**The Wonderful Geometry of Matroids**

**Lecture 2: Deletion, contraction, and dualization**

In this lecture we defined the matroid operations of deletion and contraction, and we interpreted these operations for vector configurations and graphs. We also defined the dual of a vector configuration, and showed that bases for the dual configuration are exactly complements of bases for the original configuration.

1. Let $M = (E, \mathcal{I})$ be a matroid and suppose that $e \in E$ is not a loop. Show that $(M/e)^* = M^* \setminus e$.

**Bonus Exercises**

2. Go through the proof that the dual of a matroid really is a matroid. That is, let $M = (E, \mathcal{I})$ be a matroid. Let $\mathcal{B}$ be the set of bases (maximal independent sets) of $M$. Let

$$\mathcal{B}^* := \{S \subset E \mid E \setminus S \in \mathcal{B}\}$$

and

$$\mathcal{I}^* := \{T \subset E \mid T \subset S \text{ for some } S \in \mathcal{B}^*\}.$$

Show that $M^* := (E, \mathcal{I}^*)$ satisfies the three matroid axioms.

3. Let $G$ be the graph obtained from the complete graph $K_4$ by deleting a single edge.

   (i) Draw the planar dual $G^*$ of $G$.
   (ii) Note that there is a natural bijection between the edge sets of $G$ and $G^*$. Convince yourself that this bijection induces an identification $M(G)^* = M(G^*)$.
   (iii) In general, the dual of a matroid associated with a planar graph coincides with the matroid associated with the planar dual. What happens if you start with a non-planar graph, such as the complete bipartite graph $K_{3,3}$? That is, is the dual of $M(K_{3,3})$ isomorphic to $M(G)$ for some graph $G$?