The Wonderful Geometry of Matroids

Lecture 12: The Orlik-Solomon algebra

We defined the Orlik-Solomon algebra of a matroid $M = (E, \mathcal{I})$:

$$\mathrm{OS}^*(M) := \Lambda_{\mathbb{C}}[x_e \mid e \in E] \Big/ \left\langle \sum_{i=1}^{k} (-1)^i x_{e_1} \cdots \widehat{x_{e_i}} \cdots x_{e_k} \;\middle|\; \{e_1, \ldots, e_k\} \notin \mathcal{I} \right\rangle.$$

We showed that, if M is $\mathbb{C}$-realizable, the Orlik-Solomon algebra of M comes with a natural homomorphism to the cohomology of the complement of the corresponding hyperplane arrangement. We stated the theorem that this homomorphism is an isomorphism.

At this point, we had two lingering assertions that had not yet been proved: the fact that this homomorphism is an isomorphism and the fact that the long exact sequence from the previous lecture breaks into short exact sequences. We outlined a simultaneous proof of both assertions by induction on the size of the ground set.

1. Show that, if $\{e_1, \ldots, e_k\} \notin \mathcal{I}$, then $x_{e_1} \cdots x_{e_k} = 0 \in \mathrm{OS}^*(M)$.

2. Show that the ideal of relations in the Orlik-Solomon algebra is in fact generated by elements corresponding to circuits.

Bonus Exercise

3. Write down a presentation for the cohomology ring of the configuration space of 4 distinct points in $\mathbb{C}$. Your presentation should have $\binom{4}{2} = 6$ generators and $7 = 4 + 3$ relations corresponding to circuits of size 3 and 4 for the matroid $M(K_4)$. Can you show that the first 4 relations are in fact sufficient?