The Wonderful Geometry of Matroids

Lecture 25: Maximizing correlation

For any prime power q, we constructed an $\mathbb{F}_q$-realizable matroid M and a pair of elements $e \neq f$ in the ground set of M with $\alpha(M, e, f) = 8/7$. We also defined transversal matroids, and considered the following example. Let

$$E := \{e, f, x_{ij} \mid 2 \leq i \leq d,\ 1 \leq j \leq m\}.$$

Let $A_1 = E$ and $A_j = \{f\} \cup \{x_{ij} \mid 2 \leq i \leq d\}$. Let M be the associated transversal matroid.

1. Show that $b_{ef} = (d-1)m^{d-2}$.

2. Show that $b^{ef} = (d-1)\binom{m}{2}m^{d-2}$.

3. Show that $b_e^f = m^{d-1}$.

4. Show that $b_f^e = m^{d-1} + (d-1)(d-2)\binom{m}{2}m^{d-3}$.

5. Combining the first four problems, show that

$$\lim_{m \to \infty} \alpha(M, e, f) = \frac{d^2 - 2d + 1}{d^2 - 3d + 4}.$$

This is maximized when $d = 5$, in which case it is equal to $8/7$.

Bonus Exercise

6. We did not actually prove that transversal matroids satisfy the axioms of a matroid. However, by imitating the construction in the lecture, you can prove that **this particular** transversal matroid is indeed a matroid realized by $2 + m(d-1)$ fectors in $\mathbb{F}^d$ for any sufficiently large field $\mathbb{F}$. Do it!