The Wonderful Geometry of Matroids

Lecture 14: Log concavity

Let M be a matroid. Recall that the **Poincaré polynomial**

$$\pi_M(t) := (-t)^{\operatorname{rk} M} \chi_M(-t^{-1})$$

and **reduced Poincaré polynomial**

$$\bar{\pi}_M(t) := (-t)^{\operatorname{rk} M - 1} \bar{\chi}_M(-t^{-1})$$

have positive coefficients (this was a previous exercise). Today we defined the notion of log concavity and stated the theorem of Adiprasito-Huh-Katz, which says that $\bar{\pi}_M(t)$ is log concave. We also explained why this implies that $\pi_M(t)$ is also log concave. This hinged on the following lemma.

Lemma: If (a_i) is a log concave sequence of positive numbers, then $a_{i-1}a_i \geq a_{i-2}a_{i+1}$ for all i.

Define a sequence (a_i) to be **strongly log concave** if, for all $i \leq j \leq k \leq l$ with $i + l = j + k$, we have $a_j a_k \geq a_i a_l$.

1. Prove that any log concave sequence is in fact strongly log concave.

Hint: First restrict to the case where $i = j - 1$. Prove it by induction on the quantity $k - j$, following the same strategy that we used in the proof of the lemma. Then do the general case using a second induction, this time on the quantity $j - i$.