan–Lusztig polynomials of braid matroids. The binomial identity for the Abel polynomials plays an important role in the proofs of these formulas.

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1 Introduction

Given a matroid M of positive rank, the **Kazhdan–Lusztig polynomial** $P_M(t) \in \mathbb{N}[t]$ is a polynomial with non-negative integer coefficients of degree strictly less than half the rank of M

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[EPW16, BHM$^+$20]. Though these polynomials are characterized by a simple recursive formula, computing them in practice for explicit families of matroids is often quite difficult. Let B_n be the **braid matroid**, associated with the complete graph on n vertices. Ferroni and Larson [FL23] gave a beautiful combinatorial interpretation of the coefficients of $P_{B_n}(t)$, which we now describe.

A matroid is called **quasi series-parallel** if it is isomorphic to a direct sum of loops and matroids associated with series-parallel graphs. Equivalently, it is quasi series-parallel if it does not contain any minor isomorphic to the braid matroid B_4 or the uniform matroid of rank two on four elements [FL23, Proposition 2.6]. For a finite set E, let $\mathcal{S}(E,k)$ denote the set of simple quasi series-parallel matroids of rank k on E, and write $\mathcal{S}(n,k) := \mathcal{S}([n],k)$, where $[n] = \{1, 2, \ldots, n\}$. Ferroni and Larson's combinatorial interpretation of $P_{B_n}(t)$ is as follows.

Theorem 1.1. [FL23, Theorem 1.1] *For any positive integer n, the coefficient of t^i in the Kazhdan–Lusztig polynomial $P_{B_n}(t)$ is equal to the cardinality of the set $\mathcal{S}(n-1, n-1-i)$.*

The matroid B_n has rank $n-1$, therefore the polynomial $P_{B_{2n}}(t)$ has degree at most $n-1$, while the polynomial $P_{B_{2n-1}}(t)$ has degree at most $n-2$. Using Theorem 1.1, Ferroni and Larson gave the following explicit formula for the leading coefficient of $P_{B_{2n}}(t)$, which was originally conjectured in [EPW16, Appendix].

Theorem 1.2. [FL23, Corollary 2.12] *For any $n > 1$, the coefficient of t^{n-1} in $P_{B_{2n}}(t)$ is equal to*

$$|\mathcal{S}(2n-1,n)| = (2n-1)^{n-2} \cdot (2n-3)!!.$$

Our first main result is the following explicit formula for the leading coefficient of $P_{B_{2n-1}}(t)$, which we regard as a complement to Theorem 1.2.

Theorem 1.3. *For any $n > 1$, the coefficient of t^{n-2} in $P_{B_{2n-1}}(t)$ is equal to*

$$|\mathcal{S}(2n-2,n)| = (2n-1)^{n-2} \cdot (2n-3)!! - \frac{(n-2)(n-1)^{n-5} \cdot (2n-1)!}{3 \cdot (n-2)!}.$$

Remark 1.4. By looking at the formulas in Theorems 1.2 and 1.3, it is apparent that the key formula is for the difference $|\mathcal{S}(2n-1,n)| - |\mathcal{S}(2n-2,n)|$, or equivalently the leading coefficient of the polynomial $P_{B_{2n}}(t) - tP_{B_{2n-1}}(t)$. It is not *a priori* obvious that this difference should be meaningful or interesting.

Let E_n denote the number of simple series-parallel matroids of rank n on the set $[2n-2]$. Ferroni and Larson [FL23] noted that the E_n can have large prime factors, and did not give a simple expression for E_n. However, they obtained the following expression of $|\mathcal{S}(2n-2,n)|$ in terms of E_n.

Theorem 1.5. [FL23, Proposition 2.13] *For any $n > 1$, the coefficient of t^{n-2} in $P_{B_{2n-1}}(t)$ is equal to*

$$|\mathcal{S}(2n-2,n)| = E_n + \frac{1}{2}\sum_{j=0}^{n-2} \binom{2n-2}{2j+1} \cdot (2j-1)!!(2j+1)^{j-1} \cdot (2n-2j-5)!!(2n-2j-3)^{n-j-3}.$$

Based on Theorems 1.3 and 1.5, we obtain the following explicit formula for E_n.

Corollary 1.6. *For any $n > 1$, we have*

$$E_n = (2n-1)^{n-2} \cdot (2n-3)!! - \frac{(n-2)(n-1)^{n-5} \cdot (2n-1)!}{3 \cdot (n-2)!} - \frac{(n+1)(n-1)^{n-3} \cdot (2n-3)!}{3 \cdot (n-1)!}.$$

It is known that the **inverse Kazhdan–Lusztig polynomial** $Q_M(t) \in \mathbb{N}[t]$ of matroid M is a polynomial with non-negative integer coefficients of degree strictly less than half the rank of M [GX21, BHM+20]. While we have no concrete combinatorial interpretation of these coefficients along the lines of Theorem 1.1,we obtain the following explicit formulas for the leading coefficients.

Theorem 1.7. *For any $n > 1$, the coefficient of t^{n-1} in $Q_{\mathrm{B}_{2n}}(t)$ is equal to*

$$(2n-1)^{n-2} \cdot (2n-3)!!,$$

and the coefficient of t^{n-2} in $Q_{\mathrm{B}_{2n-1}}(t)$ is equal to

$$\frac{(n-1)^{n-5} \cdot (2n-1)!}{3 \cdot (n-2)!}.$$

Remark 1.8. Comparing Theorem 1.7 with Theorems 1.2 and 1.3, we see that the leading coefficient of $Q_{\mathrm{B}_{2n}}(t)$ coincides with that of $P_{B_{2n}}(t)$, and the difference $|\mathcal{S}(2n-1,n)| - |\mathcal{S}(2n-2,n)|$ is the leading coefficient of $(n-2)Q_{\mathrm{B}_{2n-1}}(t)$.

By Theorem 1.3, Corollary 1.6, and Theorem 1.7, we see that understanding the difference $|\mathcal{S}(2n-1,n)| - |\mathcal{S}(2n-2,n)|$ is very important for us to determine the leading coefficients of $P_{B_n}(t)$ and $Q_{B_n}(t)$. We prove Theorem 1.3 by constructing a surjective map

$$\Phi(n) : \mathcal{S}(2n-1,n) \to \mathcal{S}(2n-2,n)$$

and studying the fibers. Many of the fibers turn out to be singletons, and therefore contribute nothing to the difference $|\mathcal{S}(2n-1,n)| - |\mathcal{S}(2n-2,n)|$. The remaining fibers can be understood via the enumeration of Husimi graphs. The construction of the map and analysis of the fibers take place in Section 2, while the discussion of Husimi graphs takes place in Section 3. The proof of Corollary 1.6 is also presented in Section 3, which will involve the binomial identity for the Abel polynomials. Section 4 will be devoted to the proof of Theorem 1.7.

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2 A surjective map from $\mathcal{S}(2n-1, n)$ to $\mathcal{S}(2n-2, n)$

The aim of this section is to express the difference between $|\mathcal{S}(2n-1, n)|$ and $|\mathcal{S}(2n-2, n)|$ in terms of the count of certain combinatorial objects. To this end, we will introduce some definitions and lemmas.

Lemma 2.1. *If $n > 1$, $M \in \mathcal{S}(2n-1, n)$, and $M' = M \setminus (2n-1)$, then $M' \in \mathcal{S}(2n-2, n)$.*

Proof. The class of simple quasi series-parallel matroids is minor closed by [FL23, Proposition 2.6(iii)], so we just need to show that the rank cannot decrease. In other words, we need to show that $2n-1$ is not a coloop of M. This follows from the fact that M is connected, which is a consequence of [FL23, Proposition 2.10]. $\square$

For any $n \geq 2$, Lemma 2.1 enables us to define a map

$$\Phi(n) : \mathcal{S}(2n-1, n) \to \mathcal{S}(2n-2, n)$$

by sending $M \in \mathcal{S}(2n-1, n)$ to $M \setminus (2n-1)$. Our next goal is to show that $\Phi(n)$ is surjective. To achieve this, it is convenient to use a connection between simple quasi series-parallel matroids and triangular cacti. Recall that a **triangular cactus** is a connected graph with the property that every edge belongs to a unique cycle and all cycles have length 3. Let E be a finite set of cardinality $2n-1$, and let $\Delta(E)$ denote the set of triangular cacti on E. In [FL23, Proposition 2.11], the authors constructed a bijection $\Psi(E)$ from $\mathcal{S}(E, n)$ to $\Delta(E)$. We will write $\Delta(n) := \Delta([2n-1])$ and $\Psi(n) := \Psi([2n-1])$, and denote the bijection by

$$\Psi(n) : \mathcal{S}(2n-1, n) \to \Delta(n).$$

Lemma 2.2. *For any $n \geq 2$, the deletion map $\Phi(n)$ is surjective.*

Proof. We will proceed by induction on n. The cases where $n = 2$ or $n = 3$ are trivial, thus we may assume that $n > 3$. Let $M' \in \mathcal{S}(2n-2, n)$ be given. By [FL23, Proposition 2.10], either $M' \cong A \oplus B$ for some $A \in \mathcal{S}(2k-1, k)$ and $B \in \mathcal{S}(2(n-k)-1, n-k)$, or M' is connected.

Suppose first that M' is not connected. That is, we have a subset $S \subset [2n-2]$ of cardinality $2k-1$, a simple quasi series-parallel matroid A of rank k on S, and a simple quasi series-parallel matroid B of rank $n-k$ on $[2n-2] \setminus S$, with the property that $M' = A \oplus B$. Choose elements $e \in S$ and $f \in [2n-2] \setminus S$, and consider the triangular cactus G on $[2n-1]$ obtained by taking the union of the cactus $\Delta(S)(A)$, the cactus $\Delta([2n-2] \setminus S)(B)$, and the triangle $\{e, f, 2n-1\}$. If we take $M \in \mathcal{S}(2n-1, n)$ to be the unique element with $\Psi(n)(M) = G$, then $\Phi(n)(M) = M \setminus (2n-1) = M'$.

Now suppose that M' is connected. We will break the argument into two cases, depending on whether or not M' is a series extension of a simple series-parallel matroid. First assume that it is, i.e. that there exists a cocircuit $\{e, f\} \subset [2n-2]$ of M' such that $M'' := M'/e$ is a simple series-parallel matroid. Let $E := [2n-2] \setminus \{e\}$. Note that M'' is a simple series-parallel matroid of rank $n-1$ on E. Consider the triangular cactus $G'' := \Psi(E)(M'')$. Let G be the triangular cactus on $[2n-1]$ obtained by taking the union of G'' and the triangle $\{e, f, 2n-1\}$ and then interchanging f and $2n-1$. If we take $M \in \mathcal{S}(2n-1, n)$ to be the unique matroid with $\Psi(n)(M) = G$, then we have $\Phi(n)(M) = M \setminus (2n-1) = M'$.

Finally, suppose that M' is connected but is not a series extension of a simple series-parallel matroid. In this case, there must exist a circuit $\{e, f, g\} \subset [2n-2]$ of M' such that $M'' := M' \setminus \{e, f\}$ is a simple series-parallel matroid of rank $n-1$ on $[2n-2] \setminus \{e, f\}$ and M' can be obtained from M'' by first adding a new element e that is parallel to g (a parallel extension) and then replacing e with a cocircuit $\{e, f\}$ (a series extension). We will refer to M' as a **triangle extension** of M'' at the element g. By our inductive hypothesis, there exists a simple series-parallel matroid M''' of rank $n-1$ on the set $[2n-1] \setminus \{e, f\}$ such that M'' is obtained by deleting the element $2n-1$ from $\tilde{M}$. Let M be the matroid on $[2n-1]$ obtained from M''' as a triangle extension at g. Then $\Phi(n)(M) = M \setminus (2n-1) = M'$. □

We proceed to study the fibers of the surjection $\Phi(n)$. This will be done by a careful analysis of the circuits of elements in $\mathcal{S}(2n-1, n)$. Let M be a matroid on the ground set E, and suppose that $C \subset E$ is a circuit. We say that an element $e \in E \setminus C$ is a **chord** for C if there exists a subset $S \subset C$ such that $S \cup \{e\}$ and $(C \setminus S) \cup \{e\}$ are both circuits. If C does not have any chords, we will say that it is **chordless**. We note that every 3-circuit in a simple matroid is chordless, and a simple matroid is determined by its chordless circuits. A simple matroid that has no chordless k-circuits for any $k \geq 4$ is called **chordal**. Note that every element of $\mathcal{S}(2n-1, n)$ is chordal.

Let m be a natural number. Let $\mathcal{S}_m(2n-1, n) \subset \mathcal{S}(2n-1, n)$ be the set of matroids in which the element $2n-1$ is contained in exactly m 3-circuits, and let

$$\mathcal{S}_m(2n-2, n) := \Phi(n)\big(\mathcal{S}_m(2n-1, n)\big).$$

If $M \in \mathcal{S}_m(2n-1, n)$, then $\Phi(n)(M)$ has exactly $\binom{m}{2}$ chordless 4-circuits, coming from pairs of 3-circuits in M that contain $2n-1$. This fact, along with Lemma 2.2, implies that $\mathcal{S}(2n-2, n)$ is equal to the disjoint union of the sets $\mathcal{S}_m(2n-2, n)$. We will write

$$\Phi_m(n) : \mathcal{S}_m(2n-1, n) \to \mathcal{S}(2n-2, n)$$

to denote the restriction of $\Phi(n)$ to $\mathcal{S}_m(2n-1, n)$.

Lemma 2.3. *If $M' \in \mathcal{S}_m(2n-2, n)$, then M' is connected if and only if $m > 1$.*

Proof. Choose $M \in \mathcal{S}_m(2n-1, n)$ such that $M' = \Phi_m(n)(M)$. We know from [FL23, Proposition 2.10] that M is connected, and we also know that all of the chordless circuits of M are 3-circuits. If $m > 1$, then $2n-1$ is contained in multiple 3-circuits of M, so deleting it does not disconnect the matroid. On the other hand, if $m = 1$, then $2n-1$ is contained in a unique 3-circuit $\{e, f, 2n-1\}$ of M. This implies that there are no circuits of M' containing both e and f, so M' is disconnected. □

The following lemme characterizes the fibers of $\Phi(n)$ over the connected elements of $\mathcal{S}(2n-2, n)$.

Lemma 2.4. *Let $n \geq 2$. The map $\Phi_m(n)$ is a bijection when $m \geq 3$, and it is 3-to-1 when $m = 2$.*

Proof. Let $M' \in \mathcal{S}_m(2n-2, n)$ be given. We want to count the matroids $M \in \mathcal{S}_m(2n-1, n)$ with $\Phi_m(n)(M) = M'$. Since M must be chordal, it is determined by its 3-circuits. The 3-circuits of M that do not contain the element $2n-1$ coincide with the 3-circuits of M', hence it is sufficient to think about the 3-circuits of M that contain $2n-1$.

Let $S \subset [2n-2]$ be the union of the $\binom{m}{2}$ chordless 4-circuits of M'. Then the restriction of M' to S is isomorphic to the matroid associated with the complete bipartite graph $K_{2,m}$. If $m \geq 3$, there is a unique chordal extension of this matroid, represented by the thagomizer graph T_m [Ged17]. This uniquely determines all of the 3-circuits of M that contain $2n-1$. If $m = 2$, then the matroid associated with $K_{2,2}$ is the uniform matroid of rank three on four elements, and there are three different extensions of this matroid to a chordal matroid on five elements, corresponding to the three ways to partition the four elements into pairs of subsets of size two. These determine three different matroids $M \in \mathcal{S}_m(2n-1, n)$ that map to M'. $\square$

The above lemma shows that only those fibers of $\Phi(n)$ over elements of $\mathcal{S}_1(2n-2, n) \cup \mathcal{S}_2(2n-2, n)$ can contribute to the difference $|\mathcal{S}(2n-1, n)| - |\mathcal{S}(2n-2, n)|$. As shown below, these contributions can be expressed in terms of the counts of some combinatorial objects constructed from triangular cacti. Define a **desert** to be a disjoint union of triangular cacti, and a **rooted desert** to be a disjoint union of rooted triangular cacti. Let $\mathrm{Des}_m(n)$ denote the set of deserts on the vertex set $[2n-2]$ with exactly $2m$ connected components, and let $\mathrm{RDes}_m(n)$ denote the set of rooted deserts on the vertex set $[2n-2]$ with exactly $2m$ connected components. We have a map

$$\Omega_m(n) : \mathrm{RDes}_m(n) \to \mathrm{Des}_m(n)$$

given by forgetting the roots.

Let $\Delta_m(n) \subset \Delta(n)$ denote the set of triangular cacti with the property that the vertex $2n-1$ has degree $2m$, or equivalently that it is contained in exactly m triangles. Then the bijection $\Psi(n)$ restricts to a bijection

$$\Psi_m(n) : \mathcal{S}_m(2n-1, n) \to \Delta_m(n)$$

for all m. We also have a map

$$\Pi_m(n) : \Delta_m(n) \to \mathrm{RDes}_m(n)$$

given by deleting the vertex $2n-1$ along with all of the triangles that passed through that vertex, and taking the roots to be the vertices from the deleted triangles. Note that $\Pi_m(n)$ is surjective for all m, and $\Pi_1(n)$ is a bijection.

Lemma 2.5. *There is a bijection $\Sigma_2(n) : \mathcal{S}_2(2n-2, n) \to \mathrm{RDes}_2(n)$ with the property that the* *following diagram commutes:*

$$\begin{tikzcd}
\mathcal{S}_2(2n-1,n) \arrow[r, "\Psi_2(n)", "\cong"'] \arrow[d, two heads, "\Phi_2(n)"'] & \Delta_2(n) \arrow[d, two heads, "\Pi_2(n)"] \\
\mathcal{S}_2(2n-2,n) \arrow[r, "\Sigma_2(n)", "\cong"'] & \operatorname{RDes}_2(n).
\end{tikzcd}$$

Proof. Given a matroid $M' \in \mathcal{S}_2(2n-2,n)$, we define $\Sigma_2(n)(M') \in \operatorname{RDes}_2(n)$ to have triangles consisting of the 3-circuits of M' and roots consisting of the unique chordless 4-circuit of M'. $\square$

Lemma 2.6. *There is a bijection $\Theta_1(n) : \operatorname{Des}_1(n) \to \mathcal{S}_1(2n-2,n)$ with the property that the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{S}_1(2n-1,n) \arrow[r, "\Psi_1(n)", "\cong"'] \arrow[d, two heads, "\Phi_1(n)"'] & \Delta_1(n) \arrow[r, "\Pi_1(n)", "\cong"'] & \operatorname{RDes}_1(n) \arrow[d, two heads, "\Omega_1(n)"] \\
\mathcal{S}_1(2n-2,n) \arrow[rr, "\Theta_1(n)", "\cong"'] & & \operatorname{Des}_1(n).
\end{tikzcd}$$

Proof. Let $M' \in \mathcal{S}_1(2n-2,n)$ be given. By Lemma 2.3, M' is disconnected, so there exists a subset $S \subset [2n-2]$ of cardinality $2k-1$, a matroid $A \in \mathcal{S}(S,k)$, and another matroid $B \in \mathcal{S}([2n-2] \setminus B, n-k)$ such that $M' = A \oplus B$. We then define $\Theta_1(n)(M')$ to be the union of $\Psi(S)(A)$ and $\Psi([2n-2] \setminus S)(B)$. $\square$

We now come to the main result of this section.

Proposition 2.7. *For any $n > 1$, we have*

$$|\mathcal{S}(2n-1,n)| - |\mathcal{S}(2n-2,n)| = 2 \cdot |\operatorname{RDes}_2(n)| + |\operatorname{RDes}_1(n)| - |\operatorname{Des}_1(n)|.$$

Proof. We have

$$|\mathcal{S}(2n-1,n)| - |\mathcal{S}(2n-2,n)| = \sum_{m \geq 1} \Big(|\mathcal{S}_m(2n-1,n)| - |\mathcal{S}_m(2n-2,n)| \Big).$$

When $m \geq 3$, Lemma 2.4 tells us that $\Phi_m(n) : \mathcal{S}_m(2n-1,n) \to \mathcal{S}_m(2n-2,n)$ is a bijection, thus the summand indexed by m vanishes. When $m = 2$, Lemma 2.4 tells us that the map $\Phi_2(n) : \mathcal{S}_2(2n-1,n) \to \mathcal{S}_2(2n-2,n)$ is 3-to-1, and Lemma 2.5 identifies $\mathcal{S}_2(2n-2,n)$ with $\operatorname{RDes}_2(n)$. This implies that

$$|\mathcal{S}_2(2n-1,n)| - |\mathcal{S}_2(2n-2,n)| = 2|\mathcal{S}_2(2n-2,n)| = 2|\operatorname{RDes}_2(n)|.$$

Finally, when $m = 1$, Lemma 2.6 identifies the map $\Phi_1(n) : \mathcal{S}_1(2n-1,n) \to \mathcal{S}_1(2n-2,n)$ with the map $\Omega_1(n) : \mathrm{RDes}_1(n) \to \mathrm{Des}_1(n)$. The result follows. $\square$

Thus, to give an explicit formula for computing $|\mathcal{S}(2n-1,n)| - |\mathcal{S}(2n-2,n)|$, it remains to determine $|\,\mathrm{RDes}_2(n)|, |\,\mathrm{RDes}_1(n)|$ and $|\,\mathrm{Des}_1(n)|$. This task will be completed in the next section, via the enumeration of Husimi graphs.

3 Proofs of Theorem 1.3 and Corollary 1.6

A **block** of a graph is a maximal 2-connected subgraph. A **Husimi graph** is a connected graph whose blocks are all isomorphic to complete graphs. We say that it is of type $(n_2, n_3, n_4, \dots)$, where n_i is the number of blocks isomorphic to K_i. For any $p \geq 1$, let $\tau_p(n_2, n_3, n_4, \dots)$ denote the number of Husimi graphs of type $(n_2, n_3, n_4, \dots)$ on the vertex set $[p]$. The following result was initially discovered by Husimi [Hus50] and later given a rigorous mathematical proof by Leroux [Ler04]. See [Oko15, Lemma 5.3.3] for a clear statement and discussion of the history.

Lemma 3.1. *For any $p \geq 1$, we have*

$$\tau_p(n_2, n_3, n_4, \dots) = \frac{p!}{\prod_{i=2}^{p}[(i-1)!]^{n_i}\, n_i!}\, p^{-2+\sum_{i=2}^{p} n_i}.$$

Note that a Husimi graph of type $(p-1, 0, 0, \dots)$ is just a tree on the vertex set $[p]$, and Lemma 3.1 specializes to the statement, due originally to Cayley, that the number of such trees is p^{p-2}. Similarly, a Husimi graph of type $(0, r-1, 0, 0, \dots)$ is a triangular cactus on the set $[2r-1]$. In this case, Lemma 3.1 says that

$$|\Delta(r)| = \frac{(2r-1)^{r-3} \cdot (2r-1)!}{2^{r-1} \cdot (r-1)!}. \tag{1}$$

Proposition 3.2. *For any $n > 1$ and $m \geq 1$, we have*

$$|\,\mathrm{RDes}_m(n)| = \frac{(n-1)^{n-m-2} \cdot (2n-2)!}{2 \cdot (2m-1)! \cdot (n-m-1)!}.$$

In particular,

$$|\,\mathrm{RDes}_1(n)| = \frac{(n-1)^{n-3} \cdot (2n-2)!}{2 \cdot (n-2)!} \qquad \text{and} \qquad |\,\mathrm{RDes}_2(n)| = \frac{(n-1)^{n-4} \cdot (2n-2)!}{12 \cdot (n-3)!}.$$

Proof. Let $\mathrm{HT}_m(n)$ denote the set of Husimi graphs on the vertex set $[2n-2]$ with $n-m-1$ triangular blocks and one block isomorphic to K_{2m}. There is a bijection from $\mathrm{HT}_m(n)$ to $\mathrm{RDes}_m(n)$ that takes a Husimi graph to the rooted desert obtained by deleting the edges of K_{2m} and taking its vertices as the roots. The result then follows from Lemma 3.1. $\square$

We proceed to determine $|\,\mathrm{Des}_1(n)|$. Before that, let us recall a result on the **Abel polynomials**

$$A_m(x; a) := x(x - am)^{m-1}.$$

Abel [Rom05, Section 2.6] showed that these polynomials satisfy the identity

$$\sum_{j=0}^{m} \binom{m}{j} A_j(x;a) A_{m-j}(y;a) = A_m(x+y;a)$$

for any integers m, x, y and a. By combining this formula with the definition of the Abel polynomials, we obtain the equation

$$\sum_{j=0}^{m} \binom{m}{j} xy(x-aj)^{j-1}(y-am+aj)^{m-j-1} = (x+y)(x+y-am)^{m-1}. \tag{2}$$

Differentiating both sides of (2) with respect to x, we get

$$\sum_{j=0}^{m} \binom{m}{j} j(x-a)y(x-aj)^{j-2}(y-am+aj)^{m-j-1} = m(x+y-a)(x+y-am)^{m-2}. \tag{3}$$

With these formulas, we are able to give the following explicit formula for $|\operatorname{Des}_1(n)|$.

Lemma 3.3. *For $n > 1$, we have*

$$|\operatorname{Des}_1(n)| = \frac{(n+1)(n-1)^{n-5} \cdot (2n-2)!}{6 \cdot (n-2)!}.$$

Proof. For any $m \geq 1$, an element of $\operatorname{Des}_m(n)$ consists of a partition of $[2n-2]$ into $2m$ parts and a triangular cactus on each of those parts. This implies that

$$(2m)!\,|\operatorname{Des}_m(n)| \quad = \sum_{(2k_1-1)+\cdots+(2k_{2m}-1)=2n-2} \binom{2n-2}{2k_1-1,\ldots,2k_{2m}-1} \prod_{i=1}^{2m} |\Delta(k_i)|,$$

where the factor of $(2m)!$ reflects the fact that the parts of the partition are unordered. When $m = 1$, the above formula simplifies to

$$|\operatorname{Des}_1(n)| = \frac{1}{2} \sum_{r=1}^{n-1} \binom{2n-2}{2r-1} |\Delta(r)| \cdot |\Delta(n-r)|.$$

Substituting (1) into the right hand side and reindexing, we obtain the formula

$$\begin{aligned} |\operatorname{Des}_1(n)| &= \frac{1}{2} \sum_{r=0}^{n-2} \binom{2n-2}{2r+1} \cdot (2r-1)!!(2r+1)^{r-1} \cdot (2n-2r-5)!!(2n-2r-3)^{n-r-3} \qquad (4) \\ &= \sum_{r=0}^{n-2} \frac{(2n-2)!(2r+1)^{r-2}(2n-2r-3)^{n-r-4}}{2^{n-1} \cdot r! \cdot (n-r-2)!}. \end{aligned}$$

Now it suffices to show that

$$\sum_{r=0}^{n-2} \frac{(2n-2)!(2r+1)^{r-2}(2n-2r-3)^{n-r-4}}{2^{n-1}\cdot r!\cdot(n-r-2)!} = \frac{(n+1)(n-1)^{n-5}\cdot(2n-2)!}{6\cdot(n-2)!},$$

or equivalently that

$$\sum_{r=0}^{n-2} 3\binom{n-2}{r}(2r+1)^{r-2}(2n-2r-3)^{n-r-4} = 2^{n-2}(n+1)(n-1)^{n-5}. \tag{5}$$

To this end, we take $m = n-2$, $a = -2$, $x = 1$, and $y = 1$ in Equations (2) and (3) to obtain the following two equations:

$$\sum_{r=0}^{n-2} \binom{n-2}{r}(2r+1)^{r-1}(2n-2r-3)^{n-r-3} = 2^{n-2}(n-1)^{n-3}, \tag{6}$$

$$\sum_{r=0}^{n-2} \binom{n-2}{r}3r(2r+1)^{r-2}(2n-2r-3)^{n-r-3} = 2^{n-2}(n-2)(n-1)^{n-4}. \tag{7}$$

Substituting r for $n-2-r$ into the left hand side of (7) yields

$$\sum_{r=0}^{n-2} \binom{n-2}{r}3(n-r-2)(2r+1)^{r-1}(2n-2r-3)^{n-r-4} = 2^{n-2}(n-2)(n-1)^{n-4}. \tag{8}$$

By subtracting (7) and (8) from (6) multiplied by 3, we obtain the desired (5). This completes the proof. $\square$

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let $g_n := |\mathcal{S}(2n-1,n)| - |\mathcal{S}(2n-2,n)|$. By Proposition 2.7, Proposition 3.2, and Lemma 3.3, we have

$$\begin{aligned}
g_n &= 2\cdot|\operatorname{RDes}_2(n)| + |\operatorname{RDes}_1(n)| - |\operatorname{Des}_1(n)| \\
&= \frac{(n-1)^{n-4}\cdot(2n-2)!}{6\cdot(n-3)!} + \frac{(n-1)^{n-3}\cdot(2n-2)!}{2\cdot(n-2)!} - \frac{(n+1)(n-1)^{n-5}\cdot(2n-2)!}{6\cdot(n-2)!} \\
&= (2n-2)!\cdot\frac{(n-2)(n-1)^{n-4} + 3(n-1)^{n-3} - (n+1)(n-1)^{n-5}}{6\cdot(n-2)!} \\
&= \frac{2(n-2)(n-1)^{n-5}\cdot(2n-1)!}{6\cdot(n-2)!} \\
&= \frac{(n-1)^{n-5}\cdot(2n-1)!}{3\cdot(n-3)!}.
\end{aligned}$$

Combining this with Theorem 1.2 gives the result. $\square$

Finally, we prove Corollary 1.6.

Proof of Corollary 1.6. By Equation (4) and Theorem 1.5, we find that

$$E_n = |\mathcal{S}(2n-2,n)| - |\operatorname{Des}_1(n)|.$$

Then combining Lemma 3.3 and Theorem 1.3 gives the desired result. □

4 Proof of Theorem 1.7

The aim of this section is to prove Theorem 1.7. To this end, we need to use a relation between $Q_{B_n}(t)$ and $P_{B_n}(t)$. Before recalling this relation, we will follow [GM12] to introduce some notation from matroid theory.

Let $M = (E, \mathcal{F})$ be a loopless matroid on ground set E with the set of flats $\mathcal{F}$. The lattice of flats of M is denoted by $\mathscr{L}(M)$. For any flat F of M, let $M|_F$ denote the restriction of M to F, and let M/F denote the matroid obtained from M by contracting F. For any subset I of E, let $\operatorname{rk} I$ denote the rank of I in the matroid M. The rank of matroid M, denoted by $\operatorname{rk} M$, is defined to be $\operatorname{rk} E$. Gao and Xie [GX21, Theorem 1.3] established the following relation between $Q_M(t)$ and $P_M(t)$:

$$P_M(t) = -\sum_{F\in\mathscr{L}(M)\setminus\{E\}} P_{M|_F}(t)\cdot(-1)^{\operatorname{rk} M/F} Q_{M/F}(t). \tag{9}$$

Let $[t^i]f(t)$ denote the coefficient t^i in the polynomial $f(t)$. Based on (9), Vecchi [Vec21, Theorem 4.1] showed that, for any matroid M of odd rank $2m-1$, we have the identity

$$[t^{m-1}]P_M(t) = [t^{m-1}]Q_M(t).$$

Since the rank of braid matroid B_{2n} is $2n-1$, we have

$$[t^{n-1}]P_{\mathrm{B}_{2n}}(t) = [t^{n-1}]Q_{\mathrm{B}_{2n}}(t). \tag{10}$$

The relationship between the leading coefficients of $P_{B_{2n-1}}(t)$ and $Q_{B_{2n-1}}(t)$ is more subtle; the precise formula appears in the following lemma.

Lemma 4.1. *For any $n > 1$, we have*

$$[t^{n-2}]P_{\mathrm{B}_{2n-1}}(t) + [t^{n-2}]Q_{\mathrm{B}_{2n-1}}(t) = \sum_{j=1}^{n-1}\binom{2n-1}{2j}[t^{j-1}]P_{\mathrm{B}_{2j}}(t)\cdot[t^{n-1-j}]Q_{\mathrm{B}_{2n-2j}}(t). \tag{11}$$

Proof. Taking M to be B_{2n-1} in Equation (9) yields

$$P_{\mathrm{B}_{2n-1}}(t) + Q_{\mathrm{B}_{2n-1}}(t) = -\sum_{F\in\mathscr{L}(\mathrm{B}_{2n-1})\setminus\{\emptyset,E\}} P_{\mathrm{B}_{2n-1}|_F}(t)\cdot(-1)^{\operatorname{rk}\mathrm{B}_{2n-1}/F} Q_{\mathrm{B}_{2n-1}/F}(t). \tag{12}$$

We now compare coefficients of t^{n-2} on both sides of Equation (12). It suffices to show that

$$\sum_{F\in\mathscr{L}(\mathrm{B}_{2n-1})\setminus\{\emptyset,E\}}[t^{n-2}]\Big(P_{\mathrm{B}_{2n-1}|_F}(t)\cdot(-1)^{\mathrm{rk}\,\mathrm{B}_{2n-1}/F}Q_{\mathrm{B}_{2n-1}/F}(t)\Big)$$
$$=-\sum_{j=1}^{n-1}\binom{2n-1}{2j}[t^{j-1}]P_{\mathrm{B}_{2j}}(t)\cdot[t^{n-1-j}]Q_{\mathrm{B}_{2n-2j}}(t). \tag{13}$$

The lattice of $\mathscr{L}(B_k)$ is isomorphic to the lattice of set-theoretic partitions of the set $[k]$, with the minimal element $\emptyset$ corresponding to the partition of $[k]$ into k singletons and the maximal element E corresponding to the partition of $[k]$ into a single part. We say that $F\in\mathscr{L}(\mathrm{B}_k)$ is of type λ if the partition λ can be obtained by arranging the sizes of the blocks of the corresponding set partition in descending order. If F is of type $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_{\ell(\lambda)})\vdash k$, then (after simplification) we have

$$\mathrm{B}_k|_F\cong\mathrm{B}_{\lambda_1}\oplus\mathrm{B}_{\lambda_2}\oplus\cdots\oplus\mathrm{B}_{\lambda_{\ell(\lambda)}}\quad\text{and}\quad\mathrm{B}_k/F\cong\mathrm{B}_{\ell(\lambda)}. \tag{14}$$

By [EPW16, Theorem 2.2 and Proposition 2.7] and [GX21, Theorem 1.2], we have

$$P_{\mathrm{B}_k|_F}(t)=P_{\mathrm{B}_{\lambda_1}}(t)P_{\mathrm{B}_{\lambda_2}}(t)\cdots P_{\mathrm{B}_{\lambda_{\ell(\lambda)}}}(t)\quad\text{and}\quad Q_{\mathrm{B}_k/F}(t)=Q_{\mathrm{B}_{\ell(\lambda)}}(t). \tag{15}$$

Let F be a nonempty proper flat of B_{2n-1}, and let $\lambda\vdash 2n-1$ be the type of F. We will analyze the summand of Equation (13) indexed by F according to the following cases.

Case I: $\lambda=(2j,1^{2n-1-2j})$ for some $1\le j\le n-1$. By Equation (15) and the fact $P_{B_1}(t)=1$, we have

$$P_{\mathrm{B}_{2n-1}|_F}(t)\cdot(-1)^{\mathrm{rk}\,\mathrm{B}_{2n-1}/F}Q_{\mathrm{B}_{2n-1}/F}(t)=-P_{\mathrm{B}_{2j}}(t)\cdot Q_{\mathrm{B}_{2n-2j}}(t).$$

Since $\deg P_{\mathrm{B}_{2j}}(t)\le j-1$ and $\deg Q_{\mathrm{B}_{2n-2j}}(t)\le n-j-1$, we have

$$[t^{n-2}]\Big(P_{\mathrm{B}_{2n-1}|_F}(t)\cdot(-1)^{\mathrm{rk}\,\mathrm{B}_{2n-1}/F}Q_{\mathrm{B}_{2n-1}/F}(t)\Big)=-[t^{j-1}]P_{\mathrm{B}_{2j}}(t)\cdot[t^{n-j-1}]Q_{\mathrm{B}_{2n-2j}}(t).$$

Case II: $\lambda=(2j-1,1^{2n-2j})$ for some $2\le j\le n-1$. This time, we have

$$P_{\mathrm{B}_{2n-1}|_F}(t)\cdot(-1)^{\mathrm{rk}\,\mathrm{B}_{2n-1}/F}Q_{\mathrm{B}_{2n-1}/F}(t)=P_{\mathrm{B}_{2j-1}}(t)\cdot Q_{\mathrm{B}_{2n-2j+1}}(t).$$

Since $\deg P_{\mathrm{B}_{2j}}(t)\le j-2$ and $\deg Q_{\mathrm{B}_{2n-2j+1}}(t)\le n-j-1$, we have

$$[t^{n-2}]\Big(P_{\mathrm{B}_{2n-1}|_F}(t)\cdot(-1)^{\mathrm{rk}\,\mathrm{B}_{2n-1}/F}Q_{\mathrm{B}_{2n-1}/F}(t)\Big)=0.$$

Case III: $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_i,1^{2n-1-\sum_{j=1}^{i}\lambda_j})$ for some $i\geq 2$ and $\lambda_i>1$. Now we have

$$P_{\mathrm{B}_{2n-1}|_F}(t)\cdot(-1)^{\mathrm{rk}\,\mathrm{B}_{2n-1}/F}Q_{\mathrm{B}_{2n-1}/F}(t)=(-1)^{2n+i-\sum_{j=1}^{i}\lambda_j}P_{\mathrm{B}_{\lambda_1}}(t)\cdots P_{\mathrm{B}_{\lambda_i}}(t)\cdot Q_{\mathrm{B}_{2n+i-1-\sum_{j=1}^{i}\lambda_j}}(t).$$

Since $\deg P_{\mathrm{B}_k}(t)\leq\frac{k-2}{2}$ and $\deg Q_{\mathrm{B}_k}(t)\leq\frac{k-2}{2}$ for any $k\geq 2$,

$$\deg(P_{\mathrm{B}_{\lambda_1}}(t)\cdots P_{\mathrm{B}_{\lambda_i}}(t))=\deg P_{\mathrm{B}_{\lambda_1}}(t)+\cdots+\deg P_{\mathrm{B}_{\lambda_i}}(t)\leq\frac{\sum_{j=1}^{i}\lambda_j-2i}{2}$$

and

$$\deg Q_{\mathrm{B}_{2n+i-1-\sum_{j=1}^{i}\lambda_j}}(t)\leq\frac{2n+i-3-\sum_{j=1}^{i}\lambda_j}{2}.$$

Since $i\geq 2$, we have

$$\deg P_{\mathrm{B}_{\lambda_1}}(t)\cdots P_{\mathrm{B}_{\lambda_i}}(t)+\deg Q_{\mathrm{B}_{2n+i-1-\sum_{j=1}^{i}\lambda_j}}(t)\leq\frac{2n-i-3}{2}\leq\frac{2n-5}{2}<n-2.$$

Thus

$$[t^{n-2}]\Big(P_{\mathrm{B}_{2n-1}|_F}(t)\cdot(-1)^{\mathrm{rk}\,\mathrm{B}_{2n-1}/F}Q_{\mathrm{B}_{2n-1}/F}(t)\Big)=0.$$

Combining the above three cases, we find that only those flats of type $\lambda=(2j,1^{2n-1-2j})$ can contribute to the left hand side of (13). Note that, for each $1\leq j\leq n-1$, there are exactly $\binom{2n-1}{2j}$ flats of type $\lambda=(2j,1^{2n-1-2j})$. This completes the proof of Equation (13), and hence that of the lemma. $\square$

Now we are ready to prove Theorem 1.7.

Proof of Theorem 1.7. By Equation (10) and Theorem 1.2, we see that

$$[t^{n-1}]P_{\mathrm{B}_{2n}}(t)=[t^{n-1}]Q_{\mathrm{B}_{2n}}(t)=(2n-1)^{n-2}\cdot(2n-3)!!=\frac{(2n-1)!(2n-1)^{n-3}}{2^{n-1}\cdot(n-1)!},\tag{16}$$

and by Theorem 1.3, we have

$$[t^{n-2}]P_{\mathrm{B}_{2n-1}}(t)=\frac{(2n-1)!(2n-1)^{n-3}}{2^{n-1}\cdot(n-1)!}-\frac{(n-2)(n-1)^{n-5}\cdot(2n-1)!}{3\cdot(n-2)!}.\tag{17}$$

It remains only to show that

$$[t^{n-2}]Q_{\mathrm{B}_{2n-1}}(t)=\frac{(2n-1)!(n-1)^{n-5}}{3\cdot(n-2)!}.$$

By Equations (16), (17), and (11), this is equivalent to the statement that

$$\frac{(2n-1)!(n-1)^{n-5}}{3\cdot(n-2)!} = \sum_{j=1}^{n-1}\binom{2n-1}{2j}\frac{(2j-1)!(2j-1)^{j-3}}{2^{j-1}\cdot(j-1)!}\cdot\frac{(2n-2j-1)!(2n-2j-1)^{n-j-3}}{2^{n-j-1}\cdot(n-j-1)!}$$
$$-\Big(\frac{(2n-1)!(2n-1)^{n-3}}{2^{n-1}\cdot(n-1)!}-\frac{(2n-1)!(n-1)^{n-5}}{3\cdot(n-3)!}\Big),$$

which simplifies further to the equation

$$\sum_{j=0}^{n-1}3\binom{n-1}{j}(2j-1)^{j-3}(2n-2j-1)^{n-j-3} = -8(n-3)(2n-2)^{n-4}. \tag{18}$$

To prove Equation (18), we will first establish the following two identities:

$$\sum_{j=0}^{m}3\binom{m}{j}(2j-1)^{j-2}(2m-2j+1)^{m-j-2} = 8(2m)^{m-2}, \tag{19}$$

$$\sum_{j=0}^{m}3\binom{m}{j}(4mj-4j+1)(2j-1)^{j-3}(2m-2j+1)^{m-j-2} = 8(2m^2+m-2)(2m)^{m-3}. \tag{20}$$

Let us first prove Equation (19). Differentiating both sides of Equation (2) with respect to y, we obtain the identity

$$\sum_{j=0}^{m}\binom{m}{j}(m-j)x(y-a)(x-aj)^{j-1}(y-am+aj)^{m-j-2} = m(x+y-a)(x+y-am)^{m-2}. \tag{21}$$

Letting $a=-2, x=-1, y=1$ in Equations (2), (3), and (21), we obtain the following:

$$\sum_{j=0}^{m}\binom{m}{j}(2j-1)^{j-1}(2m-2j+1)^{m-j-1} = 0, \tag{22}$$

$$\sum_{j=0}^{m}\binom{m}{j}j(2j-1)^{j-2}(2m-2j+1)^{m-j-1} = (2m)^{m-1}, \tag{23}$$

$$\sum_{j=0}^{m}-3\binom{m}{j}(m-j)(2j-1)^{j-1}(2m-2j+1)^{m-j-2} = (2m)^{m-1}. \tag{24}$$

Now, subtracting Equation (22) from Equation (23) multiplied by 2 yields

$$\sum_{j=0}^{m}\binom{m}{j}(2j-1)^{j-2}(2m-2j+1)^{m-j-1} = 2(2m)^{m-1}, \tag{25}$$

and adding Equation (22) multiplied by 3 to Equation (24) multiplied by 2 yields

$$\sum_{j=0}^{m} 3\binom{m}{j}(2j-1)^{j-1}(2m-2j+1)^{m-j-2} = 2(2m)^{m-1}. \tag{26}$$

Furthermore, adding Equation (25) multiplied by 3 to Equation (26) and then cancelling the common factor $2m$ lead to the desired Equation (19).

In the same manner we can prove Equation (20). Differentiating Equation (21) with respect to x, we have

$$\sum_{j=0}^{m} \binom{m}{j} j(m-j)(x-a)(y-a)(x-aj)^{j-2}(y-am+aj)^{m-j-2}$$
$$= m(m-1)(x+y-2a)(x+y-am)^{m-3}. \tag{27}$$

Differentiating again gives

$$\sum_{j=0}^{m} \binom{m}{j} j(j-1)(m-j)(x-2a)(y-a)(x-aj)^{j-3}(y-am+aj)^{m-j-2}$$
$$= m(m-1)(m-2)(x+y-3a)(x+y-am)^{m-4}. \tag{28}$$

Letting $a=-2, x=-1, y=1$ in Equations (27) and (28), we obtain

$$\sum_{j=0}^{m} 3\binom{m}{j} j(m-j)(2j-1)^{j-2}(2m-2j+1)^{m-j-2} = 2(m-1)(2m)^{m-2}, \tag{29}$$

$$\sum_{j=0}^{m} 3\binom{m}{j} j(j-1)(m-j)(2j-1)^{j-3}(2m-2j+1)^{m-j-2} = (m-1)(m-2)(2m)^{m-3}. \tag{30}$$

Then, subtracting Equation (30) multiplied by 2 from Equation (29) yields

$$\sum_{j=0}^{m} 3\binom{m}{j} j(m-j)(2j-1)^{j-3}(2m-2j+1)^{m-j-2} = 2(m-1)(m+2)(2m)^{m-3}. \tag{31}$$

Adding Equation (26) to Equation (31) multiplied by 4, we get Equation (20), as desired.

Now we can derive Equation (18) from Equations (19) and (20). By subtracting Equation (19) multiplied by $2m-2$ from Equation (20) and then cancelling the common factor $2m-1$, we find that

$$\sum_{j=0}^{m} 3\binom{m}{j}(2j-1)^{j-3}\big(2m-2j+1\big)^{m-j-2} = -8(m-2)(2m)^{m-3}.$$

Substituting m to $n-1$ in the above formula yields Equation (18). This completes the proof. $\square$

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