The Wonderful Geometry of Matroids

Lecture 17: The Chow ring

Let $M = (E, \mathcal{I})$ be a simple matroid. We defined the **Chow ring**

$$A^*(M) := \mathbb{R}[x_F \mid F \in L(M), \emptyset \neq F \neq E] \Big/ \left\langle x_F x_G \text{ if } F \text{ and } G \text{ are incomparable; } \sum_{e \in F} x_F - \sum_{f \in F} x_F \text{ if } e, f \in E \right\rangle.$$

Consider the uniform matroid $U_{3,4}$ of rank 3 on the ground set $\{1, 2, 3, 4\}$. The Chow ring $A^*(U_{3,4})$ has generators $x_1, \ldots, x_4$ and $x_{12}, \ldots, x_{34}$.

1. Show that $x_1 x_{12} = x_2 x_{12}$ and $x_3 x_{13} = x_3 x_{23}$.

Hint: Start with the linear relation with $e = 1$ and $f = 2$ and multiply by a well-chosen generator.

2. Use Problem 1 to conclude that $A^2(M)$ is at most 1-dimensional.