The Wonderful Geometry of Matroids
Lecture 15: Log concavity

Let $M$ be a matroid. Recall that the Poincaré polynomial
\[ \pi_M(t) := (-t)^{rk_M} \chi_M(-t^{-1}) \]
and reduced Poincaré polynomial
\[ \bar{\pi}_M(t) := (-t)^{rk_M-1} \bar{\chi}_M(-t^{-1}) \]
have non-negative coefficients (this was a previous exercise). Today we defined the notion of log concavity and stated the theorem of Adiprasito-Huh-Katz, which says that $\bar{\pi}_M(t)$ is log concave. We also explained why this implies that $\pi_M(t)$ is also log concave. This hinged on the following lemma.

**Lemma:** If $(a_i)$ is a log concave sequence, then $a_{i-1}a_i \geq a_{i-2}a_{i+1}$ for all $i$.

Define a sequence $(a_i)$ to be **strongly log concave** if, for all $i \leq j \leq k \leq l$ with $i + l = j + k$, we have $a_ja_k \geq a_ia_l$.

1. Prove that any log concave sequence is in fact strongly log concave.

**Hint:** First restrict to the case where $i = j - 1$. Prove it by induction on the quantity $k - j$, following the same strategy that we used in the proof of the lemma. Then do the general case using a second induction, this time on the quantity $j - i$. 