The Wonderful Geometry of Matroids

Lecture 10: The Poincaré polynomial

We stated the most fundamental theorem relating the combinatorics and topology of hyperplane arrangements. Specifically, let V be a complex vector space, $\varphi : E \to V$ any map, and $M := M(E, V)$. Let

$$U := V^* \smallsetminus \bigcup_{e \in E} H_e$$

be complement of the corresponding collection of hyperplanes and let

$$\pi(t) := \sum_{i \geq 0} t^i \dim H^i(U; \mathbb{Q})$$

be its Poincaré polynomial. Then

$$\pi(t) = (-t)^{\operatorname{rk} M} \chi_M(-t^{-1}).$$

1. Let U be the complement of a set of 6 hyperplanes in $\mathbb{C}^4$, arranged as generically as possible. Compute the Betti numbers (dimensions of cohomology groups) of U.

Hint: First quantify what it means for a set of hyperplanes to be arranged "as generically as possible".