The Wonderful Geometry of Matroids
Lecture 6: The lattice of flats

In this lecture we showed that the poset of flats is a geometric lattice.

Let \( L \) be a finite lattice. Let \( A(L) \) be the ring with \( \mathbb{Z} \)-basis \( \{ \sigma_x \mid x \in L \} \) and multiplication

\[
\sigma_x \cdot \sigma_y = \delta_{xy} \sigma_x.
\]

In other words, this algebra is just a direct sum of copies of \( \mathbb{Z} \), one for each element of \( L \). (Boring!)

Let \( B(L) \) be the ring with \( \mathbb{Z} \)-basis \( \{ \epsilon_x \mid x \in L \} \) and multiplication

\[
\epsilon_x \cdot \epsilon_y = \epsilon_{x \lor y}.
\]

This ring, which seems much more interesting, is called the \textbf{Möbius ring} of \( L \).

1. Show that the Möbius algebra \( B(L) \) is in fact isomorphic to the boring ring \( A(L) \).

Hint: Write down an explicit isomorphism from \( B(L) \) to \( A(L) \) by specifying where each \( \epsilon_x \) should go and showing that this map is compatible with multiplication.

\textbf{Bonus Exercise}

2. Can you write down the inverse isomorphism? Try it in the example where \( L \) is the lattice of flats of the matroid \( M(K_3) \).