Positivity theorems for hyperplane arrangements via intersection theory

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Let $V$ be a finite dimensional vector space, and $\mathcal{A}$ a finite set of hyperplanes in $V$ with $\bigcap_{H \in \mathcal{A}} H = \{0\}$.

**Definition**

A flat $F \subset V$ is an intersection of some hyperplanes.

**Example**

- 1 flat of dimension 2 ($V$ itself)
- 3 flats of dimension 1 (the lines)
- 1 flat of dimension 0 (the origin)
Example

Suppose $V = \mathbb{C}^6$ and $A$ consists of 8 generic hyperplanes.

- $\binom{8}{0} = 1$ flat of dimension 6 ($V$ itself)
- $\binom{8}{1} = 8$ flats of dimension 5 (the hyperplanes)
- $\binom{8}{2} = 28$ flats of dimension 4
- $\binom{8}{3} = 56$ flats of dimension 3
- $\binom{8}{4} = 70$ flats of dimension 2
- $\binom{8}{5} = 56$ flats of dimension 1
- 1 flat of dimension 0 (the origin)
Suppose $V = \mathbb{C}^7 / \mathbb{C}_\Delta = \{(z_1, \ldots, z_7) \in \mathbb{C}^7\} / \mathbb{C} \cdot (1, \ldots, 1)$ and $\mathcal{A}$ consists of the $\binom{7}{2}$ hyperplanes $H_{ij} := \{z_i = z_j\}$.

$H_{14} \cap H_{46} \cap H_{37} = \{(z_1, \ldots, z_7) \mid z_1 = z_4 = z_6, z_3 = z_7\} / \mathbb{C}_\Delta$

is a flat of dimension 3.

flats $\leftrightarrow$ partitions of the set $\{1, \ldots, 7\}$

$H_{14} \cap H_{46} \cap H_{37} \leftrightarrow \{1, 4, 6\} \sqcup \{3, 7\} \sqcup \{2\} \sqcup \{7\}$

$H_{14} \leftrightarrow \{1, 4\} \sqcup \{2\} \sqcup \{3\} \sqcup \{5\} \sqcup \{6\} \sqcup \{7\}$

$V \leftrightarrow \{1\} \sqcup \{2\} \sqcup \{3\} \sqcup \{4\} \sqcup \{5\} \sqcup \{6\} \sqcup \{7\}$

$\{0\} \leftrightarrow \{1, \ldots, 7\}$

flats of dimension $k$ $\leftrightarrow$ partitions into $k + 1$ parts
The Top-Heavy conjecture

Theorem (Top-Heavy conjecture)
If $k \leq \frac{1}{2} \dim V$, then

$$\# \text{ of flats of codimension } k \leq \# \text{ of flats of dimension } k.$$  

Furthermore, the same statement holds for matroids (combinatorial abstractions of hyperplane arrangements for which we can still make sense of flats).

- Conjectured by Dowling and Wilson, 1974
- True for arrangements: Huh and Wang, 2017
## Example

8 generic hyperplanes in $\mathbb{C}^6$

- $\binom{8}{1} = 8$ flats of codimension 1
- $\binom{8}{5} = 56$ flats of dimension 1
- $\binom{8}{2} = 28$ flats of codimension 2
- $\binom{8}{4} = 70$ flats of dimension 2

## Example

$V = \mathbb{C}^7 / \mathbb{C}_\Delta$, $A = \{ H_{ij} \}$

- # partitions into 6 parts $= \binom{7}{2} = 21$ flats of codimension 1
- # partitions into 2 parts $= \frac{2^7 - 2}{2} = 63$ flats of dimension 1
- $S(7, 5) = 140$ flats of codimension 2
- $S(7, 3) = 301$ flats of dimension 2
The **characteristic polynomial** $\chi_A(t)$ is a polynomial, depending only on the poset of flats, with the following property:

If $V$ is a vector space over $\mathbb{F}_q$, then $\chi_A(q)$ is equal to the number of points on the complement of the hyperplanes.

**Example**

\[
\begin{align*}
\chi_A(q) &= q^2 - 3q + 2 \\
\text{Since } q \text{ could be any prime power, we must have } &\chi_A(t) = t^2 - 3t + 2.
\end{align*}
\]

More generally, $\chi_A(t) = \sum_F \mu(V, F)t^{\dim F}$,

where $\mu$ is the Möbius function for the poset of flats.
The characteristic polynomial

**Example**

\[ V = \mathbb{C}^7 / \mathbb{C}_\Delta, \ A = \{ H_{ij} \} \]

- \( \chi_A(q) = (q - 1)(q - 2)(q - 3)(q - 4)(q - 5)(q - 6) \)

- Since \( q \) could be any prime power, we must have
  \[ \chi_A(t) = (t - 1)(t - 2)(t - 3)(t - 4)(t - 5)(t - 6). \]

If \( V \) is a vector space over \( \mathbb{F}_q \) and \( \dim V > 0 \), then \( \mathbb{F}_q^\times \) acts freely on the set of the points in the complement of the hyperplanes. That means that \( \chi_A(q) \) is a multiple of \( q - 1 \), and therefore \( \chi_A(t) \) is a multiple of \( t - 1 \). The **reduced characteristic polynomial** is

\[ \bar{\chi}_A(t) := \chi_A(t)/(t - 1). \]
The characteristic polynomial

**Example**

\[ V = \mathbb{C}^7 / \mathbb{C}_\Delta, \ A = \{ H_{ij} \} \]

\[ \chi_A(t) = (t - 1)(t - 2)(t - 3)(t - 4)(t - 5)(t - 6) \]

\[ \tilde{\chi}_A(t) = (t - 2)(t - 3)(t - 4)(t - 5)(t - 6) \]

\[ = t^5 - 20t^4 + 155t^3 - 580t^2 + 1044t - 720 \]

**Easy Lemma:** There exist positive integers \( a_0, a_1, \ldots \) such that

\[ \tilde{\chi}_A(t) = \sum_{i \geq 0} (-1)^i a_i t^{\dim V - 1 - i} . \]

**Example**

\( a_0 = 1, \ a_1 = 20, \ a_2 = 155, \ a_3 = 580, \ a_4 = 1044, \ a_5 = 720. \)
The sequence $a_0, a_1, \ldots$ is log concave. That is, for all $i > 0$,

$$a_i^2 \geq a_{i-1}a_{i+1}.$$ 

Furthermore, the statement holds for matroids.

- Conjectured for graphs by Hoggar, 1974
- Conjectured for all arrangements, and in fact for all matroids, by Welsh, 1976
- True for arrangements over $\mathbb{C}$: Huh, 2012
- True for all arrangements: Huh and Katz, 2012
- True for matroids: Adiprasito–Huh–Katz, 2017
**Contractions**

**Definition**

Let $F$ be a flat. The **contraction of $\mathcal{A}$ at $F$** is the arrangement

$$\mathcal{A}^F := \{ H \cap F \mid F \not\subset H \in \mathcal{A} \}$$

in the vector space $F$.

Note the special case $\mathcal{A}^V = \mathcal{A}$.
The Kazhdan–Lusztig polynomial $P_A(t)$ is uniquely determined by the following three properties:

- If $\dim V = 0$, then $P_A(t) = 1$
- If $\dim V > 0$, then $\deg P_A(t) < \frac{1}{2} \dim V$
- The $Z$-polynomial

$$Z_A(t) = \sum_F P_{A^F}(t)t^{\codim F}$$

is palindromic. That is, $t^{\dim V}Z_A(t^{-1}) = Z_A(t)$.

**Easy Lemma:** For any $A$, $P_A(0) = 1$. In particular, $P_A(t) = 1$ whenever $\dim V \leq 3$. 
### Example

Suppose that \( \dim V = 3 \) and \( \mathcal{A} \) consists of 5 hyperplanes in general position.

- We have 1 flat \( V \) of codimension 0, and \( P_{\mathcal{A}V}(t) = P_{\mathcal{A}}(t) \).
- We have 5 flats \( F \) of codimension 1, and \( P_{\mathcal{A}F}(t) = 1 \).
- We have \( \binom{5}{2} = 10 \) flats \( F \) of codimension 2, and \( P_{\mathcal{A}F}(t) = 1 \).
- We have 1 flat \( \{0\} \) of codimension 3, and \( P_{\mathcal{A}\{0\}}(t) = 1 \).

\[
Z_{\mathcal{A}}(t) = P_{\mathcal{A}}(t) + 5t + 10t^2 + t^3.
\]

Since \( \deg P_{\mathcal{A}}(t) < \frac{3}{2} \), we must have

\[
P_{\mathcal{A}}(t) = 1 + 5t.
\]
Example

Suppose that \( \dim V = 4 \) and \( \mathcal{A} \) consists of 6 hyperplanes in general position.

- We have 1 flat \( V \) of codimension 0, and \( P_{\mathcal{A}V}(t) = P_{\mathcal{A}}(t) \).
- We have 6 flats \( F \) of codimension 1, and \( P_{\mathcal{A}F}(t) = 1 + 5t \).
- We have \( \binom{6}{2} = 15 \) flats \( F \) of codimension 2, and \( P_{\mathcal{A}F}(t) = 1 \).
- We have \( \binom{6}{3} = 20 \) flats \( F \) of codimension 3, and \( P_{\mathcal{A}F}(t) = 1 \).
- We have 1 flat \( \{0\} \) of codimension 3, and \( P_{\mathcal{A}\{0\}}(t) = 1 \).

\[
Z_{\mathcal{A}}(t) = P_{\mathcal{A}}(t) + 6(1 + 5t)t + 15t^2 + 20t^3 + t^4
= P_{\mathcal{A}}(t) + 6t + 45t^2 + 20t^3 + t^4.
\]

Since \( \deg P_{\mathcal{A}}(t) < \frac{4}{2} = 2 \), we must have \( P_{\mathcal{A}}(t) = 1 + 14t \).
Theorem

The coefficients of $P_A(t)$ are non-negative. Furthermore, the statement holds for matroids.

Let $X$ be a (nonempty, connected) smooth projective variety of dimension $r > 0$ over $\mathbb{C}$, and let $\alpha \in H^2(X; \mathbb{Q})$ be the class of an ample line bundle.

Poincaré duality provides an isomorphism

$$\deg : H^{2r}(X; \mathbb{Q}) \to \mathbb{Q}.$$ 

Consider the symmetric bilinear pairing $\langle \ , \rangle_\alpha$ on $H^2(X; \mathbb{Q})$ given by the formula

$$\langle \eta, \xi \rangle_\alpha := \deg(\alpha^{r-2} \eta \xi).$$

**Theorem (Hodge–Riemann bilinear relations in degree 2)**

- The form $\langle \ , \rangle_\alpha$ is positive definite on $\mathbb{Q}\alpha$, i.e. $\langle \alpha, \alpha \rangle_\alpha > 0$.
- Let $P_\alpha := (\mathbb{Q}\alpha)^\perp$. The form $\langle \ , \rangle_\alpha$ is negative definite on $P_\alpha$. 

Corollary

Suppose that $\alpha$ and $\beta$ are two ample classes, and let

$$a_i := \deg(\alpha^{r-i} \beta^i).$$

The sequence $a_0, \ldots, a_r$ is log concave.

**Proof.** We’ll prove that $a_1^2 \geq a_0 a_2$; the other inequalities follow from this one by passing to a hyperplane section associated with $\beta$. We can assume that $\alpha$ and $\beta$ are linearly independent; otherwise the statement is trivial.

Restrict the form to $L = \mathbb{Q}\{\alpha, \beta\}$. Since $\alpha \in L$, it has at least one positive eigenvalue. But it only had one positive eigenvalue on all of $H^2(X; \mathbb{Q})$, so its other eigenvalue must be negative.
This means that

\[
0 > \det \begin{pmatrix} \langle \alpha, \alpha \rangle & \langle \alpha, \beta \rangle \\ \langle \beta, \alpha \rangle & \langle \beta, \beta \rangle \end{pmatrix} = a_0 a_2 - a_1^2.
\]

**Corollary**

*Suppose that \( \alpha \) and \( \beta \) are two nef classes, and let

\[
a_i := \deg(\alpha^{r-i} \beta^i).
\]

*The sequence \( a_0, \ldots, a_r \) is log concave. 

**Proof.** Approximate by ample classes.
Now we want to interpret the coefficients of the reduced characteristic polynomial as intersection numbers on some smooth projective variety.

Given an arrangement $\mathcal{A}$ in $V$, construct the wonderful variety $X_\mathcal{A}$ as follows:

- Start with $\mathbb{P}(V)$.
- Blow up $\mathbb{P}(F)$ for each 1-dimensional flat $F$.
- Blow up the proper transform of $\mathbb{P}(F)$ for each 2-dimensional flat $F$.
- Continue through all flats $F \subsetneq V$. 
Proposition (Adiprasito–Huh–Katz)

Let \( r := \dim V - 1 = \dim \mathcal{X}_A \). There exist nef classes \( \alpha, \beta \in H^2(X; \mathbb{Q}) \) such that

\[
\bar{\chi}_A(t) = \sum_{i \geq 0} (-1)^i \deg(\alpha^{r-i} \beta^i) t^{r-i}.
\]

For arbitrary matroids, there is no analogue of \( \mathcal{X}_A \), but there is a combinatorially defined ring that stands in for \( H^*(\mathcal{X}_A; \mathbb{Q}) \). The hard work of AHK is showing that this ring satisfies the Hodge–Riemann bilinear relations.
Let $Y$ be a (singular) projective variety of dimension $d$. We have the cohomology ring $H^*(Y)$ and the intersection cohomology module $IH^*(Y)$. There is a $H^*(Y)$-module homomorphism

$$H^*(Y) \rightarrow IH^*(Y)$$

which is an isomorphism when $Y$ is smooth, but is in general neither injective nor surjective.

Let $\alpha \in H^2(Y)$ be an ample class.

**Theorem (Hard Lefschetz)**

*For all $k \leq d$, multiplication by $\alpha^{d-k}$ gives an isomorphism*

$$IH^k(Y) \xrightarrow{\sim} IH^{2d-k}(Y).$$
The Schubert variety of \( \mathcal{A} \)

Let \( \mathcal{A} \) be an arrangement in \( V \). We have

\[
V \leftrightarrow \bigoplus_{H \in \mathcal{A}} V/H \cong \bigoplus_{H \in \mathcal{A}} \mathbb{A}^1 \subset \prod_{H \in \mathcal{A}} \mathbb{P}^1.
\]

**Definition**

We define the **Schubert variety** \( Y_{\mathcal{A}} := \tilde{V} \subset \prod_{H \in \mathcal{A}} \mathbb{P}^1 \).

For any flat \( F \), let

\[
U_F = \{ p \in Y_{\mathcal{A}} \mid p_H \neq \infty \leftrightarrow F \subset H \}.
\]

**Example**

We have \( U_V = (\infty, \ldots, \infty) \) and \( U_{\{0\}} = V \).
The Schubert variety of $\mathcal{A}$

**Proposition**

- For each $F$, $U_F \cong V/F$ (it’s a $V$-orbit with stabilizer $F$).
- We have $Y_\mathcal{A} = \bigsqcup_F U_F$ (i.e. every orbit is of this form).

The fact that $Y_\mathcal{A}$ admits a stratification by affine spaces has two consequences.

**Corollary**

- We have $H^{2k}(Y_\mathcal{A}) \cong \mathbb{Q}\{\text{codimension } k \text{ flats}\}$.
- The natural map from $H^*(Y_\mathcal{A})$ to $IH^*(Y_\mathcal{A})$ is an inclusion.
Hard Lefschetz and the Top-Heavy conjecture

\[
\begin{align*}
\text{IH}^{2k}(Y_A) & \xrightarrow{\cong} \alpha^{d-2k} \xrightarrow{\cong} \text{IH}^{2(d-k)}(Y_A) \\
\uparrow & \quad \uparrow \\
H^{2k}(Y_A) & \xleftarrow{\cong} \alpha^{d-2k} \xleftarrow{\cong} H^{2(d-k)}(Y_A) \\
\cong \uparrow & \quad \cong \uparrow \\
\mathbb{Q}\{\text{codim } k \text{ flats}\} & \xrightarrow{\cong} \mathbb{Q}\{\text{dim } k \text{ flats}\}
\end{align*}
\]

This proves the Top-Heavy conjecture!

For arbitrary matroids, there is no analogue of \(Y_A\), but one can give a combinatorial definition of a ring and a module that stand in for \(H^*(Y_A)\) and \(IH^*(Y_A)\). The hard work of BHMPW is showing that this module satisfies the hard Lefschetz theorem.
A spectral sequence for $IH^*(Y_A)$

Let $IC_{Y_A}$ be the intersection cohomology sheaf of $Y_A$; this is a sheaf (actually an object in the derived category of sheaves) with $H^*(IC_{Y_A}) = IH^*(Y_A)$.

Let $IC_{Y_A,F}$ be the stalk at a point in $U_F$.

Let $j_F : U_F \to Y_A$ be the inclusion. There is a spectral sequence converging to $IH^*(Y_A)$ with

$$E_1^{p,q} = \bigoplus_{\text{codim } F = p} H_c^{p+q}(j_F^*IC_{Y_A})$$

$$\Rightarrow \ldots$$

$$\Rightarrow H^{q-p}(IC_{Y_A,F}).$$
A spectral sequence for $IH^*(Y_A)$

**Lemma**

For any flat $F$, $H^*(IC_{Y_A,F}) \cong H^*(IC_{Y_A F},\{0\})$, and this cohomology vanishes in odd degree.

This implies that the spectral sequence degenerates at the $E_1$ page. This in turn means that $IH^*(Y_A)$ vanishes in odd degree, and we have a vector space isomorphism

$$IH^{2k}(Y_A) \cong \bigoplus_{p+q=2k} E_1^{p,q} \cong \bigoplus_{F} H^{2(k-\text{codim } F)} \left( IC_{Y_A F},\{0\} \right)$$

or equivalently

$$IH^*(Y_A) \cong \bigoplus_{F} H^* \left( IC_{Y_A F},\{0\} \right) [-2 \text{ codim } F].$$
KL positivity from the spectral sequence

Let

\[ \tilde{Z}_A(t) := \sum_{k \geq 0} \dim IH^{2k}(Y_A) t^k \quad \text{and} \quad \tilde{P}_A(t) := \sum_{k \geq 0} \dim H^{2k}(IC_{Y_A, \{0\}}) t^k. \]

The previous equation implies that

\[ \tilde{Z}_A(t) = \sum_F \tilde{P}_A^F(t) t^{\text{codim } F}. \]

Hard Lefschetz implies that \( \tilde{Z}_A(t) \) is palindromic, and general nonsense implies that \( \deg \tilde{P}_A(t) < \frac{d}{2} \) unless \( d = 0 \). Thus we must have

\[ \tilde{Z}_A(t) = Z_A(t) \quad \text{and} \quad \tilde{P}_A(t) = P_A(t). \]

In particular, \( P_A(t) \) has non-negative coefficients.
For arbitrary matroids, one can give a combinatorial definition of a vector space that stands in for $H^*(\text{IC}_{\mathcal{A},\{0\}})$. The hard work of BHMPW (in addition to hard Lefschetz) is showing that this vector space vanishes in degrees greater than or equal to $d$, i.e. that $\deg \tilde{P}_\mathcal{A}(t) < \frac{d}{2}$.

Thanks!