The Wonderful Geometry of Matroids

Lecture 26: Intersection cohomology

We discussed some basic properties of ℓ-adic étale intersection cohomology of varieties over $\mathbb{F}_q$. We always assume that all odd cohomology groups vanish and that the Frobenius endomorphism acts on IH^{2k} by multiplication by q^k. We write

$$P_X(t) := \sum_{k \geq 0} t^k \dim \mathrm{IH}^{2k}(X) \qquad \text{and} \qquad P_{X,p}(t) := \sum_{k \geq 0} t^k \dim \mathrm{IH}_p^{2k}(X),$$

where the group on the right is the cohomology of the stalk of IC_X at the point p. We stated the **Lefchetz formula**, which says that

$$q^{\dim X} P_X(q^{-1}) = \sum_{p \in X} P_{X,p}(q),$$

and we used this formula to compute the intersection cohomology of the space of $n \times n$ matrices of rank ≤ 1.

1. Suppose that X is a smooth, nonempty, connected, projective variety of dimension $n-1$. Use the Lefschetz formula to compute $P_{\mathrm{cone}(X)}(t)$ in terms of $P_X(t)$. Show that the answer that you get agrees with the answer predicted by the statement that the intersection cohomology of $\mathrm{cone}(X)$ is the primitive part of the cohomology of X.