The Wonderful Geometry of Matroids
Lecture 26: Intersection cohomology

We discussed some basic properties of $\ell$-adic étale intersection cohomology of varieties over $\mathbb{F}_q$. We always assume that all odd cohomology groups vanish and that the Frobenius endomorphism acts on $IH^{2k}$ by multiplication by $q^k$. We write

$$P_X(t) := \sum_{k \geq 0} t^k \dim IH^{2k}(X) \quad \text{and} \quad P_{X,p}(t) := \sum_{k \geq 0} t^k \dim IH^{2k}_p(X),$$

where the group on the right is the cohomology of the stalk of $IC_X$ at the point $p$. We stated the Lefschetz formula, which says that

$$q^{\dim X} P_X(q^{-1}) = \sum_{p \in X} P_{X,p}(q),$$

and we used this formula to compute the intersection cohomology of the space of $n \times n$ matrices of rank $\leq 1$.

1. Suppose that $X$ is a smooth, nonempty, connected, projective variety of dimension $n - 1$. Use the Lefschetz formula to compute $P_{\text{cone}(X)}(t)$ in terms of $P_X(t)$. Show that the answer that you get agrees with the answer predicted by the statement that the intersection cohomology of cone($X$) is the primitive part of the cohomology of $X$. 