# The Wonderful Geometry of Matroids

## Lecture 8: The characteristic polynomial

In this lecture we defined the characteristic polynomial

$$\chi_M(t) := \sum_{S \subset E} (-1)^{|S|} t^{\operatorname{crk} S}$$

and the reduced characteristic polynomial $\bar{\chi}_M(t) := \chi_M(t)/(t-1)$. We proved that

$$\chi_{M_1 \oplus M_2}(t) = \chi_{M_1}(t)\chi_{M_2}(t) \qquad \text{and} \qquad \bar{\chi}_{\operatorname{tr} M}(t) = \frac{\bar{\chi}_M(t) - \bar{\chi}_M(0)}{t}.$$

If $M$ has a loop, we have $\chi_M(t) = 0$. If $e$ is a coloop of $M$, we have $\chi_M(t) = (t-1)\chi_{M \setminus e}(t)$. If $e$ is neither a loop nor a coloop, we showed that

$$\chi_M(t) = \chi_{M \setminus e}(t) - \chi_{M/e}(t).$$

1. Recall that two elements $e$ and $f$ of the ground set of a matroid $M$ are said to be **parallel** if $\{e\}$ and $\{f\}$ are independent but $\{e, f\}$ is dependent. If this is the case, show that $\chi_M(t) = \chi_{M \setminus e}(t)$. Thus we can always reduce to a simple matroid when computing the characteristic polynomial.

2. Let $G$ be the graph obtained by deleting an edge from $K_4$. Compute $\chi_{M(G)}(t)$.

3. Prove that, if $M$ has no loops, the coefficients of $\chi_M(t)$ alternate in sign. Equivalently, the coefficients of $(-1)^{\operatorname{rk} M}\chi_{M(G)}(-t)$ are all positive. Do the same for $\bar{\chi}_M(t)$ (with the opposite sign).

Hint: Induct on the size of the ground set using the deletion/contraction formula.