The Wonderful Geometry of Matroids

Lecture 6: The lattice of flats

In this lecture we showed that the poset of flats is a geometric lattice.

Let L be a finite lattice. Let $A(L)$ be the ring with $\mathbb{Z}$-basis $\{\sigma_x \mid x \in L\}$ and multiplication

$$\sigma_x \cdot \sigma_y = \delta_{xy}\sigma_x.$$

In other words, this algebra is just a direct sum of copies of $\mathbb{Z}$, one for each element of L. (Boring!)

Let $B(L)$ be the ring with $\mathbb{Z}$-basis $\{\epsilon_x \mid x \in L\}$ and multiplication

$$\epsilon_x \cdot \epsilon_y = \epsilon_{x \vee y}.$$

This ring, which seems much more interesting, is called the **Möbius ring** of L.

1. Show that the Möbius ring $B(L)$ is in fact isomorphic to the boring ring $A(L)$.

Hint: Write down an explicit isomorphism from $B(L)$ to $A(L)$ by specifying where each ϵ_x should go and showing that this map is compatible with multiplication.

2. Can you write down the inverse isomorphism? Try it in the example where L is the lattice of flats of the matroid $M(K_3)$.

Bonus Exercise

3. Let $M = (E, \mathcal{I})$ be a matroid. Prove that $F \subset E$ is a flat if and only if, for every circuit $C \subset E$,

$$|F \cap C| \neq |C| - 1.$$