The Wonderful Geometry of Matroids

Lecture 2: Deletion, contraction, and dualization

In this lecture we defined the matroid operations of deletion and contraction, and we interpreted these operations for vector configurations and graphs. We also defined the dual of a vector configuration, and showed that bases for the dual configuration are exactly complements of bases for the original configuration.

1. Let $M = (E, \mathcal{I})$ be a matroid and suppose that $e \in E$ is not a loop. Show that $(M/e)^* = M^* \setminus e$.

2. Let G be the graph obtained from the complete graph K_4 by deleting a single edge.

(i) Draw the planar dual G^* of G.
(ii) Note that there is a natural bijection between the edge sets of G and G^*. Convince yourself that this bijection induces an identification $M(G)^* = M(G^*)$.
(iii) In general, the dual of a matroid associated with a planar graph coincides with the matroid associated with the planar dual. What happens if you start with a non-planar graph, such as the complete bipartite graph $K_{3,3}$? That is, is the dual of $M(K_{3,3})$ isomorphic to $M(G)$ for some graph G?

Bonus Exercise

3. Try to prove that the dual of a matroid really is a matroid. That is, let $M = (E, \mathcal{I})$ be a matroid. Let $\mathcal{B}$ be the set of bases (maximal independent sets) of M. Let

$$\mathcal{I}^* := \{S \subset E \mid S \cap B = \emptyset \text{ for some } B \in \mathcal{B}\}.$$

Show that $M^* := (E, \mathcal{I}^*)$ satisfies the three matroid axioms.