THE PARITY MODULE OF A MATROID

TOM BRADEN, JUNE HUH, JACOB P. MATHERNE, NICHOLAS PROUDFOOT, AND BOTONG WANG

ABSTRACT. We introduce the parity module of a matroid with coefficients in a field $\Bbbk$. When $\Bbbk$ has characteristic zero, we show that the parity module of a matroid is isomorphic to its intersection cohomology module, yielding a new intrinsic characterization of the latter. The structure and dimension of the parity module can vary depending on the characteristic, giving rise to characteristic-dependent generalizations of the Kazhdan–Lusztig and Z-polynomials of matroids. When the matroid is realizable over the complex numbers, the parity module is isomorphic to the hypercohomology of the parity sheaf supported on the arrangement Schubert variety of any such realization. We also introduce the notion of pure modules over any field $\Bbbk$ and show that they satisfy a matroid analogue of the Beilinson–Bernstein–Deligne–Gabber decomposition theorem: Every pure module over the graded Möbius algebra decomposes as a direct sum of shifts of the parity modules of matroids obtained by restricting to flats.

1. INTRODUCTION

Parity sheaves were introduced by Juteau, Mautner, and Williamson in 2014, and have since become fundamental objects in modular geometric representation theory [JMW14]. They provide a well-behaved substitute for intersection cohomology sheaves with coefficients in a field of positive characteristic, where the usual formalism can fail in important ways. In particular, parity sheaves retain many of the structural features familiar from characteristic zero, including an analogue of the Beilinson–Bernstein–Deligne–Gabber Decomposition Theorem in [BBD82].

The intersection cohomology module of a matroid was introduced in [BHM$^+$] and played a key role there in the proof of the Dowling–Wilson top-heavy conjecture [DW74], as well as in the proof of the nonnegativity of Kazhdan–Lusztig polynomials of matroids conjectured in [EPW16]. In this paper, we introduce the parity module of a matroid (Theorem A). This object is defined over any coefficient field and agrees with the intersection cohomology module when the coefficient field has characteristic zero (Theorem 1.4). Thus, in addition to giving a new characterization of the intersection cohomology module of a matroid, our construction initiates the study of positive characteristic analogues of Kazhdan–Lusztig and Z-polynomials of matroids.

1.1. **Pure modules.** Let $\mathcal{L}$ be a geometric lattice with minimal element $\hat{0}$ and maximal element $\hat{1}$. This means that $\mathcal{L}$ is a finite graded lattice that is **atomic**, meaning that every element is a join of

atoms, and **semimodular**, meaning that

$$\mathrm{rk}(F) + \mathrm{rk}(G) \geqslant \mathrm{rk}(F \vee G) + \mathrm{rk}(F \wedge G) \text{ for all } F, G \in \mathcal{L}.$$

Equivalently, $\mathcal{L}$ is the lattice of flats of a matroid [Oxl11, Theorem 1.7.5]. For any field $\Bbbk$, the **graded Möbius algebra** $\mathrm{H}(\mathcal{L};\Bbbk)$ is a $\Bbbk$-algebra with additive basis $\{y_F \mid F \in \mathcal{L}\}$, where y_F is homogeneous of degree $\mathrm{rk}\, F$, with multiplication given by the rule

$$y_F y_G \coloneqq \begin{cases} y_{F\vee G} & \text{if } \mathrm{rk}\, F + \mathrm{rk}\, G = \mathrm{rk}(F \vee G), \\ 0 & \text{if } \mathrm{rk}\, F + \mathrm{rk}\, G > \mathrm{rk}(F \vee G). \end{cases}$$

The element $y_{\hat{0}}$ is the identity element of $\mathrm{H}(\mathcal{L};\Bbbk)$. Whenever we use the word "module", we will always mean a module that is finitely generated and $\mathbb{Z}$-graded. Likewise, the word "module homomorphism" will always mean a degree-preserving homomorphism of graded modules. We write $\mathrm{rk}\,\mathcal{L}$ for the rank of the maximal element of $\mathcal{L}$.

A subset Σ of $\mathcal{L}$ is called an **upper set** if it is upwardly closed, meaning that, whenever we have $F \leqslant G$ and $F \in \Sigma$, we also have $G \in \Sigma$. Given an upper set Σ and an $\mathrm{H}(\mathcal{L};\Bbbk)$-module M, we define submodules

$$\mathrm{ann}_\Sigma(\mathrm{M}) \coloneqq \{m \in \mathrm{M} \mid y_F \cdot m = 0 \text{ for all } F \in \Sigma\} \text{ and } \mathrm{span}_\Sigma(\mathrm{M}) \coloneqq \Bbbk\{y_F \cdot m \mid F \in \Sigma \text{ and } m \in \mathrm{M}\}.$$

These two constructions allow us to define a notion of purity for $\mathrm{H}(\mathcal{L};\Bbbk)$-modules.

Definition 1.1. Let M be an $\mathrm{H}(\mathcal{L};\Bbbk)$-module.

(a) We call M **flabby** if $\mathrm{ann}_{\Sigma_1}(\mathrm{M}) + \mathrm{ann}_{\Sigma_2}(\mathrm{M}) = \mathrm{ann}_{\Sigma_1\cap\Sigma_2}(\mathrm{M})$ for any upper sets Σ_1 and Σ_2.

(b) We call M **coflabby** if $\mathrm{span}_{\Sigma_1}(\mathrm{M}) \cap \mathrm{span}_{\Sigma_2}(\mathrm{M}) = \mathrm{span}_{\Sigma_1\cap\Sigma_2}(\mathrm{M})$ for any upper sets Σ_1 and Σ_2.

(c) If M is both flabby and coflabby, we say that M is **pure**.

For an $\mathrm{H}(\mathcal{L};\Bbbk)$-module $\mathrm{M} = \bigoplus_i \mathrm{M}^i$, we write M^* for its **dual module**, whose degree i piece is $\mathrm{Hom}_\Bbbk(\mathrm{M}^{-i},\Bbbk)$. Note that the flabby and coflabby conditions are interchanged by duality: if M is flabby, then M^* is coflabby, and vice versa. Note also that we always have

$$\mathrm{ann}_{\Sigma_1}(\mathrm{M}) + \mathrm{ann}_{\Sigma_2}(\mathrm{M}) \subseteq \mathrm{ann}_{\Sigma_1\cap\Sigma_2}(\mathrm{M}) \text{ and } \mathrm{span}_{\Sigma_1}(\mathrm{M}) \cap \mathrm{span}_{\Sigma_2}(\mathrm{M}) \supseteq \mathrm{span}_{\Sigma_1\cap\Sigma_2}(\mathrm{M});$$

the opposite inclusions characterize flabbiness and coflabbiness.

Our first main results are as follows (see the beginning of Section 1.7 for a discussion of the distinction between lettered and numbered theorems).

Theorem A. Up to isomorphism of $\mathrm{H}(\mathcal{L};\Bbbk)$-modules, there is a unique $\mathrm{H}(\mathcal{L};\Bbbk)$-module $\mathrm{E}(\mathcal{L};\Bbbk)$ satisfying the following three conditions:

(a) $\mathrm{E}(\mathcal{L};\Bbbk)$ is pure,

(b) $\mathrm{E}(\mathcal{L};\Bbbk)$ is indecomposable,

(c) there is an element $x \in \mathrm{E}^0(\mathcal{L};\Bbbk)$ such that $y_{\hat{1}} \cdot x$ is nonzero.

Theorem B. This module $\mathrm{E}(\mathcal{L};\Bbbk)$ has the following additional properties:

(a) $\mathrm{E}^i(\mathcal{L};\Bbbk) = 0$ if $i < 0$ or $i > \operatorname{rk}\mathcal{L}$,

(b) $\mathrm{E}^0(\mathcal{L};\Bbbk)$ and $\mathrm{E}^{\operatorname{rk}\mathcal{L}}(\mathcal{L};\Bbbk)$ are 1-dimensional $\Bbbk$-vector spaces.

(c) If $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$, then $\mathrm{E}(\mathcal{L};\Bbbk) \cong \mathrm{E}(\mathcal{L}_1;\Bbbk) \otimes \mathrm{E}(\mathcal{L}_2;\Bbbk)$ as modules over the graded Möbius algebra $\mathrm{H}(\mathcal{L};\Bbbk) \cong \mathrm{H}(\mathcal{L}_1;\Bbbk) \otimes \mathrm{H}(\mathcal{L}_2;\Bbbk)$.

The module $\mathrm{E}(\mathcal{L};\Bbbk)$ is called the **parity module** of $\mathcal{L}$ with coefficients in $\Bbbk$. See Remark 1.32 for a more explicit way of expressing the indecomposability condition (b) in Theorem A, in terms of costalk-to-stalk homomorphisms. In addition to this abstract characterization, we will give an inductive construction of $\mathrm{E}(\mathcal{L};\Bbbk)$ using elementary linear algebra.

Remark 1.2. The terms flabby and coflabby refer to a sheaf and a cosheaf on $\mathcal{L}$ which can be associated with an $\mathrm{H}(\mathcal{L};\Bbbk)$-module M, where $\mathcal{L}$ is given the topology with upper sets as open sets. This point of view is important for the proof of Theorem A that begins in Section 6, but it is not needed for the rest of the introduction or for the applications in the earlier sections.

Remark 1.3. In [BHM$^+$], we gave a different definition of purity for modules over $\mathrm{H}(\mathcal{L};\mathbb{Q})$. See Proposition 8.16 for a proof that the two definitions agree.

Suppose that $\mathcal{L}$ is the lattice of flats of a matroid, and that this matroid is realizable by a linear subspace $V \subseteq \mathbb{C}^{\mathcal{L}^1}$, where $\mathcal{L}^1$ is the set of atoms of $\mathcal{L}$. The **arrangement Schubert variety** Y_V of V is by definition the closure of V inside of $(\mathbb{CP}^1)^{\mathcal{L}^1}$, and it has the property that there exists a degree-doubling isomorphism $\mathrm{H}(\mathcal{L};\Bbbk) \cong \mathrm{H}^\bullet(Y_V;\Bbbk)$ [HW17]. Let $\mathcal{E}(Y_V;\Bbbk)$ denote the parity sheaf of Y_V with coefficients in $\Bbbk$ [JMW14]. The following theorem justifies our use of the name "parity module".

Theorem C. If $\mathcal{L}$ is the lattice of flats of a matroid realized by a linear subspace $V \subseteq \mathbb{C}^{\mathcal{L}^1}$, then there exists a degree-doubling isomorphism $\mathrm{E}(\mathcal{L};\Bbbk) \cong \mathbb{H}^\bullet(\mathcal{E}(Y_V;\Bbbk))$ as modules over the ring $\mathrm{H}(\mathcal{L};\Bbbk) \cong \mathrm{H}^\bullet(Y_V;\Bbbk)$.

The **intersection cohomology** $\mathrm{H}(\mathcal{L};\mathbb{Q})$-module $\mathrm{IH}(\mathcal{L})$ was introduced in [BHM$^+$], where it was the key character in the proof of the Dowling–Wilson conjecture for $\mathcal{L}$. The definition of $\mathrm{IH}(\mathcal{L})$ given there is rather complicated [BHM$^+$, Section 3], and involves taking the orthogonal complement of a recursively defined submodule of the **augmented Chow ring** $\mathrm{CH}(\mathcal{L};\mathbb{Q})$ introduced in [BHM$^+$22]. (See Section 4 for a review of the augmented Chow ring.) The following theorem provides a direct and intrinsic characterization of the module $\mathrm{IH}(\mathcal{L})$.

Theorem 1.4. There is an isomorphism of $\mathrm{H}(\mathcal{L};\mathbb{Q})$-modules $\mathrm{E}(\mathcal{L};\mathbb{Q}) \cong \mathrm{IH}(\mathcal{L})$, and this isomorphism is unique up to multiplication by a nonzero scalar in $\mathbb{Q}$.

Remark 1.5. As in the setting of Theorem C, suppose that $\mathcal{L}$ comes from a linear subspace $V \subseteq \mathbb{C}^E$. If $\Bbbk = \mathbb{Q}$, the parity sheaf of Y_V with coefficients in $\Bbbk$ coincides with the intersection cohomology sheaf. Thus, combining Theorems C and 1.4, we obtain the statement that $\mathrm{IH}(\mathcal{L})$ is isomorphic to the intersection cohomology module $\mathrm{IH}^\bullet(Y_V;\Bbbk)$. This is implicit in [BHM+, Section 1.3], but it is not stated or proved explicitly.

For any $F \in \mathcal{L}$, we consider the **lower interval** $\mathcal{L}^F$ and the **upper interval** $\mathcal{L}_F$ defined by

$$\mathcal{L}^F \coloneqq \{G \in \mathcal{L} \mid G \leqslant F\} \text{ and } \mathcal{L}_F \coloneqq \{G \in \mathcal{L} \mid F \leqslant G\}.$$

The former corresponds to the restriction of the matroid of $\mathcal{L}$ to the flat F, while the latter corresponds to the contraction of the matroid of $\mathcal{L}$ by the flat F. Note that the inclusion of $\mathcal{L}^F$ into $\mathcal{L}$ is rank-preserving, while the inclusion of $\mathcal{L}_F$ into $\mathcal{L}$ increases rank by $\operatorname{rk} F$. There is a surjective homomorphism $\mathrm{H}(\mathcal{L};\Bbbk) \to \mathrm{H}(\mathcal{L}^F;\Bbbk)$ taking y_G to y_G if $G \leqslant F$ and to 0 otherwise. Thus an $\mathrm{H}(\mathcal{L}^F;\Bbbk)$-module can be given an induced $\mathrm{H}(\mathcal{L};\Bbbk)$-module structure.

Theorem D. Up to isomorphism and degree shift, the modules $\mathrm{E}(\mathcal{L}^F;\Bbbk)$ for $F \in \mathcal{L}$ give a complete set of indecomposable pure $\mathrm{H}(\mathcal{L};\Bbbk)$-modules. Consequently, every pure $\mathrm{H}(\mathcal{L};\Bbbk)$-module decomposes as a direct sum of shifts of these modules.

Example 1.6. The augmented Chow ring $\mathrm{CH}(\mathcal{L};\Bbbk)$ is a pure module over $\mathrm{H}(\mathcal{L};\Bbbk)$ (Proposition 4.5(e)), and therefore decomposes as a direct sum of shifts of the parity modules $\mathrm{E}(\mathcal{L}^F;\Bbbk)$. The shape of this decomposition reflects an interesting interaction between the combinatorics of $\mathcal{L}$ and the characteristic of $\Bbbk$. For example, when $\mathcal{L}$ is the lattice of flats of the uniform matroid $U_{2,3}$, we have

$$\mathrm{CH}(\mathcal{L};\Bbbk) \cong \begin{cases} \mathrm{E}(\mathcal{L};\Bbbk) & \text{if } \operatorname{char}(\Bbbk) = 2, \\ \mathrm{E}(\mathcal{L};\Bbbk) \oplus \mathrm{E}(\mathcal{L}^{\hat{0}};\Bbbk)[-1] & \text{if } \operatorname{char}(\Bbbk) \neq 2. \end{cases}$$

In Theorem F, we describe how an arbitrary pure module decomposes into shifts of parity modules using costalk-stalk homomorphisms.

1.2. **Polynomials.** Consider the **Z-polynomial** $Z_{\mathcal{L}}(t)$, introduced in [PXY18], and the **KL-polynomial** $P_{\mathcal{L}}(t)$, introduced in [EPW16]. The **stalk** of the parity module of $\mathcal{L}$ at $\hat{0}$ is the graded vector space

$$\underline{\mathrm{E}}(\mathcal{L};\Bbbk) \coloneqq \mathrm{E}(\mathcal{L};\Bbbk) \big/ \operatorname{span}_{\mathcal{L}\setminus\hat{0}} \mathrm{E}(\mathcal{L};\Bbbk).$$

By Theorem 1.4 along with [BHM+, Theorem 1.19], we have

$$Z_{\mathcal{L}}(t) = \sum_{i \geqslant 0} t^i \dim \mathrm{E}^i(\mathcal{L};\mathbb{Q}) \text{ and } P_{\mathcal{L}}(t) = \sum_{i \geqslant 0} t^i \dim \underline{\mathrm{E}}^i(\mathcal{L};\mathbb{Q}). \tag{1}$$

By analogy with classical Kazhdan–Lusztig theory, for any prime p, we define the ***p*-Z-polynomial** and ***p*-KL-polynomial**

$$ {}_pZ_{\mathcal{L}}(t) \coloneqq \sum_{i \geqslant 0} t^i \dim \mathrm{E}^i(\mathcal{L}; \mathbb{F}_p) \text{ and } {}_pP_{\mathcal{L}}(t) \coloneqq \sum_{i \geqslant 0} t^i \dim \underline{\mathrm{E}}^i(\mathcal{L}; \mathbb{F}_p). \tag{2} $$

We will show in Corollary 9.8 that field extensions do not affect the structure of parity modules. Thus, in fact, Equation (1) holds with $\mathbb{Q}$ replaced with any field of characteristic zero, and Equation (2) holds with $\mathbb{F}_p$ replaced with any field of characteristic p. By convention, we will write

$$ {}_0Z_{\mathcal{L}}(t) = Z_{\mathcal{L}}(t) \text{ and } {}_0P_{\mathcal{L}}(t) = P_{\mathcal{L}}(t), $$

and we will sometimes write $\mathbb{F}_0 = \mathbb{Q}$ so that Equation (2) holds when $p = 0$.

Theorem 1.7. Let p be a prime number or zero. The p-Z-polynomial and p-KL-polynomial satisfy the following properties:

(a) If $\operatorname{rk} \mathcal{L} = 0$, then ${}_pP_{\mathcal{L}}(t) = 1$.
(b) For all $\mathcal{L}$, the p-Z-polynomial is palindromic: ${}_pZ_{\mathcal{L}}(t) = t^{\operatorname{rk} \mathcal{L}} {}_pZ_{\mathcal{L}}(t^{-1})$.
(c) For all $\mathcal{L}$, we have ${}_pZ_{\mathcal{L}}(t) = \sum_{F \in \mathcal{L}} t^{\operatorname{rk} F} {}_pP_{\mathcal{L}_F}(t)$.

In [BHM+], it was shown that $\deg {}_0P_{\mathcal{L}}(t) < (\operatorname{rk} \mathcal{L})/2$ for all $\mathcal{L}$ with $\operatorname{rk} \mathcal{L} > 0$. This inequality, along with the three properties of Theorem 1.7, completely characterizes the polynomials ${}_0Z_{\mathcal{L}}(t)$ and ${}_0P_{\mathcal{L}}(t)$ [BV20, Theorem 2.2]. For positive p, the degree of ${}_pP_{\mathcal{L}}(t)$ can be larger than $(\operatorname{rk} \mathcal{L})/2$, and Theorem 1.7 is not sufficient to compute the polynomials ${}_pZ_{\mathcal{L}}(t)$ and ${}_pP_{\mathcal{L}}(t)$. We do not know of a purely combinatorial algorithm to compute these polynomials, except for the linear term — see Proposition 1.9 below.

Remark 1.8. It follows from Theorem 1.4 and the hard Lefschetz property for $\mathrm{IH}(\mathcal{L})$ of [BHM+, Theorem 1.6] that the coefficients of $Z_{\mathcal{L}}(t)$ form a unimodal sequence. Do the coefficients of ${}_pZ_{\mathcal{L}}(t)$ form a unimodal sequence for p positive?

If $f(t), g(t) \in \mathbb{N}[t]$, we write $f(t) \leqslant g(t)$ to mean that $g(t) - f(t) \in \mathbb{N}[t]$. Theorem E below asserts that working in positive characteristic can only result in larger Z- and KL-polynomials. The analogous result for classical Kazhdan–Lusztig polynomials can be found in [JW17, Proposition 4.2(3)].

Theorem E. If p is prime, we have

$$ {}_0Z_{\mathcal{L}}(t) \leqslant {}_pZ_{\mathcal{L}}(t) \text{ and } {}_0P_{\mathcal{L}}(t) \leqslant {}_pP_{\mathcal{L}}(t), $$

with equality for all but finitely many primes.

We define an element F in $\mathcal{L}$ to be ***p*-resonant** if F is not $\hat{1}$ and, for all $G > F$ with $\mathrm{rk}(G) = \mathrm{rk}(F) + 2$, the cardinality of the open interval (F, G) is congruent to 1 modulo p. The only 0-resonant elements are those of corank 1, for which there are no such G. However, there may be p-resonant elements of corank larger than 1. For example, every proper flat of the Fano matroid is 2-resonant, and every nonempty proper flat of the ternary affine plane is 3-resonant.

Proposition 1.9. The coefficient of t in ${}_pZ_{\mathcal{L}}(t)$ is equal to the number of p-resonant elements of $\mathcal{L}$. The coefficient of t in ${}_pP_{\mathcal{L}}(t)$ is equal to the number of p-resonant elements of $\mathcal{L}$ minus the number of rank 1 elements of $\mathcal{L}$.

1.3. **Relation with the augmented Chow ring.** Recall from Theorem 1.4 that $\mathrm{E}(\mathcal{L};\mathbb{Q})$ is isomorphic to $\mathrm{IH}(\mathcal{L})$, which is a submodule of the augmented Chow ring $\mathrm{CH}(\mathcal{L};\mathbb{Q})$ containing the graded Möbius algebra $\mathrm{H}(\mathcal{L};\mathbb{Q})$. The following proposition gives an analogous pair of inclusions over an arbitrary coefficient field.

Proposition 1.10. For any geometric lattice $\mathcal{L}$ and field $\Bbbk$, there are inclusions of $\mathrm{H}(\mathcal{L};\Bbbk)$-modules

$$\mathrm{H}(\mathcal{L};\Bbbk) \subseteq \mathrm{E}(\mathcal{L};\Bbbk) \subseteq \mathrm{CH}(\mathcal{L};\Bbbk).$$

The graded dimensions of $\mathrm{H}(\mathcal{L};\Bbbk)$ and $\mathrm{CH}(\mathcal{L};\Bbbk)$ depend only on $\mathcal{L}$, not on $\Bbbk$: The dimension of $\mathrm{H}^k(\mathcal{L};\Bbbk)$ is the number of rank k elements of $\mathcal{L}$, while the Hilbert series of $\mathrm{CH}(\mathcal{L};\Bbbk)$ is the **augmented Chow polynomial** $C_{\mathcal{L}}(t)$ studied in [FMSV24]. When $\Bbbk = \mathbb{Q}$, the first inclusion, together with the hard Lefschetz property of $\mathrm{E}(\mathcal{L};\Bbbk)$, was used in [BHM+] to prove the Dowling–Wilson conjecture:

$$|\mathcal{L}^k| \leqslant |\mathcal{L}^{d-k}| \text{ for all } k \leqslant d/2 = \mathrm{rk}\,\mathcal{L}/2.$$

The second inclusion is built into the construction of $\mathrm{IH}(\mathcal{L})$ in [BHM+].

In the next two theorems, we specify exactly when each of these inclusions is an equality. A simple, connected, modular matroid of rank greater than 1 is called a **projective geometry** [Wel76, Chapter 12]. Given a projective geometry, there exists an integer $q \geqslant 2$ such that every rank 2 flat has cardinality $q+1$; this number is called the **order**. Examples of projective geometries include the uniform matroid $U_{2,n}$ for $n \geqslant 3$, which has order $n-1$, the projective geometry $\mathbb{P}^{d-1}(\mathbb{F}_q)$ for a prime power q, which has order q, and all non-Desarguesian projective planes. We write $\mathbb{P}^{d-1}(\mathbb{F}_q)$ to denote both the matroid and its lattice of flats.

Theorem 1.11. The following conditions are equivalent for $\mathcal{L}$ and $\Bbbk$, with p the characteristic of $\Bbbk$:

(a) The $\mathrm{H}(\mathcal{L};\Bbbk)$-module $\mathrm{H}(\mathcal{L};\Bbbk)$ is pure;

(b) $\mathrm{E}(\mathcal{L};\Bbbk) \cong \mathrm{H}(\mathcal{L};\Bbbk)$ as $\mathrm{H}(\mathcal{L};\Bbbk)$-modules;

(c) ${}_pP_{\mathcal{L}}(t) = 1$;

(d) $\mathcal{L}$ is modular and every p-resonant element of $\mathcal{L}$ has corank 1;

(e) $\mathcal{L}$ is isomorphic to a product of a Boolean lattice and a collection of finite projective spaces, each with an order that is not a multiple of p.

The augmented Chow ring $\mathrm{CH}(\mathcal{L};\Bbbk)$ has as a quotient the **Chow ring** $\underline{\mathrm{CH}}(\mathcal{L};\Bbbk)$; see [AHK18] and [BHM+22]. The **Chow polynomial** $\underline{C}_{\mathcal{L}}(t)$, which depends only on $\mathcal{L}$ and not on $\Bbbk$, is defined to be the Poincaré polynomial of $\underline{\mathrm{CH}}(\mathcal{L};\Bbbk)$. For combinatorial descriptions of Chow and augmented Chow polynomials, see [FMSV24].

Theorem 1.12. The following conditions are equivalent for $\mathcal{L}$ and $\Bbbk$, with p the characteristic of $\Bbbk$:

(a) The $\mathrm{H}(\mathcal{L};\Bbbk)$-module $\mathrm{CH}(\mathcal{L};\Bbbk)$ is indecomposable;

(b) $\mathrm{E}(\mathcal{L};\Bbbk) \cong \mathrm{CH}(\mathcal{L};\Bbbk)$ as $\mathrm{H}(\mathcal{L};\Bbbk)$-modules;

(c) ${}_pZ_{\mathcal{L}}(t) = C_{\mathcal{L}}(t)$;

(d) ${}_pP_{\mathcal{L}}(t) = \underline{C}_{\mathcal{L}}(t)$;

(e) Every non-maximal element of $\mathcal{L}$ is p-resonant.

We say that $\mathcal{L}$ is ***p*-resonant** if $\mathcal{L}$ and p satisfy the equivalent conditions of Theorem 1.12. A matroid is p-resonant if its lattice of flats is p-resonant.

Example 1.13. We collect here some infinite families of p-resonant matroids.

(1) The uniform matroid $U_{2,n}$ is p-resonant if and only if n is congruent to 1 modulo p.

(2) A projective plane of order q is p-resonant if and only if p divides q. The **prime power conjecture** predicts that every projective plane is p-resonant for a unique p [Oxl11, Section 6.1].

(3) The projective geometry $\mathbb{P}^{d-1}(\mathbb{F}_q)$ is p-resonant if and only if q is a power of p. More generally, a q-niform matroid [Pro19] of rank at least 3 obtained by truncating a projective geometry over $\mathbb{F}_q$ is p-resonant if and only if q is a power of p.

(4) For $1 < t < k < n$, the blocks of a Steiner system $S(t,k,n)$ form the set of hyperplanes of a rank $t+1$ matroid on n elements [Wel76, Chapter 12]. A Steiner system $S(2,3,n)$ exists if and only if n is congruent to 1 or 3 modulo 6, and the corresponding matroid is 2-resonant if and only if n is congruent to 3 or 7 modulo 12. A Steiner system $S(2,4,n)$ exists if and only if n is congruent to 1 or 4 modulo 12, and the corresponding matroid is 3-resonant if and only if n is congruent to 4 or 13 modulo 36.

(5) A hyperoval is a set of $q+2$ points in $\mathbb{P}^2(\mathbb{F}_q)$ that intersects every line in either 0 or 2 points; such a set exists if and only if q is even. The restriction of $\mathbb{P}^2(\mathbb{F}_q)$ to the q^2-1 points in the complement of a hyperoval yields a 2-resonant matroid.

(6) Start with a projective space of order q, and choose a codimension 2 subspace S. Consider the pencil of hyperplanes containing S, choose k hyperplanes from the pencil, and restrict to

the union of those hyperplanes. If k is congruent to 1 modulo p, then the resulting matroid is p-resonant, and the same holds for its truncations.

1.4. **Inclusions of geometric lattices.** Consider a rank-preserving injective map of posets $\mathcal{L} \to \mathcal{M}$, where $\mathcal{L}$ and $\mathcal{M}$ are geometric lattices. In Proposition 5.2, we show that any such inclusion comes from the operation of deleting elements from the ground set of the corresponding simple matroid.

Example 1.14. If $\mathcal{L}$ is the lattice of flats of a simple matroid of rank n that is realizable over $\mathbb{F}_q$, then $\mathcal{L}$ includes into the lattice $\mathbb{P}^n(\mathbb{F}_q)$ of linear subspaces of $\mathbb{F}_q^{n+1}$, with different realizations inducing different inclusions. If $\mathcal{L}$ is the lattice of flats of a matroid associated with a simple graph with n vertices, then $\mathcal{L}$ includes into the lattice Π_n of partitions of the set $[n]$, which is the lattice of flats of the matroid of the complete graph on n vertices.

The inclusion $\varphi\colon \mathcal{L} \to \mathcal{M}$ induces an inclusion of rings $\mathrm{H}(\mathcal{L};\Bbbk) \to \mathrm{H}(\mathcal{M};\Bbbk)$, taking y_F to $y_{\varphi(F)}$.

Theorem 1.15. If M is a pure $\mathrm{H}(\mathcal{M};\Bbbk)$-module, then it is also a pure $\mathrm{H}(\mathcal{L};\Bbbk)$-module.

Recall that the category of finite-dimensional graded modules over a finite-dimensional graded algebra satisfies the Krull–Schmidt theorem [Ati56]. Therefore, when the graded $\mathrm{H}(\mathcal{L};\Bbbk)$-module $\mathrm{E}(\mathcal{M};\Bbbk)$ is decomposed into a direct sum of indecomposable graded submodules, the graded isomorphism classes of the indecomposable summands and their graded multiplicities are uniquely determined. Consequently, the Poincaré polynomials of the corresponding multiplicity spaces are well-defined invariants of the $\mathrm{H}(\mathcal{L};\Bbbk)$-module $\mathrm{E}(\mathcal{M};\Bbbk)$.

Let $\mathrm{S}(\varphi;\Bbbk) \coloneqq \mathrm{ann}_{\mathcal{L}\setminus\hat{0}}(\mathrm{E}(\mathcal{M};\Bbbk))$ be the socle of $\mathrm{E}(\mathcal{M};\Bbbk)$ as a module over $\mathrm{H}(\mathcal{L};\Bbbk)$, and let

$$\mathrm{L}(\varphi;\Bbbk) \coloneqq \mathrm{S}(\varphi;\Bbbk)\big/\big(\mathrm{span}_{\mathcal{L}\setminus\hat{0}}(\mathrm{E}(\mathcal{M};\Bbbk)) \cap \mathrm{S}(\varphi;\Bbbk)\big).$$

For each $F \in \mathcal{L}$, the injection φ induces an injection between the upper intervals $\mathcal{L}_F \to \mathcal{M}_{\varphi(F)}$, denoted φ_F. The following theorem describes a decomposition of $\mathrm{E}(\mathcal{M};\Bbbk)$ into indecomposable $\mathrm{H}(\mathcal{L};\Bbbk)$-modules in terms of the graded vector spaces $\mathrm{L}(\varphi_F;\Bbbk)$.

Theorem 1.16. There is a (noncanonical) isomorphism of $\mathrm{H}(\mathcal{L};\Bbbk)$-modules

$$\mathrm{E}(\mathcal{M};\Bbbk) \cong \bigoplus_{F\in\mathcal{L}} \mathrm{L}(\varphi_F;\Bbbk) \otimes \mathrm{E}(\mathcal{L}^F;\Bbbk).$$

In analogy with [Sta92], we define the **local *h*-polynomials** of φ by

$$\ell_\varphi(t) \coloneqq {}_0\ell_\varphi(t) \coloneqq \sum_{i\geqslant 0} t^i \dim \mathrm{L}^i(\varphi;\mathbb{Q}) \ \text{ and } \ {}_p\ell_\varphi(t) \coloneqq \sum_{i\geqslant 0} t^i \dim \mathrm{L}^i(\varphi;\mathbb{F}_p).$$

The module decomposition in Theorem 1.16 has the following polynomial interpretation. It gives a recursive method for computing ${}_p\ell_\varphi(t)$, assuming that the p-Z-polynomials of all lower intervals in $\mathcal{L}$ and all upper intervals in $\mathcal{M}$ are known.

Corollary 1.17. Let p be prime or zero. We have

$$ {}_pZ_{\mathcal{M}}(t) = \sum_{F \in \mathcal{L}} {}_p\ell_{\varphi_F}(t)\, {}_pZ_{\mathcal{L}^F}(t). $$

Remark 1.18. Suppose that $\mathcal{L}$ and $\mathcal{M}$ have the same rank and that the number of atoms of $\mathcal{M}$ is one more than that of $\mathcal{L}$. In matroid terminology, this means that $\mathcal{L}$ is obtained from $\mathcal{M}$ by deleting a single element that is not a coloop. In this case, with $p = 0$, Corollary 1.17 specializes to the second part of [BV20, Theorem 2.8].

Remark 1.19. Let $\mathcal{L}^1$ and $\mathcal{M}^1$ denote the sets of atoms of $\mathcal{L}$ and $\mathcal{M}$, respectively; we regard $\mathcal{L}^1$ as a subset of $\mathcal{M}^1$ via φ. Suppose that we have a commutative diagram of complex vector spaces

$$ \begin{tikzcd} V_{\mathcal{M}} \arrow[r, hook] \arrow[d, two heads] & \mathbb{C}^{\mathcal{M}^1} \arrow[d, two heads] \\ V_{\mathcal{L}} \arrow[r, hook] & \mathbb{C}^{\mathcal{L}^1} \end{tikzcd} $$

such that $\mathcal{L}$ is the lattice of flats of the matroid realized by the linear subspace $V_{\mathcal{L}} \subseteq \mathbb{C}^{\mathcal{L}^1}$, and similarly for $\mathcal{M}$. This induces a commutative diagram of arrangement Schubert varieties

$$ \begin{tikzcd} Y_{V_{\mathcal{M}}} \arrow[r, hook] \arrow[d, two heads] & (\mathbb{CP}^1)^{\mathcal{M}^1} \arrow[d, two heads] \\ Y_{V_{\mathcal{L}}} \arrow[r, hook] & (\mathbb{CP}^1)^{\mathcal{L}^1}. \end{tikzcd} $$

The case $p = 0$ of Theorem 1.16 gives a combinatorial description of the direct image of the intersection complex $\mathrm{IC}(Y_{V_{\mathcal{M}}}; \mathbb{Q})$ under this projection as a direct sum of intersection complexes on $Y_{V_{\mathcal{L}}}$. When $p > 0$ it describes the decomposition of the direct image of the parity sheaf $\mathcal{E}(Y_{V_{\mathcal{M}}}; \mathbb{F}_p)$ as a direct sum of parity sheaves [JMW14, Section 3.3].

The second part of the following theorem is proved by showing that the graded vector space $\mathrm{L}(\varphi; \Bbbk)$ satisfies a version of the relative hard Lefschetz theorem when $p = 0$. It follows that the polynomial ${}_p\ell_\varphi(t)$ is unimodal when $p = 0$. Do the coefficients of ${}_p\ell_\varphi(t)$ form a unimodal sequence for p positive?

Theorem 1.20. Let p be a prime or zero.

(a) The polynomial ${}_p\ell_\varphi(t)$ is palindromic: $t^{\mathrm{rk}\,\mathcal{M}}\, {}_p\ell_\varphi(t^{-1}) = {}_p\ell_\varphi(t)$.

(b) When $p = 0$, the polynomial ${}_p\ell_\varphi(t)$ is unimodal.

Let $\mathcal{M}/\mathcal{L}$ be the geometric lattice $\mathcal{M}_{\varphi(\hat{1})}$. Note that, by definition, $\mathrm{L}(\varphi_{\hat{1}}; \Bbbk)$ is isomorphic to the parity module of $\mathcal{M}/\mathcal{L}$ with coefficients in $\Bbbk$ as a graded vector space over $\Bbbk$. Thus, Proposition 1.9 and Corollary 1.17 imply the following inequalities.

Corollary 1.21. Let p be a prime or zero. If there is a rank-preserving inclusion $\varphi\colon \mathcal{L} \to \mathcal{M}$, then

$$ {}_pZ_{\mathcal{M}/\mathcal{L}}(t) \cdot {}_pZ_{\mathcal{L}}(t) \leqslant {}_pZ_{\mathcal{M}}(t). $$

In particular, the number of p-resonant elements of $\mathcal{M}$ is at least as large as the sum of the corresponding numbers for $\mathcal{L}$ and $\mathcal{M}/\mathcal{L}$.

Corollary 1.22. For any $F \in \mathcal{M}$ and any p which is prime or zero, we have

$$ {}_pZ_{\mathcal{M}_F}(t) \cdot {}_pZ_{\mathcal{M}^F}(t) \leqslant {}_pZ_{\mathcal{M}}(t). $$

The equality holds if $\mathcal{M}$ is the product of $\mathcal{M}_F$ and $\mathcal{M}^F$.

We illustrate Corollary 1.21 with a few examples.

Example 1.23. Corollary 1.21 implies that the Fano plane $\mathbb{P}^2(\mathbb{F}_2)$ is not realizable over $\mathbb{F}_3$: An $\mathbb{F}_3$-realization of $\mathbb{P}^2(\mathbb{F}_2)$ would induce a rank-preserving inclusion $\mathbb{P}^2(\mathbb{F}_2) \to \mathbb{P}^2(\mathbb{F}_3)$, but the number of 2-resonant flats of $\mathbb{P}^2(\mathbb{F}_2)$ is 15, whereas the number of 2-resonant flats of $\mathbb{P}^2(\mathbb{F}_3)$ is 13. Similarly, $\mathbb{P}^2(\mathbb{F}_5)$ is not realizable over $\mathbb{F}_7$, because the number of 5-resonant flats of $\mathbb{P}^2(\mathbb{F}_5)$ is 63 and the number of 5-resonant flats of $\mathbb{P}^2(\mathbb{F}_7)$ is 57.

Example 1.24. Let $\mathbb{A}^n(\mathbb{F}_q)$ be the lattice of affine subspaces of $\mathbb{F}_q^n$. We may regard these as linear subspaces of $\mathbb{F}_q^{n+1}$ that are not contained in the hyperplane at infinity, so we have an embedding $\mathbb{A}^n(\mathbb{F}_q) \subseteq \mathbb{P}^n(\mathbb{F}_q)$. The lattice $\mathbb{P}^n(\mathbb{F}_q)$ has $q^n + \cdots + q + 1$ elements of rank n, whereas $\mathbb{A}^n(\mathbb{F}_q)$ has one fewer such element. Thus the second inequality of Corollary 1.21 is strict when $p = 0$. If q is odd, then $\mathbb{P}^n(\mathbb{F}_q)$ has no additional 2-resonant elements, but $\mathbb{A}^n(\mathbb{F}_q)$ has one additional 2-resonant element, namely the minimal element. Thus the second inequality of Corollary 1.21 is an equality when $p = 2$. Note, however, that the inclusion of $\mathbb{A}^n(\mathbb{F}_q)$ into $\mathbb{P}^n(\mathbb{F}_q)$ does *not* take 2-resonant elements to 2-resonant elements.

Example 1.25. There exists an exotic inclusion of $\mathbb{A}^2(\mathbb{F}_3)$ into $\mathbb{P}^2(\mathbb{F}_4)$. The lattice $\mathbb{P}^2(\mathbb{F}_4)$ has 21 0-resonant flats, and the same number of 3-resonant flats, namely all flats of rank 2. The sublattice $\mathbb{A}^2(\mathbb{F}_3)$ has only 12 0-resonant elements, but it has 21 3-resonant elements. Thus the inequalities of Corollary 1.21 are equalities when $p = 3$.

Example 1.26. Let $\mathcal{L}$ be the geometric lattice associated with a Steiner system $S(2, 5, 45)$, viewed as a rank 3 paving matroid on 45 elements.[1] Let $\mathcal{M} = \mathbb{P}^2(\mathbb{F}_{11})$. Then $\mathcal{L}$ has 99 0-resonant elements and $\mathcal{M}$ has 133 0-resonant elements. However, $\mathcal{L}$ has 145 2-resonant elements, and $\mathcal{M}$ has only 133 2-resonant elements. Thus the $p = 0$ case of the second inequality of Corollary 1.21 does not provide an obstruction to embedding $\mathcal{L}$ into $\mathcal{M}$, but the $p = 2$ case does.

[1] For constructions, see for example [CZ22]. There are at least 1072 non-isomorphic $S(2, 5, 45)$ designs.

1.5. **Upper homogeneous geometric lattices.** We call $\mathcal{M}$ **upper homogeneous** if, for all F and G in $\mathcal{M}$ of the same rank, there is an isomorphism of posets $\mathcal{M}_F \cong \mathcal{M}_G$. Examples of upper homogeneous geometric lattices include the projective geometry $\mathbb{P}^n(\mathbb{F}_q)$, the affine geometry $\mathbb{A}^n(\mathbb{F}_q)$, and the partition lattice Π_n.

Recall from Theorem 1.15 that a rank-preserving inclusion $\varphi\colon \mathcal{L} \to \mathcal{M}$ gives the parity module $\mathrm{E}(\mathcal{M};\Bbbk)$ the structure of a pure module over $\mathrm{H}(\mathcal{L};\Bbbk)$. The following result follows from Corollary 1.17 via induction.

Theorem 1.27. If $\mathcal{M}$ is upper homogeneous, then the isomorphism type of $\mathrm{E}(\mathcal{M};\Bbbk)$ as an $\mathrm{H}(\mathcal{L};\Bbbk)$-module is independent of the choice of φ. In particular, the local h-polynomials for inclusions of $\mathcal{L}$ into $\mathcal{M}$ do not depend on the choice of inclusion.

The following example shows that the conclusion of Theorem 1.27 can fail if $\mathcal{M}$ is not upper homogeneous.

Example 1.28. Let Γ be the graph obtained from the complete graph K_4 by deleting a single edge, and let e be the unique edge of Γ that is not adjacent to the deleted edge. Let $\mathcal{M}$ be the lattice of flats of the corresponding graphic matroid. Then $\mathcal{M}$ is *not* upper homogeneous, because the matroid of the graph obtained by contracting e and simplifying is not isomorphic to the matroid of the graph obtained by contracting a different edge and simplifying.

Let $\mathcal{L}$ be the Boolean lattice of rank 3. Inclusions of $\mathcal{L}$ into $\mathcal{M}$ correspond to choices of a spanning tree for Γ, and there are two kinds of spanning trees: those that contain e and those that do not. If $\varphi\colon \mathcal{L} \to \mathcal{M}$ is an inclusion corresponding to a spanning tree that contains e, then we may use Corollary 1.17 to compute that ${}_0\ell_\varphi(t) = t + t^2$. On the other hand, if $\psi\colon \mathcal{L} \to \mathcal{M}$ is an inclusion corresponding to a spanning tree that does not contain e, then ${}_0\ell_\psi(t) = 0$.

For each $\mathcal{L}$ and p, we recursively define a polynomial ${}_p\ell_{\mathcal{L}}(q,t) \in \mathbb{Z}[q,t]$ as follows:

(1) If $\operatorname{rk}\mathcal{L} = 0$, then ${}_p\ell_{\mathcal{L}}(q,t) = 1$.
(2) If $\operatorname{rk}\mathcal{L} = d > 0$, then ${}_p\ell_{\mathcal{L}}(q,t) = \sum_{k=0}^{d} \left[{d \atop k}\right]_q t^k - \sum_{F \neq \hat{0}} {}_p\ell_{\mathcal{L}_F}(q,t)\, {}_pZ_{\mathcal{L}^F}(t)$.

This recursion indeed defines an element of $\mathbb{Z}[q,t]$, since the q-binomial coefficients $\left[{d \atop k}\right]_q$ are polynomials in q with integer coefficients.

The following proposition is a consequence of Corollary 1.17, as well as an illustration of Theorem 1.27 in the case where $\operatorname{rk}\mathcal{L} = d$ and $\mathcal{M} = \mathbb{P}^{d-1}(\mathbb{F}_q)$.

Proposition 1.29. If $\operatorname{rk}\mathcal{L} = d$ and there is an inclusion $\varphi\colon \mathcal{L} \to \mathbb{P}^{d-1}(\mathbb{F}_q)$, then ${}_p\ell_{\mathcal{L}}(q,t) = {}_p\ell_\varphi(t)$ for $p = 0$ and for every prime p not dividing q.

It follows that, if $\mathcal{L}$ is realizable over $\mathbb{F}_q$, then ${}_p\ell_{\mathcal{L}}(q,t) \in \mathbb{N}[t]$ for $p = 0$ and for every prime p not dividing q.

Example 1.30. Let $U_{2,4}$ be the lattice of flats for the uniform matroid of rank 2 on 4 elements. We may compute ${}_0\ell_{U_{2,4}}(q,t) = (q-3)t$. In particular, ${}_0\ell_{U_{2,4}}(2,t) = -t$. This reflects the fact that $U_{2,4}$ is not realizable over the field $\mathbb{F}_2$.

1.6. **Stalks, costalks, and decompositions.** For any $F \in \mathcal{L}$, consider the upper sets

$$\Sigma_{\geqslant F} \coloneqq \mathcal{L}_F = \{G \in \mathcal{L} \mid G \geqslant F\} \text{ and } \Sigma_{>F} \coloneqq \Sigma_{\geqslant F} \backslash \{F\} = \{G \in \mathcal{L} \mid G > F\}.$$

For any pure $\mathrm{H}(\mathcal{L};\Bbbk)$-module M and $F \in \mathcal{L}$, we define the **costalk** $\mathrm{M}_{[F]}$ and the **stalk** M_F by

$$\mathrm{M}_{[F]} \coloneqq \mathrm{ann}_{\Sigma_{>\hat{0}}}\big(\mathrm{span}_{\Sigma_{\geqslant F}}(\mathrm{M})\big)[\mathrm{rk}\, F] \text{ and } \mathrm{M}_F \coloneqq \big(\mathrm{span}_{\Sigma_{\geqslant F}}(\mathrm{M})/\mathrm{span}_{\Sigma_{>F}}(\mathrm{M})\big)[\mathrm{rk}\, F].$$

Observe that there is a natural **costalk-stalk homomorphism**

$$\beta_F(\mathrm{M})\colon \mathrm{M}_{[F]} \longrightarrow \mathrm{M}_F,$$

given by including $\mathrm{M}_{[F]}$ into $y_F\mathrm{M}[\mathrm{rk}\, F]$ and then projecting to M_F.

Example 1.31. By Theorem 1.15, a rank-preserving inclusion $\varphi\colon \mathcal{L} \to \mathcal{M}$ gives $\mathrm{E}(\mathcal{M};\Bbbk)$ the structure of a pure module over $\mathrm{H}(\mathcal{L};\Bbbk)$. Its costalk at $\hat{0}$ in $\mathcal{L}$ is the socle $\mathrm{S}(\varphi;\Bbbk)$, and the image of the costalk-stalk homomorphism at $\hat{0}$ is the quotient $\mathrm{L}(\varphi;\Bbbk)$.

Theorem D implies that the $\mathrm{H}(\mathcal{L};\Bbbk)$-module M decomposes as a direct sum of shifts of the parity modules $\mathrm{E}(\mathcal{L}^F;\Bbbk)$. Theorem F below characterizes the multiplicity spaces in terms of the costalk-stalk homomorphisms, generalizing Theorem 1.16.

Theorem F. For any pure $\mathrm{H}(\mathcal{L};\Bbbk)$-module M, there is an isomorphism of $\mathrm{H}(\mathcal{L};\Bbbk)$-modules

$$\mathrm{M} \cong \bigoplus_{F \in \mathcal{L}} \mathrm{im}(\beta_F(\mathrm{M})) \otimes \mathrm{E}(\mathcal{L}^F;\Bbbk).$$

Remark 1.32. It is typically difficult to check whether a given module over a ring is indecomposable. However, Theorem F provides us with such a procedure for pure modules over the ring $\mathrm{H}(\mathcal{L};\Bbbk)$: a pure module M is indecomposable if and only if the costalk-stalk homomorphism $\beta_F(\mathrm{M})$ is equal to zero for all but one choice of $F \in \mathcal{L}$, and the one nonvanishing homomorphism has rank 1. In particular, $\mathrm{E}(\mathcal{L};\Bbbk)$ is characterized, up to shift, by purity and the fact that

$$\mathrm{rk}\, \beta_F(\mathrm{E}(\mathcal{L};\Bbbk)) = \begin{cases} 1 & \text{if } F = \hat{1}, \\ 0 & \text{otherwise.} \end{cases}$$

1.7. **Structure of the paper.** The technical core of this paper consists of Theorems A, B, C, D, E, and F, all of which will be proved by introducing auxiliary categories of sheaves on posets. Rather than beginning with this somewhat dense material, we will instead begin by assuming these six results, and using them[2] to prove the numbered theorems, propositions, and corollaries from the introduction.

In Section 2, we define a filtration of an arbitrary $\mathrm{H}(\mathcal{L};\Bbbk)$-module and use it to prove Theorem 1.7. In Section 3, we study the Poincaré pairing on $\mathrm{H}(\mathcal{L};\Bbbk)$ and use it to prove Theorem 1.11. In Section 4, we study the augmented Chow ring $\mathrm{CH}(\mathcal{L};\Bbbk)$ as a module over $\mathrm{H}(\mathcal{L};\Bbbk)$, and we use it to prove Theorems 1.4 and 1.12, as well as Propositions 1.9 and 1.10. In Section 5, we consider inclusions of geometric lattices $\mathcal{L} \to \mathcal{M}$ and prove Theorems 1.15 , 1.16, and 1.20.

The remaining sections are devoted to the proofs of Theorems A, B, C, D, E, and F. Section 6 gives a way of replacing an $\mathrm{H}(\mathcal{L};\Bbbk)$-module M with an "exploded" version. Roughly, this means replacing M by the graded vector spaces $y_F\mathrm{M}$ for all $F \in \mathcal{L}$, together with the maps between them induced by factorizations $y_F = y_G y_H$ in $\mathrm{H}(\mathcal{L};\Bbbk)$. This generalizes to give a category $\mathcal{C}(\Sigma;\Bbbk)$ of "exploded modules" for any upwardly closed subset $\Sigma \subseteq \mathcal{L}$, in which only the spaces corresponding to elements $F \in \Sigma$ are retained. This makes it possible to construct $\mathrm{H}(\mathcal{L};\Bbbk)$-modules inductively, by adding elements to Σ one at a time.

In Section 7, we define the notion of flabbiness, coflabbiness, and purity for objects in $\mathcal{C}(\Sigma;\Bbbk)$, agreeing with the notions for $\mathrm{H}(\mathcal{L};\Bbbk)$-modules when $\Sigma = \mathcal{L}$. We also generalize the stalk, costalk, and duality functors to this setting. In order to prove Theorem E, we need to allow $\Bbbk$ to be a DVR and define the notion of purity in this setting.

In Section 8, we apply this to give an inductive construction and classification of indecomposable objects in $\mathcal{C}(\Sigma;\Bbbk)$. When applied to the case $\Sigma = \mathcal{L}$, this gives Theorems A, D, and F. In Section 9, we analyze the construction in greater detail in order to prove Theorems B and E. In Section 10, we give a formula for the dimension of the Hom space between two pure $\mathrm{H}(\mathcal{L};\Bbbk)$-modules in terms of their stalks and costalks, and use it to characterize parity modules with only scalar endomorphisms. Finally, in Section 11, we prove Theorem C by showing that the hypercohomology of the parity sheaf $\mathcal{E}(Y_V;\Bbbk)$ satisfies the conditions of Theorem A.

Acknowledgements. June Huh was partially supported by the Simons Investigator Grant MPS-SIM-00825156 and NSF grant DMS-2053308. Jacob Matherne received support from NSF Grant DMS-2452179 and Simons Foundation Travel Support for Mathematicians Award MPS-TSM00007970. Nicholas Proudfoot was supported by NSF grants DMS-1954050, DMS-2053243, and DMS-2344861, and by All Souls College. Botong Wang was partially supported by the Simons Foundation. Part

[2]Theorem C will not be needed to prove any of the numbered results.

of this work was carried out during the 2024–2025 special year on Algebraic and Geometric Combinatorics at the Institute for Advanced Study. The authors thank Matt Larson for helpful discussions during the preparation of this manuscript.

2. Filtrations

In this section, we define a filtration of an arbitrary $\mathrm{H}(\mathcal{L};\Bbbk)$-module and use it to prove Theorem 1.7, assuming Theorems A and F. We begin by proving that the parity module $\mathrm{E}(\mathcal{L};\Bbbk)$ satisfies the following version of Poincaré duality.

Proposition 2.1. There is an isomorphism of $\mathrm{H}(\mathcal{L};\Bbbk)$-modules $\mathrm{E}(\mathcal{L};\Bbbk)^*[-\operatorname{rk}\mathcal{L}] \cong \mathrm{E}(\mathcal{L};\Bbbk)$.

It follows that the polynomial ${}_pZ_{\mathcal{L}}(t)$ is palindromic: $t^{\operatorname{rk}\mathcal{L}}\,{}_pZ_{\mathcal{L}}(t^{-1}) = {}_pZ_{\mathcal{L}}(t)$.

Proof. Flabbiness of $\mathrm{E}(\mathcal{L};\Bbbk)$ implies coflabbiness of $\mathrm{E}(\mathcal{L};\Bbbk)^*$, and coflabbiness of $\mathrm{E}(\mathcal{L};\Bbbk)$ implies flabbiness of $\mathrm{E}(\mathcal{L};\Bbbk)^*$, so $\mathrm{E}(\mathcal{L};\Bbbk)^*$ is pure. Indecomposability of $\mathrm{E}(\mathcal{L};\Bbbk)$ implies indecomposability of $\mathrm{E}(\mathcal{L};\Bbbk)^*$. Let $x \in \mathrm{E}^0(\mathcal{L};\Bbbk)$ be an element such that $y_{\hat{1}} \cdot x \neq 0$, and let $\varphi \in \mathrm{E}^{\operatorname{rk}\mathcal{L}}(\mathcal{L};\Bbbk)^*$ be a linear functional that does not vanish on $y_{\hat{1}} \cdot x$. Then $y_{\hat{1}} \cdot \varphi$ does not vanish on x, and is therefore nonzero. Thus $\mathrm{E}(\mathcal{L};\Bbbk)^*[-\operatorname{rk}\mathcal{L}] \cong \mathrm{E}(\mathcal{L};\Bbbk)$ by Theorem A. □

Remark 2.2. Proposition 2.1 is equivalent to the existence of an $\mathrm{H}(\mathcal{L};\Bbbk)$-invariant nondegenerate bilinear pairing B on $\mathrm{E}(\mathcal{L};\Bbbk)$ that satisfies the orthogonality relations

$$B(\mathrm{E}^i(\mathcal{L};\Bbbk), \mathrm{E}^j(\mathcal{L};\Bbbk)) = 0 \ \text{ unless } i + j = \operatorname{rk}\mathcal{L}.$$

Proposition 1.10, which we will prove in Section 4, says that $\mathrm{E}(\mathcal{L};\Bbbk)$ embeds into $\mathrm{CH}(\mathcal{L};\Bbbk)$, and Proposition 4.5(b) says that the Poincaré pairing on $\mathrm{CH}(\mathcal{L};\Bbbk)$ is nondegenerate. Restricting the Poincaré pairing from $\mathrm{CH}(\mathcal{L};\Bbbk)$ to $\mathrm{E}(\mathcal{L};\Bbbk)$ yields a nondegenerate symmetric bilinear pairing on $\mathrm{E}(\mathcal{L};\Bbbk)$.[3]

For any $\mathrm{H}(\mathcal{L};\Bbbk)$-module M, the submodule $y_F\mathrm{M}$ is naturally an $\mathrm{H}(\mathcal{L}_F;\Bbbk)$-module. To see this, we use the surjective homomorphism of graded $\Bbbk$-algebras

$$\sigma_F\colon\ \mathrm{H}(\mathcal{L};\Bbbk) \longrightarrow \mathrm{H}(\mathcal{L}_F;\Bbbk), \qquad y_G \longmapsto y_F y_G,$$

where the product $y_F y_G$ is computed in $\mathrm{H}(\mathcal{L};\Bbbk)$ and then the resulting element, either $y_{F\vee G}$ or zero, is interpreted as an element in $\mathrm{H}(\mathcal{L}_F;\Bbbk)$. For instance, σ_F sends $y_{\hat{0}}$ to y_F, which is the identity

[3]Let x be an element of $\mathrm{E}^0(\mathcal{L};\Bbbk)$ such that $y_{\hat{1}} \cdot x$ is nonzero. Using Fitting's lemma, one can show that an $\mathrm{H}(\mathcal{L};\Bbbk)$-module endomorphism φ of $\mathrm{E}(\mathcal{L};\Bbbk)$ is an automorphism if and only if $\varphi(x) \neq 0$. It follows that an $\mathrm{H}(\mathcal{L};\Bbbk)$-invariant bilinear pairing B on $\mathrm{E}(\mathcal{L};\Bbbk)$ is nondegenerate if and only if $B(x, y_{\hat{1}} \cdot x) \neq 0$. Clearly, this condition is satisfied by the restriction of the multiplicative pairing on $\mathrm{CH}(\mathcal{L};\Bbbk)$ to its summand $\mathrm{E}(\mathcal{L};\Bbbk)$, for any splitting of $\mathrm{CH}(\mathcal{L};\Bbbk)$ by $\mathrm{E}(\mathcal{L};\Bbbk)$.

in $\mathrm{H}(\mathcal{L}_F;\Bbbk)$. Then the action of $\mathrm{H}(\mathcal{L}_F;\Bbbk)$ on $y_F\mathrm{M}$ is given by the formula

$$\sigma_F(z)\cdot y_F m = z y_F m \text{ for any } z\in \mathrm{H}(\mathcal{L};\Bbbk).$$

This is well-defined because the kernel of σ_F is the kernel of multiplication by y_F. Put another way, if we write $\widetilde{y}_G$ to denote the basis element of $\mathrm{H}(\mathcal{L}_F;\Bbbk)$ indexed by $G\in\mathcal{L}_F$, we have

$$\widetilde{y}_G\cdot y_F m = y_G m.$$

It follows from this that, for any upper set $\Sigma\subseteq\mathcal{L}_F\subseteq\mathcal{L}$ and any $\mathrm{H}(\mathcal{L};\Bbbk)$-module M, we have

$$\operatorname{span}_\Sigma y_F\mathrm{M} = \operatorname{span}_\Sigma \mathrm{M} \text{ and } \operatorname{ann}_\Sigma y_F\mathrm{M} = \operatorname{ann}_\Sigma \mathrm{M}\cap y_F\mathrm{M}, \tag{3}$$

where the left sides of these equalities use the $\mathrm{H}(\mathcal{L}_F;\Bbbk)$-module structure and the right sides use the $\mathrm{H}(\mathcal{L};\Bbbk)$-module structure.

Lemma 2.3. Let M be an $\mathrm{H}(\mathcal{L};\Bbbk)$-module, and $y_F\mathrm{M}$ the corresponding $\mathrm{H}(\mathcal{L}_F;\Bbbk)$-module.

(a) If M is a pure $\mathrm{H}(\mathcal{L};\Bbbk)$-module, then $y_F\mathrm{M}$ is a pure $\mathrm{H}(\mathcal{L}_F;\Bbbk)$-module.

(b) For all $G\in\mathcal{L}_F\subseteq\mathcal{L}$, we have canonical stalk and costalk isomorphisms

$$(y_F\mathrm{M})_G\cong \mathrm{M}_G[-\operatorname{rk} F] \text{ and } (y_F\mathrm{M})_{[G]}\cong \mathrm{M}_{[G]}[-\operatorname{rk} F],$$

along with an identification of the costalk-stalk homomorphisms $\beta_G(y_F\mathrm{M})=\beta_G(\mathrm{M})$.

(c) If M is an indecomposable pure $\mathrm{H}(\mathcal{L};\Bbbk)$-module, then $y_F\mathrm{M}$ is either zero or an indecomposable $\mathrm{H}(\mathcal{L}_F;\Bbbk)$-module.

Proof. Parts (a) and (b) follow from Equation (3). For Part (c), suppose that M is indecomposable. By Theorem F and Remark 1.32, the homomorphism $\beta_G(\mathrm{M})$ is equal to zero for all but one element G, and the one nonvanishing homomorphism has rank 1. If G lies in $\mathcal{L}_F$, then the same property will hold for the $\mathrm{H}(\mathcal{L}_F;\Bbbk)$-module $y_F\mathrm{M}$, and hence $y_F\mathrm{M}$ is an indecomposable $\mathrm{H}(\mathcal{L}_F;\Bbbk)$-module. Otherwise, we have $y_F\mathrm{M}=0$. □

Proposition 2.4. There is an isomorphism of $\mathrm{H}(\mathcal{L}_F;\Bbbk)$-modules $y_F\mathrm{E}(\mathcal{L};\Bbbk)[\operatorname{rk} F]\cong \mathrm{E}(\mathcal{L}_F;\Bbbk)$.

Proof. Lemma 2.3 implies that $y_F\mathrm{E}(\mathcal{L};\Bbbk)[\operatorname{rk} F]$ is a pure and indecomposable $\mathrm{H}(\mathcal{L}_F;\Bbbk)$-module. If x is an element of $\mathrm{E}^0(\mathcal{L};\Bbbk)$ such that $y_{\hat{1}}\cdot x$ is nonzero, then $\widetilde{y}_{\hat{1}}\cdot(y_F x)$ is nonzero in $y_F\mathrm{E}(\mathcal{L};\Bbbk)[\operatorname{rk} F]$. Therefore, by Theorem A, we have $y_F\mathrm{E}(\mathcal{L};\Bbbk)[\operatorname{rk} F]\cong \mathrm{E}(\mathcal{L}_F;\Bbbk)$. □

Remark 2.5. This result also follows directly from our construction of the module $\mathrm{E}(\mathcal{L};\Bbbk)$; see Remark 8.15. Indeed, this isomorphism is fundamental to the inductive construction that we give.

Lemma 2.6. Let $\Sigma\subseteq\mathcal{L}$ be an upper set, and let $F\in\Sigma$ be a minimal element. If M is a coflabby $\mathrm{H}(\mathcal{L};\Bbbk)$-module, then the following natural map is an isomorphism:

$$\mathrm{M}_F[-\operatorname{rk} F] = \operatorname{span}_{\Sigma_{\geqslant F}}(\mathrm{M})/\operatorname{span}_{\Sigma_{>F}}(\mathrm{M}) \longrightarrow \operatorname{span}_\Sigma(\mathrm{M})/\operatorname{span}_{\Sigma\setminus F}(\mathrm{M}).$$

Proof. Surjectivity follows from $\Sigma_{\geqslant F} \cup (\Sigma \backslash F) = \Sigma$. Injectivity follows from coflabbiness, since

$$\mathrm{span}_{\Sigma_{\geqslant F}}(\mathrm{M}) \cap \mathrm{span}_{\Sigma \backslash F}(\mathrm{M}) = \mathrm{span}_{\Sigma_{\geqslant F} \cap (\Sigma \backslash F)}(\mathrm{M}) = \mathrm{span}_{\Sigma_{>F}}(\mathrm{M}). \qquad \square$$

Let $\mathfrak{m}$ be the unique graded maximal ideal of $\mathrm{H}(\mathcal{L};\Bbbk)$, the span of all y_F of positive degree. For any $\mathrm{H}(\mathcal{L};\Bbbk)$-module M, we have a canonical filtration $\mathfrak{m}^0\mathrm{M} \supseteq \mathfrak{m}^1\mathrm{M} \supseteq \mathfrak{m}^2\mathrm{M} \supseteq \cdots$ of M. The **associated graded** $\mathrm{gr}\,\mathrm{M}$ is the vector space

$$\mathrm{gr}\,\mathrm{M} \coloneqq \bigoplus_{k \geqslant 0} \mathfrak{m}^k\mathrm{M}/\mathfrak{m}^{k+1}\mathrm{M}.$$

Note that $\mathrm{gr}\,\mathrm{M}$ is (noncanonically) isomorphic to M as a graded vector space, where the grading is induced by the original grading on M. The proof of the following proposition is adapted from the proof of [BHM+, Theorem 1.8].

Proposition 2.7. For any coflabby $\mathrm{H}(\mathcal{L};\Bbbk)$-module M, there is a canonical isomorphism of graded vector spaces

$$\mathrm{gr}\,\mathrm{M} \cong \bigoplus_{F \in \mathcal{L}} \mathrm{M}_F[-\operatorname{rk} F].$$

Proof. Let $\Pi \coloneqq \{F \in \mathcal{L} \mid \operatorname{rk} F \geqslant k\}$ and $\Sigma \coloneqq \{F \in \mathcal{L} \mid \operatorname{rk} F > k\}$. We then have

$$\mathfrak{m}^k\mathrm{M}/\mathfrak{m}^{k+1}\mathrm{M} = \mathrm{span}_{\Pi}(\mathrm{M})/\mathrm{span}_{\Sigma}(\mathrm{M}).$$

Write $\mathcal{L}^k = \{F_1, F_2, \ldots, F_r\}$. Thus, $F_1, \ldots, F_r$ are precisely the elements of Π that do not belong to Σ. For all $i \in \{1, \ldots, r\}$, let $\Sigma_i \coloneqq \Sigma \cup \{F_i\}$. Then

$$\bigoplus_{F \in \mathcal{L}^k} \mathrm{M}_F[-k] = \bigoplus_{i=1}^{r} \mathrm{M}_{F_i}[-k] \cong \bigoplus_{i=1}^{r} \mathrm{span}_{\Sigma_i}(\mathrm{M})/\mathrm{span}_{\Sigma}(\mathrm{M}),$$

where the second isomorphism follows from Lemma 2.6. Consider the natural map

$$\rho\colon \bigoplus_{i=1}^{r} \mathrm{span}_{\Sigma_i}(\mathrm{M})/\mathrm{span}_{\Sigma}(\mathrm{M}) \longrightarrow \mathrm{span}_{\Pi}(\mathrm{M})/\mathrm{span}_{\Sigma}(\mathrm{M}).$$

Clearly, the map ρ is surjective. To see that ρ is injective, consider an element of its kernel

$$(m_1, \ldots, m_r) \in \mathrm{span}_{\Sigma_1}(\mathrm{M}) \oplus \cdots \oplus \mathrm{span}_{\Sigma_r}(\mathrm{M}).$$

Let $S \coloneqq \{F_i \mid m_i \notin \mathrm{span}_{\Sigma}(\mathrm{M})\}$, and assume for the sake of contradiction that S is both nonempty and as small as possible. Since each component of ρ is injective, we have $|S| \geqslant 2$. Let $S = S_1 \cup S_2$ be a nontrivial partition of S, and define

$$n_1 \coloneqq \sum_{F_i \in S_1} m_i \in \mathrm{span}_{\Sigma \cup S_1}(\mathrm{M}) \ \text{ and } \ n_2 \coloneqq \sum_{F_i \in S_2} m_i \in \mathrm{span}_{\Sigma \cup S_2}(\mathrm{M}).$$

By minimality of S, neither n_1 nor n_2 lies in $\mathrm{span}_{\Sigma}(\mathrm{M})$. But $n_1 + n_2 \in \mathrm{span}_{\Sigma}(\mathrm{M})$, which implies that n_1 and n_2 both live in $\mathrm{span}_{\Sigma \cup S_1}(\mathrm{M}) \cap \mathrm{span}_{\Sigma \cup S_2}(\mathrm{M}) = \mathrm{span}_{\Sigma}(\mathrm{M})$. $\square$

Proof of Theorem 1.7. Part (a) is immediate from the definitions. Part (b) follows from Proposition 2.1. Propositions 2.4 and 2.7 tell us that

$$\operatorname{gr} \mathrm{E}(\mathcal{L};\Bbbk) \cong \bigoplus_{F\in\mathcal{L}} \mathrm{E}(\mathcal{L};\Bbbk)_F[-\operatorname{rk} F] \cong \bigoplus_{F\in\mathcal{L}} \mathrm{E}(\mathcal{L}_F;\Bbbk)_F[-\operatorname{rk} F].$$

Taking Poincaré polynomials of both sides gives us Part (c). □

3. The Poincaré pairing

In this section, we study the Poincaré pairing on $\mathrm{H}(\mathcal{L};\Bbbk)$ and use it to prove Theorem 1.11, assuming Theorems A, B, and E. Let p be zero or prime, and let $\Bbbk$ be a field of characteristic p.

Lemma 3.1. For all $\mathcal{L}$, we have ${}_pP_{\mathcal{L}}(0) = 1$.

Proof. By Parts (a) and (b) of Theorem B, we have $\underline{\mathrm{E}}^0(\mathcal{L};\Bbbk) = \mathrm{E}^0(\mathcal{L};\Bbbk) \cong \Bbbk$. □

Corollary 3.2. The coefficient of t^i in ${}_pZ_{\mathcal{L}}(t)$ is greater than or equal to $|\{F \in \mathcal{L} \mid \operatorname{rk} F = i\}|$.

Proof. This follows from Theorem 1.7 and Lemma 3.1. □

Corollary 3.3. If ${}_pP_{\mathcal{L}}(t) = 1$, then $\mathrm{E}(\mathcal{L};\Bbbk) \cong \mathrm{H}(\mathcal{L};\Bbbk)$.

Proof. If ${}_pP_{\mathcal{L}}(t) = 1$, then $\underline{\mathrm{E}}(\mathcal{L};\Bbbk)$ is 1-dimensional and concentrated in degree 0, which means that $\mathrm{E}(\mathcal{L};\Bbbk)$ is generated as an $\mathrm{H}(\mathcal{L};\Bbbk)$-module by a single element $x \in \mathrm{E}^0(\mathcal{L};\Bbbk)$. This gives us a surjection $\mathrm{H}(\mathcal{L};\Bbbk) \to \mathrm{E}(\mathcal{L};\Bbbk)$ sending y to $y \cdot x$. By Corollary 3.2, this must be an isomorphism. □

Consider the trace function $\tau\colon \mathrm{H}(\mathcal{L};\Bbbk) \to \Bbbk$ that picks out the coefficient of $y_{\hat{1}}$. We have a symmetric bilinear form on $\mathrm{H}(\mathcal{L};\Bbbk)$ given by the formula $\langle \eta, \xi \rangle := \tau(\eta\xi)$. We will call this the **Poincaré pairing** on $\mathrm{H}(\mathcal{L};\Bbbk)$.

Lemma 3.4. We have an isomorphism $\mathrm{H}(\mathcal{L};\Bbbk) \cong \mathrm{E}(\mathcal{L};\Bbbk)$ if and only if the Poincaré pairing is nondegenerate.

The Poincaré pairing on $\mathrm{H}(\mathcal{L};\Bbbk)$ is analogous to that on the cohomology of a singular projective variety over the complex numbers. Lemma 3.4 states that, in this case, cohomology is isomorphic to intersection cohomology if and only if the Poincaré pairing is nondegenerate.

Proof of Lemma 3.4. Let $d = \operatorname{rk} \mathcal{L}$. If the Poincaré pairing is nondegenerate, then it gives an isomorphism of $\mathrm{H}(\mathcal{L};\Bbbk)$-modules $\mathrm{H}(\mathcal{L};\Bbbk)^*[-d] \cong \mathrm{H}(\mathcal{L};\Bbbk)$. The module $\mathrm{H}(\mathcal{L};\Bbbk)$ is always coflabby, and hence $\mathrm{H}(\mathcal{L};\Bbbk)^*[-d]$ is always flabby. Consequently, if the Poincaré pairing is nondegenerate, then $\mathrm{H}(\mathcal{L};\Bbbk)$ is pure, and is therefore isomorphic to $\mathrm{E}(\mathcal{L};\Bbbk)$ by Theorem A.

Conversely, suppose that $\mathrm{H}(\mathcal{L};\Bbbk) \cong \mathrm{E}(\mathcal{L};\Bbbk)$. Proposition 2.1 implies that there is an isomorphism $\rho\colon \mathrm{H}(\mathcal{L};\Bbbk) \cong \mathrm{H}(\mathcal{L};\Bbbk)^*[-d]$. Multiplying ρ by a nonzero scalar if necessary, we may assume that $\rho(y_{\hat{0}})(y_{\hat{1}}) = 1$. Then, for any η, ξ in $\mathrm{H}(\mathcal{L};\Bbbk)$, we have

$$\rho(\eta)(\xi) = \rho(y_{\hat{0}})(\eta\xi) = \tau(\eta\xi) = \langle \eta, \xi \rangle.$$

The fact that ρ is an isomorphism means that the Poincaré pairing is nondegenerate. □

Proposition 3.5. Let $\mathcal{L}$ be a projective geometry of order q, and $\Bbbk$ a field of characteristic p. Then $\mathrm{H}(\mathcal{L};\Bbbk) \cong \mathrm{E}(\mathcal{L};\Bbbk)$ if and only if q is not a multiple of p.

Proof. Let $\mathrm{D}(\mathcal{L};\Bbbk)$ denote the **ungraded Möbius algebra** of $\mathcal{L}$ with coefficients in $\Bbbk$. This algebra has a $\Bbbk$-basis $\{u_F \mid F \in \mathcal{L}\}$ and multiplication given by $u_F u_G = u_{F \vee G}$ for all F and G. It is a filtered algebra whose associated graded is canonically isomorphic to $\mathrm{H}(\mathcal{L};\Bbbk)$.

Consider the trace function $\tau_{\mathrm{D}}\colon \mathrm{D}(\mathcal{L};\Bbbk) \to \Bbbk$ given by taking the coefficient of $u_{\hat{1}}$, along with the symmetric bilinear form on $\mathrm{D}(\mathcal{L};\Bbbk)$ given by the formula $\langle \xi, \eta \rangle_{\mathrm{D}} \coloneqq \tau_{\mathrm{D}}(\eta\,\xi)$. If we choose a total order on $\mathcal{L}$ that refines the partial order, then the matrix representing $\langle -\,,-\rangle_{\mathrm{D}}$ will be block lower-right triangular, with diagonal blocks given by the restricted pairing between $\{u_F \mid \operatorname{rk} F = i\}$ and $\{u_F \mid \operatorname{crk} F = i\}$, where $\operatorname{crk} F \coloneqq \operatorname{rk} \mathcal{L} - \operatorname{rk} F$. Setting the off-diagonal blocks to zero gives the matrix that represents the Poincaré pairing on $\mathrm{H}(\mathcal{L};\Bbbk)$, thus each of the two bilinear forms is nondegenerate if and only if the other is nondegenerate.

Consider an alternate basis $\{v_F \mid F \in \mathcal{L}\}$ for $\mathrm{D}(\mathcal{L};\Bbbk)$, where

$$v_F \coloneqq \sum_{F \leqslant G} \mu(F,G) u_G.$$

This basis has the property that $v_F v_G = \delta_{FG} v_F$ for all F and G, and therefore

$$\langle v_F, v_G \rangle_{\mathrm{D}} = \delta_{FG}\, \mu(F, \hat{1}).$$

It follows that the determinant of the bilinear form $\langle -\,,-\rangle_{\mathrm{D}}$ is equal to

$$\prod_{F \in \mathcal{L}} \mu(F, \hat{1}) = \prod_{F \in \mathcal{L}} \left((-1)^{\operatorname{crk} F} q^{\binom{\operatorname{crk} F}{2}} \right).$$

This determinant is equal to zero in $\Bbbk$ if and only if q is a multiple of p. The proposition therefore follows from the conclusion of the previous paragraph along with Lemma 3.4. □

Lemma 3.6. If $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$, then $\mathcal{L}$ has a p-resonant element of corank greater than 1 if and only if either $\mathcal{L}_1$ or $\mathcal{L}_2$ has a p-resonant element of corank greater than 1.

Proof. We begin by noting that F_1 is p-resonant in $\mathcal{L}_1$ if and only if $(F_1, \hat{1})$ is p-resonant in $\mathcal{L}$, and similarly F_2 is p-resonant in $\mathcal{L}_2$ if and only if $(\hat{1}, F_2)$ is p-resonant in $\mathcal{L}$. This proves the "if" direction. For the other direction, it is sufficient to show that every p-resonant element of $\mathcal{L}$ has

one of these two forms. Indeed, suppose that $F_1 \in \mathcal{L}_1$ and $F_2 \in \mathcal{L}_2$ with neither one maximal. Choose $G_i \in \mathcal{L}_i$ covering F_i. Then $\mathrm{rk}(G_1, G_2) = \mathrm{rk}(F_1, F_2) + 2$, and the open interval between (F_1, F_2) and (G_1, G_2) has cardinality 2, which is not congruent to 1 modulo p. □

Lemma 3.7. If $\mathcal{L}$ is Boolean, then an element of $\mathcal{L}$ is p-resonant if and only if it has corank 1. If $\mathcal{L}$ is a projective space of order q, then an element of $\mathcal{L}$ of corank greater than 1 is p-resonant if and only if q is a multiple of p.

Proof. Suppose $F < G \in \mathcal{L}$ with $\mathrm{rk}\, G = \mathrm{rk}\, F + 2$. If $\mathcal{L}$ is Boolean, then the open interval (F, G) has cardinality 2, which is never congruent to 1 modulo p. If $\mathcal{L}$ is a projective space of order q, then (F, G) has cardinality $q + 1$, which is congruent to 1 modulo p if and only if p divides q. □

Proof of Theorem 1.11. If $\mathrm{H}(\mathcal{L}; \Bbbk)$ is pure, then $\mathrm{H}(\mathcal{L}; \Bbbk)$ satisfies the three conditions of Theorem A, thus $\mathrm{E}(\mathcal{L}; \Bbbk) \cong \mathrm{H}(\mathcal{L}; \Bbbk)$. Conversely, if $\mathrm{E}(\mathcal{L}; \Bbbk) \cong \mathrm{H}(\mathcal{L}; \Bbbk)$, then $\mathrm{H}(\mathcal{L}; \Bbbk)$ is pure by definition of $\mathrm{E}(\mathcal{L}; \Bbbk)$. Thus condition (a) is equivalent to condition (b).

If $\mathrm{E}(\mathcal{L}; \Bbbk) \cong \mathrm{H}(\mathcal{L}; \Bbbk)$, then $\underline{\mathrm{E}}(\mathcal{L}; \Bbbk) \cong \Bbbk$, thus ${}_pP_{\mathcal{L}}(t) = 1$. Conversely, if ${}_pP_{\mathcal{L}}(t) = 1$, then $\mathrm{E}(\mathcal{L}; \Bbbk) \cong \mathrm{H}(\mathcal{L}; \Bbbk)$ by Corollary 3.3. Thus condition (c) is equivalent to condition (b).

We next prove the equivalence of conditions (d) and (e). Birkhoff's theorem states that a geometric lattice $\mathcal{L}$ is modular if and only if $\mathcal{L} \cong \mathcal{L}_0 \times \mathcal{L}_1 \times \cdots \times \mathcal{L}_r$, where $\mathcal{L}_0$ is Boolean and $\mathcal{L}_i$ is a projective space of order q_i for all $i \geqslant 1$, see [Wel76, Chapter 12]. By Lemmas 3.6 and 3.7, such a lattice has a p-resonant element of corank greater than 1 if and only if q_i is a multiple of p for some i. This establishes the equivalence between (d) and (e).

It remains to prove the equivalence between conditions (b) and (e). First, we assume that we have an isomorphism $\mathcal{L} \cong \mathcal{L}_0 \times \mathcal{L}_1 \times \cdots \times \mathcal{L}_r$, where $\mathcal{L}_0$ is Boolean and $\mathcal{L}_i$ is a projective space of order q_i for some q_i that is not a multiple of p for each $1 \leqslant i \leqslant r$. By Theorem B(c) and Proposition 3.5, we have

$$\mathrm{H}(\mathcal{L}; \Bbbk) \cong \mathrm{H}(\mathcal{L}_0; \Bbbk) \otimes \cdots \otimes \mathrm{H}(\mathcal{L}_r; \Bbbk) \cong \mathrm{E}(\mathcal{L}_0; \Bbbk) \otimes \cdots \otimes \mathrm{E}(\mathcal{L}_r; \Bbbk) \cong \mathrm{E}(\mathcal{L}; \Bbbk).$$

Thus condition (e) implies condition (b). Conversely, suppose that $\mathrm{H}(\mathcal{L}; \Bbbk) \cong \mathrm{E}(\mathcal{L}; \Bbbk)$. This implies that ${}_pP_{\mathcal{L}}(t) = 1$, and therefore that ${}_0P_{\mathcal{L}}(t) = 1$ by Theorem E. By [EPW16, Proposition 2.14], this implies that $\mathcal{L}$ is modular. By Birkhoff's theorem, we have an isomorphism $\mathcal{L} \cong \mathcal{L}_0 \times \mathcal{L}_1 \times \cdots \times \mathcal{L}_r$, where $\mathcal{L}_0$ is Boolean and $\mathcal{L}_i$ is a projective space of order q_i for all $i \geqslant 1$. By Theorem B(c), we have $\mathrm{H}(\mathcal{L}_i; \Bbbk) \cong \mathrm{E}(\mathcal{L}_i; \Bbbk)$ for all $i \geqslant 0$, and by Proposition 3.5, q_i is not a multiple of p for all $i \geqslant 1$. This proves that condition (b) implies condition (e), and thus completes the proof of the theorem. □

4. The augmented Chow ring

In this section, we study the augmented Chow ring $\mathrm{CH}(\mathcal{L};\Bbbk)$ as a module over $\mathrm{H}(\mathcal{L};\Bbbk)$, and we use it to prove Theorems 1.4 and 1.12 as well as Proposition 1.9.

Let $\mathcal{L}^1$ be the set of atoms of $\mathcal{L}$, and let $S_{\mathcal{L}}$ be the ring of polynomials in two sets of variables:

$$S_{\mathcal{L}} \coloneqq \Bbbk[y_i \mid i \text{ is an element of } \mathcal{L}^1] \otimes_{\Bbbk} \Bbbk[x_F \mid F \text{ is an element of } \mathcal{L}].$$

The **augmented Chow ring** of $\mathcal{L}$ with coefficients in $\Bbbk$ is the quotient algebra

$$\mathrm{CH}(\mathcal{L};\Bbbk) \coloneqq S_{\mathcal{L}}/(I_{\mathcal{L}} + J_{\mathcal{L}}),$$

where $I_{\mathcal{L}}$ is the ideal generated by the linear forms

$$\sum_{F\in\mathcal{L}} x_F \quad\text{and}\quad y_i - \sum_{i\not\leqslant F} x_F, \quad \text{for every element } i \text{ of } \mathcal{L}^1,$$

and $J_{\mathcal{L}}$ is the ideal generated by the quadratic monomials

$$x_{F_1}x_{F_2}, \text{ for every pair of incomparable elements } F_1 \text{ and } F_2 \text{ of } \mathcal{L}, \text{ and}$$
$$y_i\, x_F, \text{ for every element } i \text{ of } \mathcal{L}^1 \text{ and every element } F \text{ of } \mathcal{L} \text{ satisfying } i \not\leqslant F.$$

By [BHM+22, Proposition 2.18], the $\Bbbk$-subalgebra of the augmented Chow ring generated by the elements y_i is isomorphic to the graded Möbius algebra $\mathrm{H}(\mathcal{L};\Bbbk)$. The **Chow ring** of $\mathcal{L}$ with coefficients in $\Bbbk$ is defined to be the stalk

$$\underline{\mathrm{CH}}(\mathcal{L};\Bbbk) \coloneqq \mathrm{CH}(\mathcal{L};\Bbbk)_{\hat{0}} = \mathrm{CH}(\mathcal{L};\Bbbk)/\langle y_i \mid i \in \mathcal{L}^1\rangle.$$

The Chow ring of a matroid is precisely the Chow ring of its associated Bergman fan, while the augmented Chow ring is the Chow ring of the augmented Bergman fan. For a comprehensive treatment of these constructions and their properties, we refer the reader to [BHM+22].

We will be particularly interested in the class $\underline{\beta}_{\mathcal{L}} \coloneqq -x_{\hat{0}}$ in $\underline{\mathrm{CH}}(\mathcal{L};\Bbbk)$.

Lemma 4.1.

(a) The space $\mathrm{CH}^1(\mathcal{L};\Bbbk)$ has $\Bbbk$-basis $\{x_F \mid \mathrm{rk}\, F \geqslant 1\}$.

(b) The space $\underline{\mathrm{CH}}^1(\mathcal{L};\Bbbk)$ has $\Bbbk$-basis $\{x_F \mid \mathrm{rk}\, F \geqslant 2\}$.

Proof. The relation $\sum_{F\in\mathcal{L}} x_F$ allows us to express $x_{\hat{0}}$ in terms of x_F for F of positive rank in $\mathrm{CH}(\mathcal{L};\Bbbk)$. In the Chow ring, where $y_i = 0$ for all i, we have $x_i = -\sum_{i<F} x_F$ for any $i \in \mathcal{L}^1$. □

For any $G \in \mathcal{L}\backslash\{\hat{0}\}$, we let m_G be the number of atoms of $\mathcal{L}^G$.

Lemma 4.2. In the Chow ring $\underline{\mathrm{CH}}(\mathcal{L};\Bbbk)$, we have the identity

$$\underline{\beta}_{\mathcal{L}} = \sum_{\mathrm{rk}\, G\geqslant 2} (1 - m_G)x_G.$$

Proof. For any element F of rank 1, we have

$$\sum_{F \leqslant G} x_G = \sum_G x_G - \sum_{F \nleqslant G} x_G = 0 - 0 = 0.$$

It follows that

$$\sum_{\operatorname{rk} G \geqslant 1} m_G x_G = \sum_{\substack{F \leqslant G \\ \operatorname{rk} F = 1}} x_G = \sum_{\operatorname{rk} F = 1} \sum_{F \leqslant G} x_G = 0.$$

We therefore have

$$\sum_{\operatorname{rk} G \geqslant 2} (1 - m_G) x_G = \sum_{\operatorname{rk} G \geqslant 1} (1 - m_G) x_G = \sum_{\operatorname{rk} G \geqslant 1} x_G - \sum_{\operatorname{rk} G \geqslant 1} m_G x_G = \sum_{\operatorname{rk} G \geqslant 1} x_G.$$

Since $\sum_G x_G = 0$, this is equal to $-x_{\hat{0}} = \underline{\beta}_{\mathcal{L}}$. □

Lemma 4.3. If m_G is congruent to 1 modulo p for every element $G \in \mathcal{L}$ of rank 2, then m_G is congruent to 1 modulo p for every element $G \in \mathcal{L}$ of rank at least 2.

Proof. Let G be given of rank at least 2, and fix a rank 1 element $F \leqslant G$. There is a surjective map

$$\nu \colon (\mathcal{L}^G)^1 \smallsetminus \{F\} \longrightarrow (\mathcal{L}^G_F)^1 = \mathcal{L}^F_G \cap \mathcal{L}^2$$

that takes F' to $F \vee F'$, and the cardinality of $\nu^{-1}(H)$ is equal to $m_H - 1$. We therefore have

$$m_G - 1 = \left|(\mathcal{L}^G)^1 \smallsetminus \{F\}\right| = \sum_{H \in (\mathcal{L}^G_F)^1} (m_H - 1).$$

Thus, if m_H is congruent to 1 for all H of rank 2, then m_G is congruent to 1, as well. □

Recall that an element $F \in \mathcal{L}$ is p-resonant if and only if $F \neq \hat{1}$ and, for all $G > F$ with $\operatorname{rk}(G) = \operatorname{rk}(F) + 2$, the cardinality of the open interval (F, G) is congruent to 1 modulo p.

Corollary 4.4. We have $\underline{\beta}_{\mathcal{L}} = 0 \in \underline{\mathrm{CH}}^1(\mathcal{L}; \Bbbk)$ if and only if $\hat{0} \in \mathcal{L}$ is p-resonant or $\operatorname{rk} \mathcal{L} = 0$.

Proof. We have $\hat{0} \neq \hat{1}$ if and only if $\operatorname{rk} \mathcal{L} > 0$, and the second condition for p-resonance of $\hat{0}$ is equivalent to the statement that $\underline{\beta}_{\mathcal{L}} = 0$ by Lemmas 4.1, 4.2, and 4.3. □

The following statements are proved in [BHM+22] and [BHM+] when $\Bbbk = \mathbb{Q}$, and the proofs go through for arbitrary $\Bbbk$ without modification.

Proposition 4.5. Let $\mathrm{CH}(\mathcal{L}; \Bbbk)$ be the augmented Chow ring of $\mathcal{L}$ with coefficients in a field $\Bbbk$.

(a) The $\mathrm{H}(\mathcal{L}; \Bbbk)$-module homomorphism from $\mathrm{H}(\mathcal{L}; \Bbbk)$ to $\mathrm{CH}(\mathcal{L}; \Bbbk)$ sending 1 to 1 is injective [BHM+22, Proposition 2.18].

(b) The Poincaré pairing on $\mathrm{CH}(\mathcal{L}; \Bbbk)$ is nondegenerate [BHM+22, Theorem 1.6(4)].

(c) For all $F \in \mathcal{L}$, there is a canonical isomorphism

$$y_F \operatorname{CH}(\mathcal{L};\Bbbk) \cong \operatorname{CH}(\mathcal{L}_F;\Bbbk)[-\operatorname{rk} F]$$

of $\mathrm{H}(\mathcal{L}_F;\Bbbk)$-modules [BHM+, Propositions 2.15 and 2.17].

(d) If $\operatorname{rk}\mathcal{L} > 0$, the ideals $\langle x_{\hat{0}}\rangle$ and $\langle y_i \mid i \in \mathcal{L}^1\rangle$ are mutual annihilators [BHM+, Lemma 5.3].

(e) The $\mathrm{H}(\mathcal{L};\Bbbk)$-module $\operatorname{CH}(\mathcal{L};\Bbbk)$ is pure [BHM+, Proposition 5.14].

The following proposition is proved in [BHM+, Proposition 5.9] without explicit reference to the commutative diagram; we provide the full proof here for completeness.

Proposition 4.6. We have $\operatorname{CH}(\mathcal{L};\Bbbk)_{\hat{1}} \cong \Bbbk \cong \operatorname{CH}(\mathcal{L};\Bbbk)_{[\hat{1}]}$, and the costalk-stalk homomorphism $\beta_{\hat{1}}(\operatorname{CH}(\mathcal{L};\Bbbk))$ is an isomorphism. For all $F \neq \hat{1}$, we have isomorphisms

$$\operatorname{CH}(\mathcal{L};\Bbbk)_{[F]} \cong \underline{\operatorname{CH}}(\mathcal{L}_F;\Bbbk)[-1] \quad\text{and}\quad \operatorname{CH}(\mathcal{L};\Bbbk)_F \cong \underline{\operatorname{CH}}(\mathcal{L}_F;\Bbbk)$$

such that the following diagram commutes:

$$\begin{tikzcd}
\operatorname{CH}(\mathcal{L};\Bbbk)_{[F]} \arrow[r, "\beta_F(\operatorname{CH}(\mathcal{L};\Bbbk))"] \arrow[d, "\cong"] & \operatorname{CH}(\mathcal{L};\Bbbk)_F \arrow[d, "\cong"] \\
\underline{\operatorname{CH}}(\mathcal{L}_F;\Bbbk)[-1] \arrow[r, "\cdot\underline{\beta}_{\mathcal{L}_F}"] & \underline{\operatorname{CH}}(\mathcal{L}_F;\Bbbk).
\end{tikzcd}$$

Proof. The first statement is very general: for any $\mathrm{H}(\mathcal{L};\Bbbk)$-module M, we have

$$\mathrm{M}_{\hat{1}} = y_{\hat{1}}\mathrm{M}[\operatorname{rk}\mathcal{L}] = \mathrm{M}_{[\hat{1}]},$$

and the costalk-stalk map $\beta_{\hat{1}}(\mathrm{M})$ is an isomorphism.

We prove the second statement for $F = \hat{0}$. By part (d) of Proposition 4.5, we have

$$\operatorname{CH}(\mathcal{L};\Bbbk)_{[\hat{0}]} = x_{\hat{0}} \operatorname{CH}(\mathcal{L};\Bbbk) \cong \underline{\operatorname{CH}}(\mathcal{L};\Bbbk)[-1],$$

where the isomorphism sends $x_{\hat{0}}x_F$ to $-x_F$. Using this isomorphism, the composition

$$\underline{\operatorname{CH}}(\mathcal{L};\Bbbk)[-1] \cong \operatorname{CH}(\mathcal{L};\Bbbk)_{[\hat{0}]} \xrightarrow{\beta_{\hat{0}}(\operatorname{CH}(\mathcal{L};\Bbbk))} \operatorname{CH}(\mathcal{L};\Bbbk)_{\hat{0}} \cong \underline{\operatorname{CH}}(\mathcal{L};\Bbbk)$$

is given by multiplication by $-x_{\hat{0}} = \underline{\beta}_{\mathcal{L}}$. The second statement for $\hat{0} < F < \hat{1}$ follows from the statement for $F = \hat{0}$ using part (c) of Proposition 4.5 and Lemma 2.3. □

Proof of Theorem 1.4. By [BHM+, Theorem 3.21] and [BHM+, Proposition 6.4], $\operatorname{IH}(\mathcal{L})$ is an indecomposable direct summand of $\operatorname{CH}(\mathcal{L};\mathbb{Q})$ that contains $\operatorname{CH}^0(\mathcal{L};\mathbb{Q})$, and its endomorphism ring as an $\mathrm{H}(\mathcal{L};\mathbb{Q})$-module consists only of scalars. In particular, it is pure by Proposition 4.5(e), and it has an element $1 \in \operatorname{IH}^0(\mathcal{L}) = \operatorname{CH}^0(\mathcal{L};\mathbb{Q})$ with the property that $y_{\hat{1}} \cdot 1 \neq 0$. Therefore, by Theorem A, $\operatorname{IH}(\mathcal{L})$ is isomorphic to $\mathrm{E}(\mathcal{L};\mathbb{Q})$ as $\mathrm{H}(\mathcal{L};\mathbb{Q})$-modules, and this isomorphism is unique up to a nonzero scalar. □

Proof of Proposition 1.9. By Proposition 4.5(e), Proposition 4.6, and Theorem F, there is an isomorphism

$$\mathrm{CH}(\mathcal{L};\Bbbk) \cong \mathrm{E}(\mathcal{L};\Bbbk) \oplus \bigoplus_{F<\hat{1}} \underline{\beta}_{\mathcal{L}_F} \underline{\mathrm{CH}}(\mathcal{L}_F;\Bbbk) \otimes \mathrm{E}(\mathcal{L}^F;\Bbbk). \tag{4}$$

By Corollary 4.4, $\underline{\beta}_{\mathcal{L}_F} = 0$ if and only if F is p-resonant, and hence

$$\mathrm{CH}^1(\mathcal{L};\Bbbk) \cong \mathrm{E}^1(\mathcal{L};\Bbbk) \oplus \Bbbk^r, \tag{5}$$

where r is the number of $F < \hat{1}$ which are not p-resonant. Similarly, by taking stalks at $\hat{0}$ of both sides of Equation (4), we obtain the equation

$$\underline{\mathrm{CH}}^1(\mathcal{L};\Bbbk) \cong \underline{\mathrm{E}}^1(\mathcal{L};\Bbbk) \oplus \Bbbk^r. \tag{6}$$

By parts (a) and (b) of Theorem B, $\mathrm{E}^0(\mathcal{L};\Bbbk)$ and $\underline{\mathrm{E}}^0(\mathcal{L};\Bbbk)$ are both 1-dimensional. By Lemma 4.1, the dimension of $\mathrm{CH}^1(\mathcal{L};\Bbbk)$ is equal to the cardinality of $\mathcal{L}$ minus one, and the dimension of $\underline{\mathrm{CH}}^1(\mathcal{L};\Bbbk)$ is $|\mathcal{L}| - |\mathcal{L}^1| - 1$. The proposition therefore follows from Equations (5) and (6). □

Remark 4.7. In Section 9.2, we give an alternative proof of Proposition 1.9 that does not use the augmented Chow ring.

Proof of Proposition 1.10. Theorem F implies that $\mathrm{E}(\mathcal{L};\Bbbk)$ is a direct summand of $\mathrm{CH}(\mathcal{L};\Bbbk)$, yielding the second inclusion. Since $\mathrm{E}^0(\mathcal{L};\Bbbk) = \mathrm{CH}^0(\mathcal{L};\Bbbk)$, the first inclusion follows from part (a) of Proposition 4.5. □

Remark 4.8. The inclusion $\mathrm{H}(\mathcal{L};\Bbbk) \subseteq \mathrm{E}(\mathcal{L};\Bbbk)$ can alternatively be deduced from the injectivity of the map ρ appearing in the proof of Proposition 2.7.

Proof of Theorem 1.12. By Equation (4), $\mathrm{E}(\mathcal{L};\Bbbk)$ is isomorphic to a direct summand of $\mathrm{CH}(\mathcal{L};\Bbbk)$, and hence conditions (a), (b), and (c) are equivalent. By Equation (4) and Corollary 4.4, conditions (b) and (e) are equivalent. By taking stalks at $\hat{0}$, we see that condition (b) implies condition (d). Finally, condition (d) implies condition (e) by Equation (6). □

5. Inclusions of lattices

In this section, we prove Theorems 1.15, 1.16, and 1.20. As in Section 1.4, we consider a rank-preserving inclusion of geometric lattices $\varphi\colon \mathcal{L} \to \mathcal{M}$. We begin by explaining why such an inclusion always comes from the operation of matroid restriction.

We assume that $\varphi\colon \mathcal{L} \to \mathcal{M}$ is an inclusion of posets satisfying

$$\mathrm{rk}_{\mathcal{M}}(\varphi(F)) = \mathrm{rk}_{\mathcal{L}}(F) \text{ for every } F \in \mathcal{L}.$$

Then φ preserves joins. Indeed, since every element of $\mathcal{L}$ is a join of atoms, it is enough to show that $\varphi(F \vee G) = \varphi(F) \vee \varphi(G)$ for any $F \in \mathcal{L}$ and $G \in \mathcal{L}^1$. If $G \leqslant F$, both sides are equal to $\varphi(F)$. If $G \nleqslant F$, then, since φ is order-preserving, $\varphi(F \vee G) \geqslant \varphi(F) \vee \varphi(G)$, and

$$\operatorname{rk}_{\mathcal{M}} \varphi(F \vee G) = \operatorname{rk}_{\mathcal{L}}(F \vee G) = \operatorname{rk}_{\mathcal{L}}(F) + 1 = \operatorname{rk}_{\mathcal{M}}(\varphi(F)) + 1 = \operatorname{rk}_{\mathcal{M}}\big(\varphi(F) \vee \varphi(G)\big).$$

It follows that $\varphi(F \vee G) = \varphi(F) \vee \varphi(G)$, as claimed.

Recall that the geometric lattice $\mathcal{M}$ may be viewed as the lattice of flats of a simple matroid with ground set $\mathcal{M}^1$ by identifying $F \in \mathcal{M}$ with the set

$$a_{\mathcal{M}}(F) \coloneqq \{G \in \mathcal{M}^1 \mid G \leqslant F\}.$$

In the same way, $\mathcal{L}$ may be viewed as the lattice of flats of a simple matroid on the ground set $\mathcal{L}^1$.

Lemma 5.1. For any $F \in \mathcal{L}$, we have $\varphi^{-1}(a_{\mathcal{M}}(\varphi(F))) = a_{\mathcal{L}}(F)$.

Proof. This is obvious when $\operatorname{rk}(F) \leqslant 1$. If the lemma holds for a particular $F \in \mathcal{L}$, then, by the partition axiom for flats of a matroid, it also holds for every G that covers F. Thus, by induction on rank, the statement holds for all F. □

Define a map $\delta\colon \mathcal{M} \to \mathcal{L}$ by letting $\delta(F)$ be the join of the elements of $\varphi^{-1}(a_{\mathcal{M}}(F))$ for each $F \in \mathcal{M}$. The map δ is clearly order-preserving, although it need not be injective. The composition $\delta \circ \varphi$ is the identity on $\mathcal{L}$: indeed, the previous lemma shows that $\delta\big(\varphi(F)\big)$ is the join of $a_{\mathcal{L}}(F)$, which is equal to F.

The composition $\varphi \circ \delta$ need not be the identity. However, since φ preserves joins, $\varphi\big(\delta(F)\big)$ is the join of the elements of $\varphi\big(\varphi^{-1}(a_{\mathcal{M}}(F))\big) \subseteq a_{\mathcal{M}}(F)$. It follows that $\varphi\big(\delta(F)\big) \leqslant F$ for every $F \in \mathcal{M}$.

Proposition 5.2. The geometric lattice $\mathcal{L}$ is naturally isomorphic to the lattice of flats of the restriction of $\mathcal{M}$ to $S \coloneqq \varphi(\mathcal{L}^1)$, or equivalently, to the lattice of flats of the matroid obtained from $\mathcal{M}$ by deleting all elements of $\mathcal{M}^1 \backslash S$.

Proof. Considering every geometric lattice as a matroid, the lattice of flats of the restriction $\mathcal{M}|S$ is equal to the set of all intersections $a_{\mathcal{M}}(F) \cap S$ for $F \in \mathcal{M}$. Since φ is injective, it induces a bijection between $\mathcal{L}^1$ and S, and so as posets,

$$\mathcal{M}|S \cong \{\varphi^{-1}(a_{\mathcal{M}}(F)) \mid F \in \mathcal{M}\}.$$

Since δ is surjective, it suffices to prove that

$$a_{\mathcal{L}}(\delta(F)) = \varphi^{-1}(a_{\mathcal{M}}(F)) \text{ for all } F \in \mathcal{M}.$$

Since $\delta(F)$ is defined to be the join of the elements on the right hand side, we have

$$\varphi^{-1}(a_{\mathcal{M}}(F)) \subseteq a_{\mathcal{L}}(\delta(F)).$$

On the other hand, using Lemma 5.1 and the fact that $\varphi\big(\delta(F)\big) \leqslant F$, we get

$$a_{\mathcal{L}}(\delta(F)) = \varphi^{-1}\left(a_{\mathcal{M}}\left(\varphi\left(\delta(F)\right)\right)\right) \subseteq \varphi^{-1}(a_{\mathcal{M}}(F)). \qquad \square$$

We now study how the maps φ and δ interact with upper sets of $\mathcal{L}$ and $\mathcal{M}$. For an upper set $\Sigma \subseteq \mathcal{L}$, define its **$\mathcal{M}$-saturation** by

$$\Sigma^+ := \{F \in \mathcal{M} \mid F \geqslant \varphi(G) \text{ for some } G \in \Sigma\}.$$

It is the smallest upper set containing $\varphi(\Sigma)$. For any $\mathrm{H}(\mathcal{M})$-module M and any upper set $\Sigma \subseteq \mathcal{L}$, we have

$$\mathrm{ann}_{\Sigma}(\mathrm{M}) = \mathrm{ann}_{\Sigma^+}(\mathrm{M}) \ \text{ and } \ \mathrm{span}_{\Sigma}(\mathrm{M}) = \mathrm{span}_{\Sigma^+}(\mathrm{M}),$$

where on the left-hand sides of the two equations we regard M as a module over $\mathrm{H}(\mathcal{L};\Bbbk)$, and on the right-hand sides of the two equations we regard M as a module over $\mathrm{H}(\mathcal{M};\Bbbk)$.

Lemma 5.3. For any upper set $\Sigma \subseteq \mathcal{L}$, we have $\Sigma^+ = \delta^{-1}(\Sigma)$.

Proof. Suppose first that $F \in \delta^{-1}(\Sigma)$, so that $\delta(F) \in \Sigma$. Since $F \geqslant \varphi\big(\delta(F)\big)$, we have $F \in \Sigma^+$. Conversely, suppose that $F \in \Sigma^+$. Then there is $G \in \Sigma$ such that $F \geqslant \varphi(G)$. It follows that $\delta(F) \geqslant \delta(\varphi(G)) = G$, and hence $\delta(F) \in \Sigma$. $\square$

Corollary 5.4. For any pair of upper sets $\Sigma_1, \Sigma_2 \subseteq \mathcal{L}$, we have

$$(\Sigma_1 \cap \Sigma_2)^+ = \Sigma_1^+ \cap \Sigma_2^+.$$

Proof of Theorem 1.15. For any pair of upper sets $\Sigma_1, \Sigma_2 \subseteq \mathcal{L}$, we have

$$\begin{aligned}\mathrm{ann}_{\Sigma_1}(\mathrm{M}) + \mathrm{ann}_{\Sigma_2}(\mathrm{M}) &= \mathrm{ann}_{\Sigma_1^+}(\mathrm{M}) + \mathrm{ann}_{\Sigma_2^+}(\mathrm{M}) = \mathrm{ann}_{\Sigma_1^+ \cap \Sigma_2^+}(\mathrm{M}) \\ &= \mathrm{ann}_{(\Sigma_1 \cap \Sigma_2)^+}(\mathrm{M}) = \mathrm{ann}_{\Sigma_1 \cap \Sigma_2}(\mathrm{M}),\end{aligned}$$

and hence M is flabby as a module over $\mathrm{H}(\mathcal{L};\Bbbk)$. The proof of coflabbiness is similar. $\square$

Proof of Theorem 1.16. By Theorem F, there is an $\mathrm{H}(\mathcal{L};\Bbbk)$-module isomorphism

$$\mathrm{E}(\mathcal{M};\Bbbk) \cong \bigoplus_{F \in \mathcal{L}} \mathrm{im}(\beta_F(\mathrm{E}(\mathcal{M};\Bbbk))) \otimes \mathrm{E}(\mathcal{L}^F;\Bbbk).$$

Thus we need only show that $\mathrm{im}(\beta_F(\mathrm{E}(\mathcal{M};\Bbbk))) \cong \mathrm{L}(\varphi_F;\Bbbk)$ for all $F \in \mathcal{L}$. When $F = \hat{0}$, this is immediate from the definition of $\mathrm{L}(\varphi;\Bbbk)$; see Example 1.31. For arbitrary F, by Lemma 2.3 and Proposition 2.4, we have

$$\beta_F(\mathrm{E}(\mathcal{M};\Bbbk)) \cong \beta_F(\mathrm{E}(\mathcal{M}_{\varphi(F)};\Bbbk)).$$

Under this isomorphism, F is the minimal element of the geometric lattice $\mathcal{L}_F$, so the latter agrees with $\mathrm{L}(\varphi_F;\Bbbk)$. $\square$

Proof of Theorem 1.20. The palindromicity of the local h-polynomials follows from Corollary 1.17, given the palindromicity of the Z-polynomials from Theorem 1.7. To be precise, say that a polynomial $f(t)$ is d-palindromic if $f(t) = t^d f(t^{-1})$. We claim that the local h-polynomial of φ is d-palindromic with $d = \mathrm{rk}(\mathcal{M})$.

The statement is proved by induction on $c = \mathrm{rk}(\mathcal{L})$. When $c = 0$, the local h-polynomial is $Z_{\mathcal{M}}(t)$, which is d-palindromic by Theorem 1.7. When $c > 0$, the left-hand side of the equation of Corollary 1.17 is d-palindromic, and all of the terms with $F \neq \hat{0}$ are a product of a $\mathrm{rk}(\mathrm{M}_{\varphi_F}) = \mathrm{crk}(\varphi(F))$-palindromic polynomial and a $\mathrm{rk}(\mathcal{L}^F) = \mathrm{rk}(F) = \mathrm{rk}(\varphi(F))$-palindromic polynomial. Since multiplying palindromic polynomials adds the degree of palindromicity, the result follows.

The proof of the unimodality of ${}_p\ell_\varphi(t)$ when $p = 0$ relies on the decomposition theorem for Lefschetz modules in [AHL25]. We will show in Proposition 9.7 that extending the coefficient field does not change the structure of parity modules. Thus, when $p = 0$, we may assume without loss of generality that $\Bbbk = \mathbb{R}$. By [BHM+, Theorem 1.6], the module $\mathrm{E}(\mathcal{M};\mathbb{R}) \cong \mathrm{IH}(\mathcal{M};\mathbb{R})$ satisfies Poincaré duality, the hard Lefschetz property, and the Hodge–Riemann relations with respect to the interior of the cone generated by $\{y_i \mid i \in \mathcal{M}^1\}$. It is therefore a Lefschetz module in the sense of [AHL25]. We apply the decomposition package of [AHL25] to the subalgebra of $\mathrm{H}(\mathcal{M};\mathbb{R})$ generated by $\{y_i \mid i \in \mathcal{L}^1\}$, which is naturally identified with $\mathrm{H}(\mathcal{L};\mathbb{R})$. In particular, [AHL25, Corollary 1.8] yields the relative hard Lefschetz theorem for $\mathrm{L}(\varphi;\mathbb{R})$, from which it follows that the polynomial $\ell_\varphi(t)$ is palindromic and unimodal. □

6. Exploding $\mathrm{H}(\mathcal{L};\Bbbk)$-modules

We now begin working toward the proofs of Theorems A, B, C, D, E, and F. As a first step, we introduce a new category $\mathcal{C}_{\mathrm{reg}}(\mathcal{L};\Bbbk)$ and prove that it is equivalent to the category of finitely generated graded $\mathrm{H}(\mathcal{L};\Bbbk)$-modules.

Though we have assumed until now that $\Bbbk$ is a field, we will now relax this assumption and allow $\Bbbk$ to be any local principal ideal domain, meaning that $\Bbbk$ is either a field or a discrete valuation ring. We will be particularly interested in the case of the p-adic integers, which will provide a framework for comparing parity modules in characteristics 0 and p, leading ultimately to a proof of Theorem E.

6.1. **The maps φ and ψ.** Let M be an $\mathrm{H}(\mathcal{L};\Bbbk)$-module. For any $F \in \mathcal{L}$, let $\mathrm{M}(F) \coloneqq y_F\mathrm{M}[\operatorname{rk} F]$. We can factor the multiplication by y_F as follows:

$$\begin{tikzcd} \mathrm{M} \arrow[rr, "y_F"] \arrow[dr, "\varphi^F"'] & & \mathrm{M}[\operatorname{rk} F] \\ & \mathrm{M}(F) \arrow[ur, "\psi^F"'] & \end{tikzcd}$$

Note that φ^F is surjective and ψ^F is injective, and $\mathrm{M}(F)$ is an $\mathrm{H}(\mathcal{L}_F;\Bbbk)$-module via the construction in Section 2. More generally, given $G \leqslant F \in \mathcal{L}$, we choose an element $H \in \mathcal{L}^{\operatorname{rk} F - \operatorname{rk} G}$ with $G \vee H = F$, and define a commutative diagram

$$\begin{tikzcd} \mathrm{M}(G) \arrow[rr, "y_H"] \arrow[dr, "\varphi^F_G"'] & & \mathrm{M}(G)[\operatorname{rk} F - \operatorname{rk} G] \\ & \mathrm{M}(F) \arrow[ur, "\psi^F_G"'] & \end{tikzcd}$$

by setting $\varphi^F_G(y_G m) \coloneqq y_H y_G m = y_F m$ and $\psi^F_G(y_F m) \coloneqq y_F m$ for all $m \in \mathrm{M}$. When $G = \hat{0}$, we have $\varphi^F_{\hat{0}} = \varphi^F$ and $\psi^F_{\hat{0}} = \psi^F$. In general, the element H with $G \vee H = F$ is not unique, but these maps do not depend on the choice.

Proposition 6.1. The maps $\{\varphi^F_G \mid G \leqslant F\}$ and $\{\psi^F_G \mid G \leqslant F\}$ satisfy the following relations:

(a) We have $\varphi^F_F = \mathrm{id}_{\mathrm{M}(F)} = \psi^F_F$.

(b) If $H \leqslant G \leqslant F$, then $\varphi^F_G \varphi^G_H = \varphi^F_H$ and $\psi^G_H \psi^F_G = \psi^F_H$.

(c) For any elements $F, G, H \in \mathcal{L}$ with $H \leqslant F \wedge G$, we have

$$\varphi^G_H \psi^F_H = \begin{cases} \psi^{F\vee G}_G \, \varphi^{F\vee G}_F & \text{if } F \text{ and } G \text{ form a modular pair and } H = F \wedge G, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the map $\varphi^G_H \psi^F_H$ sends $\mathrm{M}(F)$ to $\mathrm{M}(G)[\operatorname{rk} F - \operatorname{rk} H]$, while $\psi^{F\vee G}_G \varphi^{F\vee G}_F$ sends $\mathrm{M}(F)$ to $\mathrm{M}(G)[\operatorname{rk}(F \vee G) - \operatorname{rk} G]$. The condition that F and G form a modular pair and $H = F \wedge G$ is exactly what is needed for these maps to have the same shift.

Proof of Proposition 6.1. Statements (a) and (b) are straightforward. To prove statement (c), choose $K \in \mathcal{L}^{\operatorname{rk} F - \operatorname{rk} H}$ with $y_F = y_H y_K$ and $L \in \mathcal{L}^{\operatorname{rk} G - \operatorname{rk} H}$ with $y_G = y_H y_L$. For an element $y_F m \in \mathrm{M}(F)$, we have

$$\varphi^G_H \psi^F_H (y_F m) = \varphi^G_H (y_F m) = y_L y_F m.$$

Note that $F \vee L = (H \vee K) \vee L = (H \vee K) \vee (H \vee L) = F \vee G$, and

$$\operatorname{rk} F + \operatorname{rk} L = \operatorname{rk} H + \operatorname{rk} K + \operatorname{rk} L = \operatorname{rk} F + \operatorname{rk} G - \operatorname{rk} H.$$

If F and G form a modular pair and $H = F \wedge G$, this implies that $\operatorname{rk} F + \operatorname{rk} L = \operatorname{rk}(F \vee G) = \operatorname{rk}(F \vee L)$, and so $y_L y_F = y_{L \vee F} = y_{F \vee G}$. This implies that

$$\varphi_H^G \psi_H^F(y_F m) = y_L y_F m = y_{F \vee G} m = \psi_G^{F \vee G}(y_{F \vee G} m) = \psi_G^{F \vee G} \varphi_F^{F \vee G}(y_F m).$$

On the other hand, if F and G do not form a modular pair or $H < F \wedge G$, then $\operatorname{rk} L + \operatorname{rk} F > \operatorname{rk}(L \vee F)$, and hence $y_L y_F m = 0$. □

Remark 6.2. To improve readability, we will sometimes use an alternative notation for compositions of the maps $\{\varphi_G^F \mid G \leqslant F\}$ and $\{\psi_G^F \mid G \leqslant F\}$ by listing a sequence of elements, from right to left and with each one comparable with the next, separated by $\swarrow$ or $\nwarrow$ to indicate the order relation. For instance, the relation (b) says that

$$(F \nwarrow G \nwarrow H) = (F \nwarrow H) \text{ and } (H \swarrow G \swarrow F) = (H \swarrow F),$$

while relation (c) implies that

$$(G \nwarrow (F \wedge G) \swarrow F) = (G \swarrow (F \vee G) \nwarrow F)$$

whenever F and G form a modular pair.

6.2. **The category $\mathcal{C}(\mathcal{L}; \Bbbk)$.** Proposition 6.1 motivates the following definition. Consider the category $\mathcal{C}(\mathcal{L}; \Bbbk)$ whose objects are tuples $\mathcal{V} = \big(\{\mathcal{V}(F)\}, \{\varphi_G^F\}, \{\psi_G^F\}\big)$ consisting of a graded $\Bbbk$-module $\mathcal{V}(F)$ for each $F \in \mathcal{L}$ along with graded $\Bbbk$-linear maps

$$\varphi_G^F \colon \mathcal{V}(G) \to \mathcal{V}(F) \text{ and } \psi_G^F \colon \mathcal{V}(F) \to \mathcal{V}(G)[\operatorname{rk} F - \operatorname{rk} G]$$

for each $G \leqslant F$, satisfying the three conditions of Proposition 6.1. A morphism from $\mathcal{V}$ to $\mathcal{W}$ is a system of linear maps $\{\mathcal{V}(F) \to \mathcal{W}(F) \mid F \in \mathcal{L}\}$ commuting with the maps $\{\varphi_G^F \mid G \leqslant F\}$ and $\{\psi_G^F \mid G \leqslant F\}$.

Let $\mathcal{C}_{\mathrm{reg}}(\mathcal{L}; \Bbbk)$ be the full subcategory of $\mathcal{C}(\mathcal{L}; \Bbbk)$ consisting of objects such that the maps φ_G^F are all surjective and the maps ψ_G^F are all injective. We call such objects **regular**. There is a functor $\Lambda \colon \mathrm{H}(\mathcal{L}; \Bbbk)\text{–mod} \to \mathcal{C}_{\mathrm{reg}}(\mathcal{L}; \Bbbk)$ taking M to $(\{\mathrm{M}(F)\}, \{\varphi_G^F\}, \{\psi_G^F\})$ as defined in Section 6.1. We refer to $\Lambda\mathrm{M}$ as the **explosion** of M.

Proposition 6.3. The explosion functor $\Lambda \colon \mathrm{H}(\mathcal{L}; \Bbbk)\text{–mod} \to \mathcal{C}_{\mathrm{reg}}(\mathcal{L}; \Bbbk)$ is an equivalence of categories.

Proof. Define a functor $\Theta \colon \mathcal{C}(\mathcal{L}; \Bbbk) \to \mathrm{H}(\mathcal{L}; \Bbbk)\text{–mod}$ by sending an object $(\{\mathcal{V}(F)\}, \{\varphi_G^F\}, \{\psi_G^F\})$ to $\mathcal{V}(\hat{0})$, with the $\mathrm{H}(\mathcal{L}; \Bbbk)$-module structure defined by letting y_F act by $\psi_{\hat{0}}^F \varphi_{\hat{0}}^F$. Let us check that this defines an $\mathrm{H}(\mathcal{L}; \Bbbk)$-module. Let F and G be given elements of $\mathcal{L}$. If $\operatorname{rk}(F \vee G) = \operatorname{rk} F + \operatorname{rk} G$, then

(F, G) is a modular pair and $F \wedge G = \hat{0}$. In this case, by properties (b) and (c) of Proposition 6.1, for any $a \in \mathcal{V}(\hat{0})$, we have

$$\begin{aligned} y_F y_G a &= (\hat{0} \swarrow F \nwarrow \hat{0} \swarrow G \nwarrow \hat{0})(a) \\ &= (\hat{0} \swarrow F \swarrow (F \vee G) \nwarrow G \nwarrow \hat{0})(a) \\ &= (\hat{0} \swarrow (F \vee G) \nwarrow \hat{0})(a) = y_{F \vee G} a. \end{aligned}$$

On the other hand, if $\operatorname{rk}(F \vee G) < \operatorname{rk} F + \operatorname{rk} G$, then either F and G do not form a modular pair or $F \wedge G \neq \hat{0}$, so $(F \nwarrow \hat{0} \swarrow G) = 0$. This implies that $y_F y_G$ annihilates $\mathcal{V}(\hat{0})$, as required.

It is clear that, for any $\mathrm{H}(\mathcal{L}; \Bbbk)$-module M, we have a natural isomorphism $\Theta\Lambda\mathrm{M} \cong \mathrm{M}$. In the other direction, if $\mathcal{V} = (\{\mathcal{V}(F)\}, \{\varphi_G^F\}, \{\psi_G^F\})$ is regular and $\mathrm{M} = \Theta\mathcal{V}$, then there is a natural isomorphism

$$y_F \mathrm{M} = \operatorname{im}(\psi_{\hat{0}}^F \varphi_{\hat{0}}^F) \cong \mathcal{V}(F)[-\operatorname{rk} F],$$

since $\psi_{\hat{0}}^F$ is injective and $\varphi_{\hat{0}}^F$ is surjective. Putting these maps together gives a natural isomorphism $\mathcal{V} \cong \Lambda\Theta\mathcal{V}$. □

Example 6.4. Consider the $\mathrm{H}(\mathcal{L}; \Bbbk)$-module $\mathrm{CH}(\mathcal{L}; \Bbbk)$. For each $F \in \mathcal{L}$, we have

$$\Lambda(\mathrm{CH}(\mathcal{L}; \Bbbk))(F) = y_F \, \mathrm{CH}(\mathcal{L}; \Bbbk)[\operatorname{rk} F] \cong \mathrm{CH}(\mathcal{L}_F; \Bbbk),$$

where the last isomorphism comes from Proposition 4.5(c). For any $G \leqslant F$, the $\Bbbk$-module homomorphism $\varphi_G^F \colon \mathrm{CH}(\mathcal{L}_G; \Bbbk) \to \mathrm{CH}(\mathcal{L}_F; \Bbbk)$ agrees with the homomorphism $\varphi_{F \setminus G}^{\mathcal{L}_G}$ from [BHM+22, Proposition 2.28], while ψ_G^F agrees with the homomorphism $\psi_{F \setminus G}^{\mathcal{L}_G}$ from [BHM+22, Proposition 2.29].

Remark 6.5. The category $\mathcal{C}(\mathcal{L}; \Bbbk)$ can also be viewed as the category of finitely generated graded modules for the $\Bbbk$-path algebra C of a quiver with relations. The quiver has vertices given by elements of $\mathcal{L}$, and arrows in both directions between F and G whenever F covers G. If F covers G, then we have the relation $(F \nwarrow G \swarrow F) = 0$. If $F \neq F'$ both cover G and $H = F \vee F'$, then we have the relations

$$(F' \nwarrow G \swarrow F) = (F' \swarrow H \nwarrow F)$$

and

$$(H \swarrow G \swarrow F) = (H \swarrow G' \swarrow F); \quad (F \nwarrow G \nwarrow H) = (F \nwarrow G' \nwarrow H).$$

The algebra C is graded by assigning degree zero to the upward arrows and degree one to the downward arrows. If e is the idempotent corresponding to the path of length 0 at the vertex $\hat{0}$, then $eCe \cong \mathrm{H}(\mathcal{L}; \Bbbk)$, and the functor Θ of Proposition 6.3 is identified with the functor from C–mod to eCe–mod taking M to $e\mathrm{M}$. Note, however, that the explosion functor Λ is *not* identified either with the left adjoint $\mathrm{N} \mapsto Ce \otimes_{eCe} \mathrm{N}$ nor with the right adjoint $\mathrm{N} \mapsto \operatorname{Hom}_{eCe}(eC, \mathrm{N})$. Indeed, Λ is

neither left nor right exact. The nice feature of the functor Λ is that it interacts well with duality; see Section 7.3.

6.3. **Localization.** One of the main reasons to consider the category $\mathcal{C}_{\mathrm{reg}}(\mathcal{L};\Bbbk)$ instead of the equivalent category $\mathrm{H}(\mathcal{L};\Bbbk)$–mod is that we can consider "local" categories on subsets of $\mathcal{L}$. Given an upper set $\Sigma \subseteq \mathcal{L}$, let $\mathcal{C}(\Sigma;\Bbbk)$ denote the category whose objects are tuples $(\{\mathcal{V}(F)\},\{\varphi_G^F\},\{\psi_G^F\})$ satisfying the same properties as before, but with $F, G \in \Sigma$. Similarly, we let $\mathcal{C}_{\mathrm{reg}}(\Sigma;\Bbbk)$ be the full subcategory of $\mathcal{C}(\Sigma;\Bbbk)$ consisting of objects such that the maps φ_G^F are all surjective and the maps ψ_G^F are all injective. If $\Sigma' \subseteq \Sigma$ is another upper set, there is a forgetful functor $\mathcal{C}(\Sigma;\Bbbk) \to \mathcal{C}(\Sigma';\Bbbk)$, which we denote $\mathcal{V} \mapsto \mathcal{V}|_{\Sigma'}$.

If we consider $\mathcal{L}$ as a topological space whose open sets are the upper sets, then the categories $\mathcal{C}(\Sigma;\Bbbk)$ behave like sheaves on $\mathcal{L}$, in the sense that objects and morphisms glue. That is, given upper sets Σ_1 and Σ_2, objects $\mathcal{V}_i$ of $\mathcal{C}(\Sigma_i;\Bbbk)$, and an isomorphism $\mathcal{V}_1|_{\Sigma_1\cap\Sigma_2} \cong \mathcal{V}_2|_{\Sigma_1\cap\Sigma_2}$, they glue to give an object of $\mathcal{C}(\Sigma_1 \cup \Sigma_2)$ that restricts to both $\mathcal{V}_1$ and $\mathcal{V}_2$. Furthermore, given objects $\mathcal{V}$ and $\mathcal{W}$ of $\mathcal{C}(\Sigma_1 \cup \Sigma_2)$, we have an exact sequence of $\Bbbk$-modules

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_{\mathcal{C}(\Sigma_1\cup\Sigma_2;\Bbbk)}(\mathcal{V},\mathcal{W}) \longrightarrow \mathrm{Hom}_{\mathcal{C}(\Sigma_1;\Bbbk)}(\mathcal{V}|_{\Sigma_1},\mathcal{W}|_{\Sigma_1}) \oplus \mathrm{Hom}_{\mathcal{C}(\Sigma_2;\Bbbk)}(\mathcal{V}|_{\Sigma_2},\mathcal{W}|_{\Sigma_2}) \\ \longrightarrow \mathrm{Hom}_{\mathcal{C}(\Sigma_1\cap\Sigma_2;\Bbbk)}(\mathcal{V}|_{\Sigma_1\cap\Sigma_2},\mathcal{W}|_{\Sigma_1\cap\Sigma_2}). \end{aligned}$$

The inclusion $\mathcal{L}_F = \Sigma_{\geqslant F} \subseteq \mathcal{L}$ induces a functor from $\mathcal{C}(\mathcal{L};\Bbbk)$ to $\mathcal{C}(\mathcal{L}_F;\Bbbk)$, which restricts to a functor from $\mathcal{C}_{\mathrm{reg}}(\mathcal{L};\Bbbk)$ to $\mathcal{C}_{\mathrm{reg}}(\mathcal{L}_F;\Bbbk)$. Under the equivalence given by Proposition 6.3, this corresponds to the functor

$$\mathrm{H}(\mathcal{L};\Bbbk)\text{–mod} \longrightarrow \mathrm{H}(\mathcal{L}_F;\Bbbk)\text{–mod}, \qquad \mathrm{M} \longmapsto y_F\mathrm{M}[\mathrm{rk}\, F].$$

Remark 6.6. We also have a relation for lower intervals: an object $\mathcal{V}$ of $\mathcal{C}(\mathcal{L}^F;\Bbbk)$ can be extended to an object of $\mathcal{C}(\mathcal{L};\Bbbk)$ by letting $\mathcal{V}(G) = 0$ for all $G \nleqslant F$. This corresponds to pulling back by the quotient homomorphism $\mathrm{H}(\mathcal{L};\Bbbk) \to \mathrm{H}(\mathcal{L}^F;\Bbbk)$ described at the end of Section 1.1.

7. Purity, (co)stalks, and duality

In this section, we revisit various notions that we have already discussed for $\mathrm{H}(\mathcal{L};\Bbbk)$-modules, such as purity (flabbiness and coflabbiness), stalk and costalk functors, and duality. For each of these, we study the analogous properties and constructions for objects of the category $\mathcal{C}(\Sigma;\Bbbk)$.

7.1. **Flabbiness and coflabbiness.** An object $\mathcal{V} = (\{\mathcal{V}(F)\},\{\varphi_G^F\},\{\psi_G^F\})$ in $\mathcal{C}(\Sigma;\Bbbk)$ has a dual nature. The vector spaces $\mathcal{V}(F)$ together with the maps φ_G^F can be viewed as a sheaf on Σ, viewed as a finite topological space whose open sets are upper sets [Bac75]. Given an upper set $\Sigma' \subseteq \Sigma$, we define the sections on Σ' by

$$\Gamma_*(\Sigma',\mathcal{V}) \coloneqq \varprojlim_{F\in\Sigma'} \mathcal{V}(F),$$

where the limit is taken using the maps φ_G^F. In other words, $\Gamma_*(\Sigma', \mathcal{V})$ is the submodule of $\prod_{F \in \Sigma'} \mathcal{V}(F)$ given by tuples (x_F) for which $\varphi_G^F(x_G) = x_F$ for all $G < F$ in Σ'. There is a restriction map $\Gamma_*(\Sigma', \mathcal{V}) \to \Gamma_*(\Sigma'', \mathcal{V})$ for any upper sets $\Sigma'' \subseteq \Sigma' \subseteq \Sigma$, and these maps define a sheaf on Σ.

Dually, we can view $(\{\mathcal{V}(F)\}, \{\psi_G^F\})$ as a cosheaf by defining its cosections on Σ' to be

$$\Gamma_!(\Sigma', \mathcal{V}) = \varinjlim_{F \in \Sigma'} \mathcal{V}(F)[-\operatorname{rk} F],$$

with the colimit taken using the maps ψ_G^F. In other words, it is the quotient of $\bigoplus_{F \in \Sigma'} \mathcal{V}(F)[-\operatorname{rk} F]$ by the relations $\psi_G^F(x) - x$ for all $G < F$ in Σ' and $x \in \mathcal{V}(F)$. We have a corestriction map $\Gamma_!(\Sigma'', \mathcal{V}) \to \Gamma_!(\Sigma', \mathcal{V})$ for any upper sets $\Sigma'' \subseteq \Sigma' \subseteq \Sigma$, and these maps define a cosheaf on Σ.

For any element $F \in \Sigma$, the sections and cosections on $\Sigma_{\geqslant F}$ are particularly simple. Since $\Sigma_{\geqslant F}$ is the smallest open set containing F, we have

$$\Gamma_*(\Sigma_{\geqslant F}, \mathcal{V}) \cong \mathcal{V}(F) \quad \text{and} \quad \Gamma_!(\Sigma_{\geqslant F}, \mathcal{V}) \cong \mathcal{V}(F)[-\operatorname{rk} F]. \tag{7}$$

Definition 7.1. We say that $\mathcal{V}$ is **flabby** if the restriction maps $\Gamma_*(\Sigma', \mathcal{V}) \to \Gamma_*(\Sigma'', \mathcal{V})$ are all surjective. We say that $\mathcal{V}$ is **coflabby** if the corestrictions $\Gamma_!(\Sigma'', \mathcal{V}) \to \Gamma_!(\Sigma', \mathcal{V})$ are all injective.

Remark 7.2. Based on Definition 1.1, the reader might expect us to define $\mathcal{V}$ to be **pure** if it is both flabby and coflabby. This is correct if $\Bbbk$ is a field, but when $\Bbbk$ is a DVR we will want to impose an additional condition, and similarly for $\mathrm{H}(\mathcal{L}; \Bbbk)$-modules; see Definition 7.10.

We next show that the explosion functor Λ takes flabby (respectively coflabby) $\mathrm{H}(\mathcal{L}; \Bbbk)$-modules to flabby (respectively coflabby) objects of $\mathcal{C}(\mathcal{L}; \Bbbk)$. For any $\mathrm{H}(\mathcal{L}; \Bbbk)$-module M and any upper set Σ, we have natural maps

$$\Gamma_!(\Sigma, \Lambda\mathrm{M}) \twoheadrightarrow \operatorname{span}_\Sigma(\mathrm{M}) \quad \text{and} \quad \mathrm{M}/\operatorname{ann}_\Sigma(\mathrm{M}) \hookrightarrow \Gamma_*(\Sigma, \Lambda\mathrm{M})$$

induced by the maps $(\Lambda\mathrm{M})(F)[-\operatorname{rk} F] = y_F\mathrm{M} \to \operatorname{span}_\Sigma(\mathrm{M})$ and $\mathrm{M}/\operatorname{ann}_\Sigma(\mathrm{M}) \to (\Lambda\mathrm{M})(F)$ for all $F \in \Sigma$.

Proposition 7.3. Let M be an $\mathrm{H}(\mathcal{L}; \Bbbk)$-module. The following are equivalent:

(a) M is coflabby.

(b) $\Lambda\mathrm{M}$ is coflabby.

(c) The map $\Gamma_!(\Sigma, \Lambda\mathrm{M}) \twoheadrightarrow \operatorname{span}_\Sigma(\mathrm{M})$ is an isomorphism for all upper sets $\Sigma \subseteq \mathcal{L}$.

Dually, the following are equivalent:

(d) M is flabby.

(e) $\Lambda\mathrm{M}$ is flabby.

(f) The map $\mathrm{M}/\operatorname{ann}_\Sigma(M) \hookrightarrow \Gamma_*(\Sigma, \Lambda\mathrm{M})$ is an isomorphism for all upper sets $\Sigma \subseteq \mathcal{L}$.

Proof. We begin by observing that $\Lambda\mathrm{M}$ is coflabby if and only if, for all upper sets $\Sigma \subseteq \mathcal{L}$, the corestriction map

$$\Gamma_!(\Sigma, \Lambda\mathrm{M}) \longrightarrow \Gamma_!(\mathcal{L}, \Lambda\mathrm{M}) = \mathrm{M}$$

is injective. The image of this map is $\operatorname{span}_\Sigma(\mathrm{M})$, so (b) and (c) are equivalent.

Suppose that (c) holds, and consider two upper sets Σ_1 and Σ_2. The fact that $\Sigma \mapsto \Gamma_!(\Sigma, \Lambda\mathrm{M})$ is a cosheaf gives an exact sequence

$$\Gamma_!(\Sigma_1 \cap \Sigma_2, \Lambda\mathrm{M}) \longrightarrow \Gamma_!(\Sigma_1, \Lambda\mathrm{M}) \oplus \Gamma_!(\Sigma_2, \Lambda\mathrm{M}) \longrightarrow \Gamma_!(\Sigma_1 \cup \Sigma_2, \Lambda\mathrm{M}).$$

Applying (c) we obtain an exact sequence

$$\operatorname{span}_{\Sigma_1 \cap \Sigma_2}(\mathrm{M}) \longrightarrow \operatorname{span}_{\Sigma_1}(\mathrm{M}) \oplus \operatorname{span}_{\Sigma_2}(\mathrm{M}) \longrightarrow \operatorname{span}_{\Sigma_1 \cup \Sigma_2}(\mathrm{M}),$$

where the maps are induced by the natural inclusions, up to a sign. It follows that we have $\operatorname{span}_{\Sigma_1}(\mathrm{M}) \cap \operatorname{span}_{\Sigma_2}(\mathrm{M}) = \operatorname{span}_{\Sigma_1 \cap \Sigma_2}(\mathrm{M})$, so (a) holds.

Conversely, suppose that (a) holds. We show that $\Gamma_!(\Sigma, \Lambda\mathrm{M}) \to \operatorname{span}_\Sigma \mathrm{M}$ is an isomorphism by induction on $|\Sigma|$. If Σ is empty, then both sides vanish. If $|\Sigma| \geqslant 1$, there are two cases. If $\Sigma = \Sigma_{\geqslant F}$ for some F, then both sides are equal to $y_F\mathrm{M}$. Otherwise, choose a minimal element F and let $\Sigma' = \Sigma \backslash \{F\}$, so that $\Sigma = \Sigma' \cup \Sigma_{\geqslant F}$ and $\Sigma' \cap \Sigma_{\geqslant F} = \Sigma_{>F}$. Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Gamma_!(\Sigma_{>F}, \Lambda\mathrm{M}) \arrow[r] \arrow[d] & \Gamma_!(\Sigma', \Lambda\mathrm{M}) \oplus \Gamma_!(\Sigma_{\geqslant F}, \Lambda\mathrm{M}) \arrow[r] \arrow[d] & \Gamma_!(\Sigma, \Lambda\mathrm{M}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{span}_{\Sigma_{>F}}(\mathrm{M}) \arrow[r] & \operatorname{span}_{\Sigma'}(\mathrm{M}) \oplus \operatorname{span}_{\Sigma_{\geqslant F}}(\mathrm{M}) \arrow[r] & \operatorname{span}_\Sigma(\mathrm{M}) \arrow[r] & 0
\end{tikzcd}$$

Since Σ has more than one minimal element, both Σ' and $\Sigma_{\geqslant F}$ have fewer elements than Σ. The induction hypothesis implies that the first and second vertical maps are isomorphisms, so the result will follow by the five-lemma if we can show that both rows are exact.

The bottom row is exact on the left because the components of the map are inclusions, it is exact on the right since $\operatorname{span}_\Sigma(\mathrm{M}) = \operatorname{span}_{\Sigma'}(\mathrm{M}) + \operatorname{span}_{\Sigma_{\geqslant F}}(\mathrm{M})$, and it is exact in the middle by applying property (a) with $\Sigma_1 = \Sigma'$ and $\Sigma_2 = \Sigma_{\geqslant F}$. For the top row, exactness on the left follows from the injectivity of the left vertical map and exactness of the bottom row. Exactness in the other two places follows from the fact that $\Sigma \mapsto \Gamma_!(\Sigma, \Lambda\mathrm{M})$ is a cosheaf.

This completes the proof that (a), (b), (c) are equivalent. The proof that (d), (e), (f) are equivalent is analogous, with $\Gamma_!$ replaced by Γ_*, $\operatorname{span}_\Sigma \mathrm{M}$ replaced by $\mathrm{M}/\operatorname{ann}_\Sigma \mathrm{M}$, and all arrows reversed. □

Example 7.4. The regular $\mathrm{H}(\mathcal{L}; \Bbbk)$-module $\mathrm{H}(\mathcal{L}; \Bbbk)$ is coflabby because $\operatorname{span}_\Sigma \mathrm{H}(\mathcal{L}; \Bbbk)$ has a basis $\{y_F \mid F \in \Sigma\}$. On the other hand, $\mathrm{H}(\mathcal{L}; \Bbbk)$ is in general not flabby. For example, consider the lattice of flats $\mathcal{L}$ of the rank 3 uniform matroid on $E = \{1, 2, 3, 4\}$, so that the elements of $\mathcal{L}$ are E itself

along with all subsets $F \subseteq E$ with $|F| \leqslant 2$. Then $y_3 - y_4 \in \operatorname{ann}_{\Sigma_{\geqslant 12}} \mathrm{H}(\mathcal{L};\Bbbk)$, since $y_3 y_{12} = y_E = y_4 y_{12}$, but $y_3 - y_4 \notin \operatorname{ann}_{\Sigma_{\geqslant 1}} \mathrm{H}(\mathcal{L};\Bbbk) + \operatorname{ann}_{\Sigma_{\geqslant 2}} \mathrm{H}(\mathcal{L};\Bbbk)$. Since $\mathcal{L}$ is not modular, this also follows from Theorem 1.11.

Example 7.5. Let $\mathcal{L}$ be the Boolean lattice on the ground set $\{1,2\}$, so $\mathrm{H}(\mathcal{L};\Bbbk)$ is isomorphic to $\Bbbk[y_1,y_2]/\langle y_1^2, y_2^2\rangle$, with $y_{12} = y_1 y_2$. Then the maximal ideal $\mathfrak{m}$ is not coflabby: we have $y_1\mathfrak{m} \cap y_2\mathfrak{m} = \Bbbk y_{12}$, but $y_{12}\mathfrak{m} = 0$. On the other hand, since $\operatorname{ann}_{\Sigma_{\geqslant 1}} \mathfrak{m} + \operatorname{ann}_{\Sigma_{\geqslant 2}} \mathfrak{m} = \mathfrak{m} = \operatorname{ann}_{\Sigma_{\geqslant 1}\cap\Sigma_{\geqslant 2}} \mathfrak{m}$ and $(\Sigma_{\geqslant 1}, \Sigma_{\geqslant 2})$ is the only pair of incomparable upper sets of $\mathcal{L}$, it follows that $\mathfrak{m}$ is flabby.

7.2. **Stalks and costalks.** We have already discussed the stalk and costalk functors

$$(\ )_F, (\ )_{[F]} \colon \mathrm{H}(\mathcal{L};\Bbbk)\text{–mod} \longrightarrow \Bbbk\text{–mod}.$$

We define analogous functors on the category $\mathcal{C}(\Sigma;\Bbbk)$ as follows. For any $F \in \Sigma$, and any object $\mathcal{V} = (\{\mathcal{V}(F)\}, \{\varphi_G^F\}, \{\psi_G^F\})$ of $\mathcal{C}(\Sigma;\Bbbk)$, we define the **stalk**

$$\mathcal{V}_F \coloneqq \operatorname{coker}\left(\sum_{G>F} \psi_F^G \colon \bigoplus_{G>F} \mathcal{V}(G)[\operatorname{rk} F - \operatorname{rk} G] \longrightarrow \mathcal{V}(F)\right)$$

and the **costalk**

$$\mathcal{V}_{[F]} \coloneqq \ker\left(\sum_{G>F} \varphi_F^G \colon \mathcal{V}(F) \longrightarrow \bigoplus_{G>F} \mathcal{V}(G)\right).$$

These functors can also be described in terms of sections and cosections. Using the isomorphisms in Equation (7), the restriction and corestriction maps for the inclusion $\Sigma_{>F} \subseteq \Sigma_{\geqslant F}$ give maps

$$\Gamma_!\big(\Sigma_{>F}, \mathcal{V}\big)[\operatorname{rk} F] \xrightarrow{\Psi_F} \mathcal{V}(F) \xrightarrow{\Phi_F} \Gamma_*\big(\Sigma_{>F}, \mathcal{V}\big).$$

Then, we have $\mathcal{V}_F = \operatorname{coker} \Psi_F$ and $\mathcal{V}_{[F]} = \ker \Phi_F$. The following lemma is an immediate consequence of the definitions.

Lemma 7.6. For an $\mathrm{H}(\mathcal{L};\Bbbk)$-module M, we have natural isomorphisms

$$(\Lambda \mathrm{M})_F \cong \mathrm{M}_F \ \text{ and } \ (\Lambda \mathrm{M})_{[F]} \cong \mathrm{M}_{[F]}.$$

Remark 7.7. Lemma 7.6 illustrates why we have chosen to use the words "stalk" and "costalk" for $\mathcal{V}_F$ and $\mathcal{V}_{[F]}$. However, the reader is warned that $\mathcal{V}_F$ is *not* isomorphic to the stalk of $\mathcal{V}$ in the sense of sheaf theory. If we regard $\mathcal{V}$ as a sheaf on the finite topological space Σ, then $\Sigma_{\geqslant F}$ is the minimal open set containing F, so the stalk of $\mathcal{V}$ at F is equal to $\Gamma_*(\Sigma_{\geqslant F}, \mathcal{V}) \cong \mathcal{V}(F)$, which is typically not isomorphic to $\mathcal{V}_F$.

The following lemma is tautological when $\Sigma' = \Sigma_{\geqslant F}$, but we will need it for arbitrary Σ'.

Lemma 7.8. Suppose we have upper sets $\Sigma' \subseteq \Sigma$, and let $\Sigma'' \coloneqq \Sigma' \backslash \{F\}$, where F is a minimal element in Σ'. Let $\mathcal{V}$ be an object of $\mathcal{C}(\Sigma;\Bbbk)$.

(a) There is an exact sequence

$$0 \longrightarrow \mathcal{V}_{[F]} \longrightarrow \Gamma_*(\Sigma', \mathcal{V}) \longrightarrow \Gamma_*(\Sigma'', \mathcal{V}),$$

and the last map is surjective if $\mathcal{V}$ is flabby.

(b) There is an exact sequence

$$\Gamma_!(\Sigma'', \mathcal{V})[\operatorname{rk} F] \longrightarrow \Gamma_!(\Sigma', \mathcal{V})[\operatorname{rk} F] \longrightarrow \mathcal{V}_F \longrightarrow 0,$$

and the first map is injective if $\mathcal{V}$ is coflabby.

Proof. The fact that $\mathcal{V}$ is a sheaf on Σ implies that we have an exact sequence

$$0 \longrightarrow \Gamma_*(\Sigma', \mathcal{V}) \longrightarrow \Gamma_*(\Sigma'', \mathcal{V}) \oplus \Gamma_*(\Sigma_{\geqslant F}, \mathcal{V}) \longrightarrow \Gamma_*(\Sigma_{>F}, \mathcal{V}).$$

Thus, the kernel of the restriction map $\Gamma_*(\Sigma', \mathcal{V}) \to \Gamma_*(\Sigma'', \mathcal{V})$ is isomorphic to the kernel of the map $\Phi_F \colon \mathcal{V}(F) = \Gamma_*(\Sigma_{\geqslant F}, \mathcal{V}) \to \Gamma_*(\Sigma_{>F}, \mathcal{V})$. This gives the first exact sequence, and flabbiness of $\mathcal{V}$ implies the surjectivity of the last map. The proof of (b) is analogous. □

Lemma 7.9. An object $\mathcal{V}$ of $\mathcal{C}(\Sigma; \Bbbk)$ is flabby if and only if $\Phi_F \colon \mathcal{V}(F) \to \Gamma_*\big(\Sigma_{>F}, \mathcal{V}\big)$ is surjective for all $F \in \Sigma$, and coflabby if and only if $\Psi_F \colon \Gamma_!\big(\Sigma_{>F}, \mathcal{V}\big)[\operatorname{rk} F] \to \mathcal{V}(F)$ is injective for all $F \in \Sigma$.

Proof. We prove the first statement; the proof of the second statement is similar. The map Φ_F is by definition the restriction of sections from $\Sigma_{\geqslant F}$ to $\Sigma_{>F}$, so it is surjective when $\mathcal{V}$ is flabby. Now assume that Φ_F is surjective for all $F \in \Sigma$, and let $\Sigma'' \subseteq \Sigma' \subseteq \Sigma$ be given. We wish to show that the restriction $\Gamma_*(\Sigma', \mathcal{V}) \to \Gamma_*(\Sigma'', \mathcal{V})$ is a surjection.

By induction on $|\Sigma' \backslash \Sigma''|$, it suffices to prove surjectivity when $\Sigma' = \Sigma'' \cup \{F\}$ for some $F \in \Sigma$. Given a section $s \in \Gamma_*(\Sigma'', \mathcal{V})$, the restriction $s|_{\Sigma_{>F}}$ extends to a section $t \in \Gamma_*(\Sigma_{\geqslant F}, \mathcal{V}) = \mathcal{V}(F)$, since Φ_F is surjective. The sections s and t agree on $\Sigma'' \cap \Sigma_{\geqslant F} = \Sigma_{>F}$, so they glue to give a section on Σ' whose restriction to Σ'' is s. □

7.3. **Freeness and Duality.** When $\Bbbk$ is a field, the endofunctor of $\mathrm{H}(\mathcal{L}; \Bbbk)$–$\mathrm{mod}$ that sends M to $\mathrm{M}^* := \mathrm{Hom}_{\Bbbk}(\mathrm{M}, \Bbbk)$ is exact, and the natural map $\mathrm{M} \to (\mathrm{M}^*)^*$ is an isomorphism. (Recall that all of our modules are finitely generated, and that any finitely generated $\mathrm{H}(\mathcal{L}; \Bbbk)$-module is also finitely generated as a $\Bbbk$-module.)

When $\Bbbk$ is a DVR, however, we only have $V \cong (V^*)^*$ when V is a free $\Bbbk$-module, and the dual of a *split* exact sequence is exact, but the duals of exact sequences are only left exact in general. Because of this, we need to add an additional freeness hypothesis when $\Bbbk$ is a DVR. The case of a DVR is needed for the proof of Theorem E, which appears in Section 9.4. For all other results in the paper, nothing is lost by assuming that $\Bbbk$ is a field.

Definition 7.10. An $\mathrm{H}(\mathcal{L};\Bbbk)$-module M is called **locally free** if the stalk M_F is a free $\Bbbk$-module for all $F \in \mathcal{L}$. An object $\mathcal{V}$ of $\mathcal{C}(\Sigma;\Bbbk)$ is called **locally free** if the stalk $\mathcal{V}_F$ is a free $\Bbbk$-module for all $F \in \Sigma$. An object of either $\mathrm{H}(\mathcal{L};\Bbbk)$–$\mathrm{mod}$ or $\mathcal{C}(\Sigma;\Bbbk)$ is called **pure** if it is flabby, coflabby, and locally free.

Remark 7.11. Lemma 7.6 and Proposition 7.3 imply that M is locally free if and only if $\Lambda\mathrm{M}$ is locally free, and M is pure if and only if $\Lambda\mathrm{M}$ is pure.

Lemma 7.12. Let $\mathcal{V}$ be an object of $\mathcal{C}(\Sigma;\Bbbk)$. If $\mathcal{V}$ is pure, then it is regular.

Proof. We have $\Gamma_*(\Sigma_{\geqslant F},\mathcal{V}) \cong \mathcal{V}(F)$, and for any $G \leqslant F$, the restriction map from $\Gamma_*(\Sigma_{\geqslant G},\mathcal{V})$ to $\Gamma_*(\Sigma_{\geqslant F},\mathcal{V})$ is given by φ_G^F. Thus flabbiness implies these maps are surjective. A dual argument shows that coflabbiness implies the maps ψ_G^F are injective. □

Proposition 7.13. If an object $\mathcal{V}$ of $\mathcal{C}(\Sigma;\Bbbk)$ is coflabby and locally free, the following hold:

(a) For any upper set $\Sigma' \subseteq \Sigma$, the space $\Gamma_!(\Sigma',\mathcal{V})$ of cosections is a free $\Bbbk$-module.
(b) The corestriction homomorphisms $\Gamma_!(\Sigma'',\mathcal{V}) \to \Gamma_!(\Sigma',\mathcal{V})$ are split injections.
(c) Every $\mathcal{V}(F)$ is a free $\Bbbk$-module, and each map ψ_G^F is a split injection.
(d) For any upper set $\Sigma' \subseteq \Sigma$, the space $\Gamma_*(\Sigma',\mathcal{V})$ of sections is a free $\Bbbk$-module.

Proof. Consider upper sets $\Sigma'' \subseteq \Sigma' \subseteq \Sigma$, and suppose that $\Sigma'' = \Sigma'\backslash\{F\}$ for a minimal element $F \in \Sigma'$. Lemma 7.8 gives an exact sequence of $\Bbbk$-modules

$$0 \longrightarrow \Gamma_!(\Sigma'',\mathcal{V}) \longrightarrow \Gamma_!(\Sigma',\mathcal{V}) \longrightarrow \mathcal{V}_F[-\operatorname{rk} F] \longrightarrow 0.$$

Since $\mathcal{V}_F$ is free, the sequence splits, and if $\Gamma_!(\Sigma'',\mathcal{V})$ is free, then so is $\Gamma_!(\Sigma',\mathcal{V})$. Induction on the quantity $|\Sigma'|$ shows that $\Gamma_!(\Sigma',\mathcal{V})$ is free, and induction on the quantity $|\Sigma'\backslash\Sigma''|$ shows that the corestriction map $\Gamma_!(\Sigma'',\mathcal{V}) \to \Gamma_!(\Sigma',\mathcal{V})$ is a composition of split injections, hence also a split injection. The statement (c) is a special case of (a) and (b), since $\mathcal{V}(F) = \Gamma_!(\Sigma_{\geqslant F},\mathcal{V})[\operatorname{rk} F]$ and the maps ψ_G^F are corestrictions.

To prove (d), we note that $\Gamma_*(\Sigma',\mathcal{V})$ is a $\Bbbk$-submodule of the free module $\bigoplus_{F\in\Sigma'}\mathcal{V}(F)$, hence it is torsion-free, and a finitely generated torsion-free module over a DVR is free. □

We next define the duality functor on $\mathcal{C}(\Sigma;\Bbbk)$. For an object $\mathcal{V} = (\{\mathcal{V}(F)\},\{\varphi_G^F\},\{\psi_G^F\})$, let

$$\mathcal{W}(F) = (\mathcal{V}(F))^*[\operatorname{rk} F].$$

For any $G \leqslant F \in \Sigma$, let $\bar{\varphi}_G^F\colon \mathcal{W}(G) \to \mathcal{W}(F)$ be the dual of the $\Bbbk$-module homomorphism

$$\psi_G^F[-\operatorname{rk} F]\colon \mathcal{V}(F)[-\operatorname{rk} F] \longrightarrow \mathcal{V}(G)[-\operatorname{rk} G].$$

Similarly, let $\bar{\psi}^F_G \colon \mathcal{W}(F) \to \mathcal{W}(G)[\operatorname{rk} F - \operatorname{rk} G]$ be the dual of the homomorphism

$$\varphi^F_G[-\operatorname{rk} F] \colon \mathcal{V}(G)[-\operatorname{rk} F] \longrightarrow \mathcal{V}(F)[-\operatorname{rk} F].$$

Then we define $\mathcal{V}^*$ to be the triple $(\{\mathcal{W}(F)\}, \{\bar{\varphi}^F_G\}, \{\bar{\psi}^F_G\})$.

Lemma 7.14. Let M be an $\mathrm{H}(\mathcal{L};\Bbbk)$-module. If $\Bbbk$ is a field or if M is coflabby and locally free, then there is a natural isomorphism $\Lambda(\mathrm{M}^*) \cong (\Lambda \mathrm{M})^*$.

Proof. By Proposition 7.13, M and $\mathrm{M}(F)$ are free $\Bbbk$-modules, and $\psi^F = \psi^F_0 \colon \mathrm{M}(F) \to \mathrm{M}[\operatorname{rk} F]$ is a split injection. Dualizing and shifting gives a factorization of multiplication by y_F on M^*:

$$\begin{tikzcd}
\mathrm{M}^* \arrow[rr, "y_F"] \arrow[dr, "(\psi^F)^*"'] & & \mathrm{M}^*[\operatorname{rk} F] \\
& \mathrm{M}(F)^*[\operatorname{rk} F] \arrow[ur, "(\varphi^F)^*"'] &
\end{tikzcd}$$

Since $(\psi^F)^*$ is a surjection and $(\varphi^F)^*$ is an injection, we have an isomorphism $\mathrm{M}^*(F) \cong \mathrm{M}(F)^*[\operatorname{rk} F]$. Dualizing the maps φ^F_G and ψ^F_G is handled similarly. □

Lemma 7.15. Let $\Sigma \subseteq \mathcal{L}$ be an upper set and $\mathcal{V}$ an object of $\mathcal{C}_{\mathrm{reg}}(\Sigma)$. If $\Bbbk$ is a field or if $\mathcal{V}$ is coflabby and locally free, then $\mathcal{V}^*$ is regular.

Proof. Since coflabbiness, local freeness, and regularity are all preserved under restriction from $\mathcal{C}(\Sigma;\Bbbk)$ to $\mathcal{C}(\Sigma';\Bbbk)$, it is enough to prove the assertion when $\Sigma = \Sigma_{\geqslant F}$ is a principal open set. But $\Sigma_{\geqslant F} = \mathcal{L}_F$ is a geometric lattice, so it is enough to prove the statement when $\Sigma = \mathcal{L}$. Let $\mathrm{M} \coloneqq \Theta\mathcal{V}$; then Proposition 6.3 tells us that $\mathcal{V} \cong \Lambda\mathrm{M}$. Since $\Bbbk$ is a field or M is coflabby and locally free, Lemma 7.14 says that

$$\mathcal{V}^* \cong (\Lambda\mathrm{M})^* \cong \Lambda(\mathrm{M}^*),$$

which is regular by Proposition 6.3. □

When $\Bbbk$ is a field, the duality functor on $\mathcal{C}(\Sigma;\Bbbk)$ interchanges the section and cosection functors. This also holds when $\Bbbk$ is a DVR, but additional hypotheses are needed.

Proposition 7.16. Let $\mathcal{V}$ be an object of $\mathcal{C}(\Sigma;\Bbbk)$.

(a) There is a natural isomorphism $\Gamma_*(\Sigma, \mathcal{V}^*) \cong \Gamma_!(\Sigma, \mathcal{V})^*$. For any $\Sigma' \subseteq \Sigma$, this isomorphism intertwines restriction to Σ' for $\mathcal{V}^*$ with the dual of corestriction for $\mathcal{V}$.

(b) If $\Bbbk$ is a field or $\mathcal{V}$ is pure, then there is a natural isomorphism $\Gamma_!(\Sigma, \mathcal{V}^*) \cong \Gamma_*(\Sigma, \mathcal{V})^*$. For any $\Sigma' \subseteq \Sigma$, this isomorphism intertwines corestriction for $\mathcal{V}^*$ with the dual of restriction for $\mathcal{V}$.

Proof. By the definition of the cosection functor, we have an exact sequence

$$\bigoplus_{G \geqslant F \in \Sigma} \mathcal{V}(G)[-\operatorname{rk} G] \longrightarrow \bigoplus_{F \in \Sigma} \mathcal{V}(F)[-\operatorname{rk} F] \longrightarrow \Gamma_!(\Sigma, \mathcal{V}) \to 0.$$

Dualizing gives an exact sequence

$$0 \to \Gamma_!(\Sigma, \mathcal{V})^* \longrightarrow \bigoplus_{F \in \Sigma} \mathcal{V}^*(F) \longrightarrow \bigoplus_{G \geqslant F \in \Sigma} \mathcal{V}^*(G).$$

This is the sequence that defines the sections $\Gamma_*(\Sigma, \mathcal{V}^*)$, which proves (a).

When $\Bbbk$ is a field, dualizing is an exact functor, and (b) can be proved in the same way. When $\Bbbk$ is a DVR and $\mathcal{V}$ is pure, we proceed by induction on $|\Sigma|$. If $\Sigma = \Sigma_{\geqslant F}$, the result follows by the identification of sections of $\mathcal{V}$ with $\mathcal{V}(F)$ and cosections of $\mathcal{V}^*$ with $\mathcal{V}^*(F)[-\operatorname{rk} F] = \mathcal{V}(F)^*$. In particular, the result holds when $|\Sigma| = 1$. When $|\Sigma| > 1$, choose a minimal element $F \in \Sigma$ and let $\Sigma' = \Sigma \backslash \{F\}$. Consider the sequence

$$0 \longrightarrow \Gamma_*(\Sigma, \mathcal{V}) \longrightarrow \Gamma_*(\Sigma', \mathcal{V}) \oplus \mathcal{V}(F) \longrightarrow \Gamma_*(\Sigma_{>F}, \mathcal{V}) \longrightarrow 0.$$

This sequence is exact on the left and in the middle by the sheaf property of Γ_*, and it is exact on the right because $\mathcal{V}$ is flabby. Furthermore, $\Gamma_*(\Sigma_{>F}, \mathcal{V})$ is free by Proposition 7.13. Thus the sequence splits, so dualizing it and using the inductive hypothesis gives an exact sequence

$$0 \longrightarrow \Gamma_!(\Sigma_{>F}, \mathcal{V}^*) \longrightarrow \Gamma_!(\Sigma', \mathcal{V}^*) \oplus \mathcal{V}^*(F)[-\operatorname{rk} F] \longrightarrow \Gamma_*(\Sigma, \mathcal{V})^* \to 0,$$

where the first map is induced by the corestrictions. The isomorphism $\Gamma_!(\Sigma, \mathcal{V}^*) \cong \Gamma_*(\Sigma, \mathcal{V})^*$ follows. □

Example 7.17. This example illustrates the need for the purity hypothesis in Proposition 7.16(b). Let $\Bbbk = \mathbb{Z}_2$ be the ring of 2-adic integers, and let $\mathcal{L}$ be a Boolean lattice of rank 2. Let $\mathrm{M} = \mathrm{H}(\mathcal{L}; \Bbbk)$, endowed not with the structure of the regular module, but rather with the module structure given by the map $\mathrm{H}(\mathcal{L}; \Bbbk) \to \mathrm{H}(\mathcal{L}; \Bbbk)$ that sends y_i to $2y_i$. Let $\mathcal{W} \coloneqq \Lambda \mathrm{M}$, which is flabby but not coflabby and not locally free. Let $\mathcal{V} \coloneqq \mathcal{W}^*$, and note that $\mathcal{V}^* \cong \mathcal{W}$. Thus $\Gamma_!(\mathcal{L} \backslash \{\hat{0}\}, \mathcal{V}^*) = \Gamma_!(\mathcal{L} \backslash \{\hat{0}\}, \mathcal{W})$ has torsion, and therefore cannot be isomorphic to the dual of a $\Bbbk$-module.

Proposition 7.18. Suppose that $\mathcal{V}$ is a pure object of $\mathcal{C}(\Sigma; \Bbbk)$. Then the dual $\mathcal{V}^*$ is pure. For any $F \in \Sigma$, we have natural isomorphisms

$$(\mathcal{V}^*)_F \cong (\mathcal{V}_{[F]})^*[\operatorname{rk} F] \text{ and } (\mathcal{V}^*)_{[F]} \cong (\mathcal{V}_F)^*[\operatorname{rk} F].$$

Proof. By Proposition 7.13, the corestriction homomorphisms of $\mathcal{V}$ are split injections. Proposition 7.16(a) then implies that the restriction homomorphisms for $\mathcal{V}^*$ are surjections, which means that $\mathcal{V}^*$ is flabby. Similarly, the flabbiness of $\mathcal{V}$ together with Proposition 7.16(b) shows that $\mathcal{V}^*$ is coflabby.

Lemma 7.8(a) gives an exact sequence

$$0 \to \mathcal{V}_{[F]} \longrightarrow \Gamma_*(\Sigma_{\geqslant F}, \mathcal{V}) \longrightarrow \Gamma_*(\Sigma_{>F}, \mathcal{V}) \longrightarrow 0,$$

which is split since Proposition 7.13 implies that $\Gamma_*(\Sigma_{>F}, \mathcal{V})$ is free. Dualizing this sequence and applying Proposition 7.16 gives the exact sequence

$$0 \longrightarrow \Gamma_!(\Sigma_{>F}, \mathcal{V}^*) \longrightarrow \Gamma_!(\Sigma_{\geqslant F}, \mathcal{V}^*) \longrightarrow (\mathcal{V}_{[F]})^* \to 0.$$

Then Lemma 7.8(b) gives the isomorphism $(\mathcal{V}^*)_F \cong (\mathcal{V}_{[F]})^*[\operatorname{rk} F]$. Since duals are always free, this shows that $\mathcal{V}^*$ is locally free. The isomorphism $(\mathcal{V}^*)_{[F]} \cong (\mathcal{V}_F)^*[\operatorname{rk} F]$ is proved similarly. □

8. Construction and decomposition of pure objects

In this section we construct and classify indecomposable pure objects in $\mathcal{C}_{\mathrm{reg}}(\Sigma; \Bbbk)$. Specializing our results to $\Sigma = \mathcal{L}$ will allow us to prove Theorems A, D, and F from the introduction.

8.1. **Extending objects.** We start by describing an inductive schema for building objects in $\mathcal{C}(\Sigma; \Bbbk)$.

Consider an upper set $\Sigma \subseteq \mathcal{L}$, and suppose that $\Sigma_{>F} \subseteq \Sigma$. Let $\mathcal{V}$ be an object of $\mathcal{C}(\Sigma; \Bbbk)$. If $F \in \Sigma$, we have the maps

$$\Gamma_!\big(\Sigma_{>F}, \mathcal{V}\big)[\operatorname{rk} F] \xrightarrow{\Psi_F} \mathcal{V}(F) \xrightarrow{\Phi_F} \Gamma_*\big(\Sigma_{>F}, \mathcal{V}\big).$$

We will see in Lemma 8.1 that the composition $\Phi_F \Psi_F$ depends only on the restriction of $\mathcal{V}$ to $\Sigma_{>F}$.

Consider the system of maps $\Xi_F(G, H) \colon \mathcal{V}(G)[\operatorname{rk} F - \operatorname{rk} G] \to \mathcal{V}(H)$ for all elements $G, H \in \Sigma_{>F}$ defined by the formula

$$\Xi_F(G, H) = \begin{cases} \psi_H^{G \vee H} \varphi_G^{G \vee H} & \text{if } F = G \wedge H \text{ and } G \text{ and } H \text{ form a modular pair,} \\ 0 & \text{otherwise.} \end{cases}$$

If $F \in \Sigma$, then Proposition 6.1(c) tells us that $\Xi_F(G, H) = \varphi_F^H \psi_F^G$, but the map is defined even if $F \notin \Sigma$.

Lemma 8.1. The maps $\Xi_F(G, H)$ induce a map $\Xi_F \colon \Gamma_!(\Sigma_{>F}, \mathcal{V})[\operatorname{rk} F] \to \Gamma_*(\Sigma_{>F}, \mathcal{V})$. If $F \in \Sigma$, then $\Xi_F = \Phi_F \Psi_F$.

Proof. To see that the maps $\Xi_F(G, H)$ pass to the limit, we need to show that

$$\Xi_F(G, H)\psi_G^{G'} = \Xi_F(G', H) \ \text{ and } \ \varphi_G^{G'} \Xi_F(H, G) = \Xi_F(H, G') \tag{8}$$

for any $G, G', H \in \Sigma_{>F}$ with $G' > G$. We have $G' \vee (G \vee H) = G' \vee H$, and

$$\begin{aligned} &\operatorname{rk} G' + \operatorname{rk} H - \operatorname{rk}(G' \vee H) - \operatorname{rk} F \\ &\qquad = (\operatorname{rk} G' + \operatorname{rk}(G \vee H) - \operatorname{rk}(G' \vee H) - \operatorname{rk} G) + (\operatorname{rk} G + \operatorname{rk} H - \operatorname{rk}(G \vee H) - \operatorname{rk} F). \end{aligned} \tag{9}$$

If the left side of (9) vanishes, then (G', H) is a modular pair and $F = G' \wedge H$. Since both summands on the right side of (9) are nonnegative, they must also vanish, and so $(G', G \vee H)$ and (G, H) are modular pairs and $G = G' \wedge (G \vee H)$, $F = G \wedge H$. Then we have

$$\begin{aligned}\Xi_F(G,H)\psi_G^{G'} &= \psi_H^{G\vee H}\varphi_G^{G\vee H}\psi_G^{G'}\\ &= (H \swarrow G \vee H \nwarrow G \swarrow G')\\ &= (H \swarrow G \vee H \swarrow G' \vee H \nwarrow G')\\ &= (H \swarrow G' \vee H \nwarrow G') = \psi_H^{G'\vee H}\varphi_{G'}^{G'\vee H}\\ &= \Xi_F(G',H).\end{aligned}$$

On the other hand, if the left side of (9) is nonzero, then $\Xi_F(G', H) = 0$. At least one of the summands on the right side must be positive. If the first one is nonzero, then

$$\varphi_G^{G\vee H}\psi_G^{G'} = \Xi_G(G', G \vee H) = 0,$$

while if the second is nonzero, then $\Xi_F(G, H) = 0$. In either case we get $\Xi_F(G, H)\psi_G^{G'} = 0$, proving the first part of (8). The proof of the second part is similar.

Finally, if $F \in \Sigma$, we have $\Xi_F(G, H) = \varphi_F^H \psi_F^G$ for all $G, H > F$, and it follows immediately that $\Xi_F = \Phi_F \Psi_F$. □

Consider a pair of upper sets $\Sigma' \subseteq \Sigma$ with the property that F is a minimal element in Σ and $\Sigma' = \Sigma \backslash \{F\}$. We wish to understand how to extend an object $\mathcal{V}'$ of $\mathcal{C}(\Sigma'; \Bbbk)$ to Σ. From the preceding discussion, giving an object $\mathcal{V}$ of $\mathcal{C}(\Sigma; \Bbbk)$ and an isomorphism $\mathcal{V}|_{\Sigma'} \cong \mathcal{V}'$ is equivalent to giving a graded $\Bbbk$-module $\mathcal{V}(F)$ and maps Ψ_F, Φ_F making the triangle

$$\begin{tikzcd}\Gamma_!(\Sigma_{>F}, \mathcal{V}')[\operatorname{rk} F] \arrow[rr, "\Xi_F"] \arrow[dr, "\Psi_F"'] & & \Gamma_*(\Sigma_{>F}, \mathcal{V}') \\ & \mathcal{V}(F) \arrow[ur, "\Phi_F"'] & \end{tikzcd} \tag{10}$$

commute. Indeed, given such a diagram we let $\mathcal{V}(G) = \mathcal{V}'(G)$ for $G \neq F$, and for $G > F$, we let ψ_F^G be the composition

$$\mathcal{V}(G) = \Gamma_!\big(\Sigma_{\geqslant G}, \mathcal{V}'\big)[\operatorname{rk} G] \longrightarrow \Gamma_!\big(\Sigma_{>F}, \mathcal{V}'\big)[\operatorname{rk} G] \xrightarrow{\Psi_F} \mathcal{V}(F)[\operatorname{rk} G - \operatorname{rk} F].$$

Similarly, let φ_F^G be the composition of Φ_F with the restriction $\Gamma_*(\Sigma_{>F}, \mathcal{V}') \to \Gamma_*(\Sigma_{\geqslant G}, \mathcal{V}') = \mathcal{V}(G)$. This defines an object of $\mathcal{C}(\Sigma; \Bbbk)$ with the required properties.

Lemma 8.2. If $\mathcal{V}$ is the extension of $\mathcal{V}'$ determined by the diagram (10), then

(a) $\mathcal{V}$ is flabby if and only if $\mathcal{V}'$ is flabby and Φ_F is surjective.

(b) $\mathcal{V}$ is coflabby if and only if $\mathcal{V}'$ is coflabby and Ψ_F is injective.

(c) $\mathcal{V}$ is coflabby and locally free if and only if $\mathcal{V}'$ is coflabby and locally free, $\mathcal{V}(F)$ is a free $\Bbbk$-module, and Ψ_F is a split injection.

(d) $\mathcal{V}$ is pure if and only if $\mathcal{V}'$ is pure, $\mathcal{V}(F)$ is a free $\Bbbk$-module, Φ_F is surjective, and Ψ_F is a split injection.

Proof. Statements (a) and (b) follow immediately from Lemma 7.9. The statement (d) follows by combining statements (a) and (c), so it is enough to prove (c).

If $\mathcal{V}$ is coflabby and locally free, then its restriction $\mathcal{V}'$ to Σ' is again coflabby and locally free, and Proposition 7.13 parts (b) and (c) show that $\mathcal{V}(F)$ is locally free and Ψ_F is a split injection.

Conversely, suppose that $\mathcal{V}'$ is coflabby and locally free, $\mathcal{V}(F)$ is free, and Ψ_F is a split injection. Then Lemma 7.9 shows that $\mathcal{V}$ is coflabby, so we just need to check that $\mathcal{V}$ is locally free. For $G \in \Sigma'$, the stalk $\mathcal{V}_G = \mathcal{V}'_G$ is a free $\Bbbk$-module because $\mathcal{V}'$ is locally free, and since Ψ_F is a split injection, its cokernel $\mathcal{V}_F$ is isomorphic to a direct summand of the free $\Bbbk$-module $\mathcal{V}(F)$, and is therefore free. This completes the proof. $\square$

Example 8.3. Let $\mathcal{L}$ be the lattice of flats of the rank 2 uniform matroid on $E = \{1, 2, 3\}$, so $\mathcal{L} = \{\hat{0}, \{1\}, \{2\}, \{3\}, E\}$. Let $\Bbbk = \mathbb{Z}_2$, the 2-adic integers. If we let $\mathcal{V} = \Lambda\,\mathrm{H}(\mathcal{L}; \Bbbk)$, then the restriction $\mathcal{V}|_{\Sigma_{>\hat{0}}}$ is pure, but $\mathcal{V}$ itself is not pure. Indeed, $\mathrm{H}(\mathcal{L}; \Bbbk)$ is coflabby, and $\Gamma_!(\Sigma_{>\hat{0}}, \mathcal{V})$ is isomorphic to the graded maximal ideal $\mathfrak{m} \subseteq \mathrm{H}(\mathcal{L}; \Bbbk)$. In degree 1, the diagram (10) becomes

$$\begin{tikzcd}
\mathrm{H}^1(\mathcal{L}; \Bbbk) \arrow[rr, "\Xi_{\hat{0}}"] \arrow[dr, "\Psi_{\hat{0}}=\mathrm{id}"'] & & \Gamma_*\left(\Sigma_{>\hat{0}}, \mathcal{V}\right)^1 = \bigoplus_{F \in \mathcal{L}^1} \mathrm{H}^2(\mathcal{L}; \Bbbk) \cong \Bbbk^3, \\
& \mathrm{H}^1(\mathcal{L}; \Bbbk) \arrow[ur, "\Phi_{\hat{0}}"'] &
\end{tikzcd}$$

where the component of $\Xi_{\hat{0}}$ corresponding to $F \in \mathcal{L}^1$ is given by multiplication by y_F. So with the usual bases of the left and right sides the matrix representing $\Xi_{\hat{0}} = \Phi_{\hat{0}}$ is

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

which has determinant 2. This map is not surjective, thus $\mathcal{V}$ is not flabby.

Example 8.4. Let $\mathcal{L}$ and $\Bbbk$ be as in Example 8.3, but this time consider the object

$$\mathcal{W} := \mathcal{V}^* = \Lambda(\mathrm{H}(\mathcal{L}; \Bbbk)^*).$$

For the purposes of this example, we will decorate our maps with a subscript of $\mathcal{W}$ to distinguish them from the maps in Example 8.3. The restrictions of $\mathcal{W}$ and $\mathcal{V}$ to $\Sigma_{>\hat{0}}$ are isomorphic up to a shift, but the map $\Phi_{\hat{0},\mathcal{W}}$ is the identity, thus $\mathcal{W}$ is flabby, while $\Psi_{\hat{0},\mathcal{W}} = \Xi_{\hat{0},\mathcal{W}}$ is given by the same

matrix above. So $\Psi_{\hat{0},\mathcal{W}}$ is injective, thus $\mathcal{W}$ is coflabby, but its cokernel has torsion, thus $\mathcal{W}$ is not locally free. On the other hand, if we took $\Bbbk$ to be a field of characteristic 2, then $\Psi_{\hat{0},\mathcal{W}}$ would not be injective, and $\mathcal{W}$ would not be coflabby.

8.2. **Extending morphisms.** We now study how morphisms in our category extend when a new element is added. We continue with the notation of the previous section: Σ and Σ' are upper sets with $\Sigma = \Sigma' \cup \{F\}$. Given objects $\mathcal{V}, \mathcal{V}'$ of $\mathcal{C}(\Sigma; \Bbbk)$, a homomorphism $f\colon \mathcal{V} \to \mathcal{V}'$ is given uniquely by its restrictions $f|_{\Sigma'}$ and $f|_{\Sigma_{\geqslant F}}$, subject to the condition that they agree when restricted to $\Sigma_{>F}$ (in other words, homomorphisms form a sheaf on $\mathcal{L}$ considered as a topological space whose open sets are upper sets). So, to extend a morphism from Σ' to Σ, it is enough to consider the case where $\Sigma = \Sigma_{\geqslant F}$ and $\Sigma' = \Sigma_{>F}$.

Our main results will be consequences of the following simple linear algebra lemma. Let $\overline{\Bbbk}$ denote the residue field of $\Bbbk$. If A is a $\Bbbk$-module, we put $\bar{A} = A \otimes_{\Bbbk} \overline{\Bbbk}$, and if $f\colon A \to B$ is a map of $\Bbbk$-modules, we let $\bar{f}\colon \bar{A} \to \bar{B}$ be the induced map of $\overline{\Bbbk}$-modules. If A and B are free $\Bbbk$-modules, then $\bar{f}$ is surjective if and only if f is surjective, and $\bar{f}$ is injective if and only if f is split injective.

Lemma 8.5. Suppose that we are given a commutative rectangle of free $\Bbbk$-modules and maps of the following form:

$$\begin{tikzcd} A_1 \arrow[r, "\Psi_1"] \arrow[d, "f_A"'] & B_1 \arrow[r, "\Phi_1"] & C_1 \arrow[d, "f_C"] \\ A_2 \arrow[r, "\Psi_2"'] & B_2 \arrow[r, "\Phi_2"'] & C_2 \end{tikzcd}$$

Then the following two statements hold:

(a) If Ψ_1 is split injective and Φ_2 is surjective, there exists a map $f_B\colon B_1 \to B_2$ so that both squares commute.

(b) If in addition Ψ_2 is split injective, Φ_1 is surjective, $\ker \bar{\Phi}_1 \subseteq \operatorname{im} \bar{\Psi}_1$, $\ker \bar{\Phi}_2 \subseteq \operatorname{im} \bar{\Psi}_2$, and f_A and f_C are isomorphisms, then any map f_B as in (a) must be an isomorphism.

Proof. Let us prove the statement (a). Since Ψ_1 is split injective, we can find a complement B_0 to $\Psi_1(A_1)$ in B_1, and there is a unique way to define the restriction $f_B|_{\Psi_1(A_1)}$ so that the left square commutes, by letting

$$f_B(\Psi_1(a_1)) = \Psi_2(f_A(a_1))$$

for $a_1 \in A_1$. This also makes the right square commute on $\Psi_1(A_1)$, since

$$\Phi_2 f_B(\Psi_1(a_1)) = \Phi_2\Psi_2(f_A(a_1)) = f_C(\Phi_1(\Psi_1(a_1)))$$

for all $a_1 \in A_1$. Finally, since Φ_2 is surjective and B_0 is free we can find $f_B|_{B_0}$ so that the right square commutes on B_0.

To prove (b), first note that f_B is an isomorphism if and only if $\bar{f}_B$ is an isomorphism. Thus it is enough to consider the case where $\Bbbk$ is a field.

Suppose that b_1 is in the kernel of f_B. Since f_C is an isomorphism, we have $b_1 \in \ker \Phi_1$, so there exists $a_1 \in A_1$ with $\Psi_1(a_1) = b_1$. Then $\Psi_2 f_A(a_1) = f_B \Psi_1(a_1) = f_B(b_1) = 0$, and since Ψ_2 is injective we have $a_1 = 0$ and thus $b_1 = 0$.

Next, take $b_2 \in B_2$. Since Φ_1 and f_C are surjective, there exists $b_1 \in B_1$ with $f_C \Phi_1(b_1) = \Phi_2(b_2)$. Then $b_2 - f_B(b_1)$ is in $\ker \Phi_2 \subseteq \operatorname{im} \Psi_2$. Since f_A is surjective, there exists $a_1 \in A_1$ with $\Psi_2 f_A(a_1) = b_2 - f_B(b_1)$. Then we have

$$f_B(b_1 + \Psi_1(a_1)) = f_B(b_1) + \Psi_2 f_A(a_1) = f_B(b_1) + b_2 - f_B(b_1) = b_2,$$

so f_B is surjective. □

Proposition 8.6. Given upper sets $\Sigma' \subseteq \Sigma$ and objects $\mathcal{V}, \mathcal{W}$ in $\mathcal{C}(\Sigma; \Bbbk)_{\mathrm{pure}}$, any morphism $\mathcal{V}|_{\Sigma'} \to \mathcal{W}|_{\Sigma'}$ extends to a morphism $\mathcal{V} \to \mathcal{W}$.

Proof. By induction on $|\Sigma \backslash \Sigma'|$, it is enough to consider the case $\Sigma' = \Sigma \backslash \{F\}$. A morphism $f \colon \mathcal{V}|_{\Sigma'} \to \mathcal{W}|_{\Sigma'}$ gives rise to the following commutative diagram, where we include $\mathcal{V}$ and $\mathcal{W}$ in the subscripts to distinguish between maps associated with different objects:

$$\begin{tikzcd}
\Gamma_!(\Sigma_{>F}, \mathcal{V})[\operatorname{rk} F] \arrow[r, "\Psi_{F,\mathcal{V}}"] \arrow[d] & \mathcal{V}(F) \arrow[r, "\Phi_{F,\mathcal{V}}"] & \Gamma_*(\Sigma_{>F}, \mathcal{V}) \arrow[d] \\
\Gamma_!(\Sigma_{>F}, \mathcal{W})[\operatorname{rk} F] \arrow[r, "\Psi_{F,\mathcal{W}}"'] & \mathcal{W}(F) \arrow[r, "\Phi_{F,\mathcal{W}}"'] & \Gamma_*(\Sigma_{>F}, \mathcal{W})
\end{tikzcd}$$

Lemma 7.9 and Proposition 7.13 imply that all of these $\Bbbk$-modules are free, $\Psi_{F,\mathcal{V}}$ is split injective, and $\Phi_{F,\mathcal{W}}$ is surjective, so Lemma 8.5 says that there is a map $\mathcal{V}(F) \to \mathcal{W}(F)$ making the squares commute. This defines the required extension of f to Σ. □

8.3. **Indecomposable pure objects.** The results of Sections 8.1 and 8.2 allow us to construct pure objects in $\mathcal{C}(\Sigma; \Bbbk)$ and morphisms between them inductively by adding one element at a time to Σ. Our next objective is to give a similar inductive description of indecomposability. The key definition is the following.

Definition 8.7. For an object $\mathcal{V}$ of $\mathcal{C}(\Sigma; \Bbbk)$ with $F \in \Sigma$, we say that $\mathcal{V}$ is **taut at** F if $\ker \overline{\Phi_F} \subseteq \operatorname{im} \overline{\Psi_F}$.

Lemma 8.8. Let M be a pure $\mathrm{H}(\mathcal{L}; \Bbbk)$-module. Then $\Lambda \mathrm{M}$ is taut at F if and only if the $\overline{\Bbbk}$-reduction $\overline{\beta_F(\mathrm{M})}$ of the costalk-stalk homomorphism $\beta_F(\mathrm{M}) \colon \mathrm{M}_{[F]} \to \mathrm{M}_F$ vanishes.

Proof. The exact sequences in Lemma 7.8 are split by Proposition 7.13. So after tensoring with $\bar{\Bbbk}$ they remain exact, and so we have $\ker \overline{\Phi_F} = \overline{\mathrm{M}_{[F]}}$ and $\operatorname{coker} \overline{\Psi_F} = \overline{\mathrm{M}_F}$. Hence, $\ker \overline{\Phi_F} \subseteq \operatorname{im} \overline{\Psi_F}$ if and only if $\overline{\beta_F(\mathrm{M})} = 0$. □

Proposition 8.9. Given an object $\mathcal{V}'$ of $\mathcal{C}(\Sigma'; \Bbbk)_{\mathrm{pure}}$, there exists an object $\mathcal{V}$ of $\mathcal{C}(\Sigma; \Bbbk)_{\mathrm{pure}}$ with $\mathcal{V}|_{\Sigma'} \cong \mathcal{V}'$ that is taut at F. It is unique up to a (non-unique) isomorphism.

Proof. By the results of Section 8.1, to construct $\mathcal{V}$ we need to define $\mathcal{V}(F)$ along with maps Ψ_F, Φ_F factoring Ξ_F as in (10), so that Ψ_F is a split injection, Φ_F is surjective, and $\ker \overline{\Phi_F} \subseteq \operatorname{im} \overline{\Psi_F}$. To do this, choose a complement $\overline{W} \subseteq \overline{\Gamma_*(\Sigma_{>F}, \mathcal{V}')}$ to the image of $\bar{\Xi}_F$. Since $\Gamma_*(\Sigma_{>F}, \mathcal{V}')$ is a free $\Bbbk$-module by Proposition 7.13(d), we can lift $\overline{W}$ to a direct summand W of $\Gamma_*(\Sigma_{>F}, \mathcal{V}')$. Define $\mathcal{V}(F) = \Gamma_!(\Sigma_{>F}, \mathcal{V}')[\operatorname{rk} F] \oplus W$, let Ψ_F be the inclusion of $\Gamma_!(\Sigma_{>F}, \mathcal{V}')[\operatorname{rk} F]$, and let Φ_F equal Ξ_F on the first summand and the inclusion on the second. The required properties are now easy to verify.

Lemma 8.5 implies that, for any two such extensions, the identity map on their restrictions to Σ' extends to an isomorphism, so the extension $\mathcal{V}$ is unique up to isomorphism. □

Example 8.10. Continuing with the notation of Example 8.3, we illustrate the result of applying Proposition 8.9 to the pure object $\mathcal{V}' = (\Lambda\, \mathrm{H}(\mathcal{L}; \Bbbk))|_{\Sigma_{>\hat{0}}}$ with coefficient ring $\Bbbk = \mathbb{Z}_2$. The degree 1 part of the map $\overline{\Xi_{\hat{0},\mathcal{V}'}}$ is a rank 2 map $(\mathbb{F}_2)^3 \to (\mathbb{F}_2)^3$, and a complement to its image is given by the subspace $W = \Bbbk \cdot (1,1,1) \subseteq (\mathbb{F}_2)^3$. If $\mathcal{V}$ is the object constructed in Proposition 8.9, then the degree one part of $\mathcal{V}(\hat{0})$ is a free $\Bbbk$-module of rank four, and one can check that $\mathcal{V}$ is isomorphic to $\Lambda(\mathrm{CH}(\mathcal{L}; \Bbbk))$. Note that this agrees with the statement of Theorem 1.12.

Proposition 8.11. Let $\mathcal{V} \in \mathcal{C}(\Sigma; \Bbbk)_{\mathrm{pure}}$ be an object with $\mathcal{V}|_{\Sigma'} \neq 0$. Then $\mathcal{V}$ is indecomposable if and only if $\mathcal{V}|_{\Sigma'}$ is indecomposable and $\mathcal{V}$ is taut at F.

Proof. First suppose that $\mathcal{V}$ is not taut at F, so $\ker \overline{\Phi_F}$ is not contained in $\operatorname{im} \overline{\Psi_F}$. We will show that $\mathcal{V}$ is decomposable. Since Ψ_F is a split injection by Proposition 7.13, tensoring the quotient map $\pi \colon \mathcal{V}(F) \to \mathcal{V}(F)/\operatorname{im} \Psi_F$ with $\bar{\Bbbk}$ gives the quotient

$$\bar{\pi} \colon \overline{\mathcal{V}(F)} \to \overline{\mathcal{V}(F)/\operatorname{im} \Psi_F} \cong \overline{\mathcal{V}(F)}/\operatorname{im} \overline{\Psi_F}.$$

By lifting an element of $\ker \overline{\Phi_F}$ that is not in $\operatorname{im} \overline{\Psi_F}$, we obtain $v \in \ker \Phi_F$ whose image

$$\bar{v} \in \overline{\ker \Phi_F} = \ker \overline{\Phi_F} \subseteq \overline{\mathcal{V}(F)}$$

is not in the kernel of $\bar{\pi}$. This implies that $\pi(\Bbbk \cdot v)$ is a direct summand of the free $\Bbbk$-module $\mathcal{V}(F)/\operatorname{im} \Psi_F = \mathcal{V}_F$. Choose a complement $W \subseteq \mathcal{V}(F)/\operatorname{im} \Psi_F$ to $\pi(\Bbbk \cdot v)$.

We define a nontrivial direct sum decomposition $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$ as follows:

(1) For $G \neq F$, let $\mathcal{V}_1(G) = \mathcal{V}(G)$ and $\mathcal{V}_2(G) = 0$.

(2) Let $\mathcal{V}_1(F) = \pi^{-1}(W)$, $\mathcal{V}_2(F) = \Bbbk \cdot v$.

Because $\operatorname{im} \Psi_F \subseteq \mathcal{V}_1(F)$ and $\mathcal{V}_2(F) \subseteq \ker \Phi_F$, restricting the φ and ψ maps of $\mathcal{V}$ gives objects $\mathcal{V}_1$ and $\mathcal{V}_2$ of $\mathcal{C}(\Sigma; \Bbbk)$. Since $\mathcal{V}(G) = \mathcal{V}_1(G) \oplus \mathcal{V}_2(G)$ for all $G \in \Sigma$, we have constructed the required direct sum decomposition; it is nontrivial since $\mathcal{V}_1|_{\Sigma'} = \mathcal{V}|_{\Sigma'}$ is nonzero and $\mathcal{V}_2(F)$ is nonzero.

Next, suppose that $\mathcal{V}$ is indecomposable but there is a non-trivial direct sum decomposition $\mathcal{V}|_{\Sigma'} = \mathcal{V}_1' \oplus \mathcal{V}_2'$. By Proposition 8.9, there are pure extensions $\mathcal{V}_1$ and $\mathcal{V}_2$ of $\mathcal{V}_1'$ and $\mathcal{V}_2'$ that are taut at F. But we have just seen that $\mathcal{V}$ must be taut at F, and $\mathcal{V}_1 \oplus \mathcal{V}_2$ is also taut at F, so the uniqueness part of Proposition 8.9 implies that $\mathcal{V} \cong \mathcal{V}_1 \oplus \mathcal{V}_2$, contradicting the indecomposability of $\mathcal{V}$.

Finally, suppose that we are given a non-trivial decomposition $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$ but $\mathcal{V}|_{\Sigma'}$ is indecomposable. Then one of $\mathcal{V}_1|_{\Sigma'}$ or $\mathcal{V}_2|_{\Sigma'}$ must vanish; suppose without loss of generality that $\mathcal{V}_2|_{\Sigma'} = 0$. Then we must have $\mathcal{V}_2(F) \neq 0$, and so $\overline{\mathcal{V}_2(F)} \neq 0$. The vanishing of $\mathcal{V}_2|_{\Sigma'}$ implies that $\overline{\mathcal{V}_2(F)} \subseteq \ker \overline{\Phi_F}$ and $\operatorname{im} \overline{\Psi_F} \subseteq \overline{\mathcal{V}_1(F)}$, so it follows that $\mathcal{V}$ is not taut at F. □

Theorem 8.12. For each element $F \in \Sigma$, there is an object $\mathcal{Q}^F = \mathcal{Q}^F_{\Sigma,\Bbbk}$ of $\mathcal{C}(\Sigma; \Bbbk)_{\mathrm{pure}}$ for which the following properties hold:

(a) $\mathcal{Q}^F(G) = 0$ unless $G \leqslant F$,

(b) $\mathcal{Q}^F(F) \cong \Bbbk$, concentrated in degree 0,

(c) $\mathcal{Q}^F$ is taut at all elements in $\Sigma \backslash \{F\}$.

This object is indecomposable and unique up to isomorphism. The collection of objects $\mathcal{Q}^F[k]$ over all $F \in \Sigma$ and all $k \in \mathbb{Z}$ gives a complete set of indecomposable objects in $\mathcal{C}(\Sigma; \Bbbk)_{\mathrm{pure}}$.

Proof. We proceed by induction on $|\Sigma|$. Since the dependence on Σ is important in this proof but the dependence on $\Bbbk$ is not, for the remainder of the proof we will denote the indecomposables described in the theorem by $\mathcal{Q}^F_\Sigma$.

When $|\Sigma| = 1$, $\mathcal{C}(\Sigma; \Bbbk)_{\mathrm{pure}}$ is equivalent to the category of free graded $\Bbbk$-modules, and the result is immediate. Now assume that $|\Sigma| > 1$ and the result is known for all upper sets that are properly contained in Σ. Let F be a minimal element in Σ and let $\Sigma' = \Sigma \backslash \{F\}$. By the inductive hypothesis, we have modules $\mathcal{Q}^G_{\Sigma'}$ for $G \in \Sigma'$ representing all isomorphism classes of indecomposable objects in $\mathcal{C}(\Sigma'; \Bbbk)_{\mathrm{pure}}$ up to shift. By Proposition 8.9, each object $\mathcal{Q}^G_{\Sigma'}$ has an extension $\mathcal{Q}^G_\Sigma$ that is pure and taut at F, and Proposition 8.11 implies that this extension is indecomposable. We have one more indecomposable object $\mathcal{Q}^F_\Sigma$ in $\mathcal{C}(\Sigma; \Bbbk)_{\mathrm{pure}}$ given by setting $\mathcal{Q}^F_\Sigma(F) = \Bbbk$ in degree 0 and $\mathcal{Q}^F_\Sigma(G) = 0$ for all $G \neq F$. It is easy to see that these objects satisfy the required conditions.

Now let $\mathcal{V}$ be any nonzero indecomposable object of $\mathcal{C}(\Sigma; \Bbbk)_{\mathrm{pure}}$. If $\mathcal{V}|_{\Sigma'}$ is nonzero, then Proposition 8.11 implies that $\mathcal{V}|_{\Sigma'}$ is indecomposable and $\mathcal{V}$ is taut at F. Thus we must have $\mathcal{V}|_{\Sigma'} \cong \mathcal{Q}^G_{\Sigma'}[k]$ for some $G \neq F$ and some $k \in \mathbb{Z}$. But then $\mathcal{V} \cong \mathcal{Q}^G_\Sigma[k]$ by the uniqueness part of Proposition 8.9.

On the other hand, if $\mathcal{V}|_{\Sigma'} = 0$, then $\mathcal{V}(F)$ must be a free $\Bbbk$-module of rank 1, which means that $\mathcal{V} \cong \mathcal{Q}^F_\Sigma[k]$ for some $k \in \mathbb{Z}$. □

8.4. **Parity modules.** In this section, we prove Theorems A, D, and F.

Definition 8.13. Let $\mathrm{E}(\mathcal{L};\Bbbk) := \Theta\mathcal{Q}^{\hat{1}}_{\mathcal{L},\Bbbk}$, where $\Theta\colon \mathcal{C} \to \mathrm{H}(\mathcal{L};\Bbbk)\text{–mod}$ is the functor introduced in the proof of Proposition 6.3 and $\mathcal{Q}^{\hat{1}}_{\mathcal{L},\Bbbk}$ is the indecomposable pure object from Theorem 8.12.

Recall from the introduction that, for any $F \in \mathcal{L}$, we have a natural surjective homomorphism $\mathrm{H}(\mathcal{L};\Bbbk) \to \mathrm{H}(\mathcal{L}^F;\Bbbk)$, and this means that $\mathrm{E}(\mathcal{L}^F;\Bbbk)$ is naturally a module over $\mathrm{H}(\mathcal{L};\Bbbk)$. The following lemma follows from Remark 6.6.

Lemma 8.14. For any $F \in \mathcal{L}$, we have $\mathrm{E}(\mathcal{L}^F;\Bbbk) \cong \Theta\mathcal{Q}^F_{\mathcal{L},\Bbbk}$.

Proof of Theorem D. By Proposition 6.3, Remark 7.11, and Lemma 7.12, a complete set of indecomposable pure $\mathrm{H}(\mathcal{L};\Bbbk)$-modules (up to shift and isomorphism) is obtained from a complete set of indecomposable pure objects of $\mathcal{C}(\mathcal{L};\Bbbk)$ (up to shift and isomorphism) via the functor Θ. By Theorem 8.12 and Lemma 8.14, one obtains the modules $\mathrm{E}(\mathcal{L}^F;\Bbbk)$ for all $F \in \mathcal{L}$. □

Proof of Theorem A. By Theorem D, $\mathrm{E}(\mathcal{L};\Bbbk)$ is pure and indecomposable. Furthermore, we have

$$y_{\hat{1}}\mathrm{E}(\mathcal{L};\Bbbk) = \mathcal{Q}^{\hat{1}}_{\mathcal{L},\Bbbk}(\hat{1})[-\operatorname{rk}\mathcal{L}] \cong \Bbbk[-\operatorname{rk}\mathcal{L}],$$

which implies the existence of an element $x \in \mathrm{E}^0(\mathcal{L};\Bbbk)$ such that $y_{\hat{1}} \cdot x$ is nonzero.

Conversely, let M be an indecomposable pure module and $x \in \mathrm{M}^0$ an element with $y_{\hat{1}}x \neq 0$. By Theorem D, there exists an element $F \in \mathcal{L}$ and an integer $k \in \mathbb{Z}$ such that $\mathrm{M} \cong \mathrm{E}(\mathcal{L}^F;\Bbbk)[k]$. By Lemma 8.14 and parts (a) and (b) of Theorem 8.12, we must have $F = \hat{1}$ and $k = 0$. □

Remark 8.15. Our construction of parity modules also gives an alternative proof of Proposition 2.4. To see this, note that the restriction of $\mathcal{Q}^{\hat{1}}_{\mathcal{L},\Bbbk}$ to the upper set $\mathcal{L}_F$ is clearly isomorphic to $\mathcal{Q}^{\hat{1}}_{\mathcal{L}_F,\Bbbk}$. Applying the functor $\Theta_F\colon \mathcal{C}(\mathcal{L}_F;\Bbbk) \to \mathrm{H}(\mathcal{L}_F;\Bbbk)\text{–mod}$ to this isomorphism gives an isomorphism

$$y_F\mathrm{E}(\mathcal{L};\Bbbk)[\operatorname{rk} F] \cong \mathrm{E}(\mathcal{L}_F;\Bbbk).$$

In [BHM+], a graded $\mathrm{H}(\mathcal{L};\mathbb{Q})$-module is called **pure** if it is isomorphic to a direct sum of direct summands of modules of the form $\mathrm{CH}(\mathcal{L}^F;\mathbb{Q})[k]$, where $F \in \mathcal{L}$ and $k \in \mathbb{Z}$. The next result says that this definition agrees with our definition of pure modules.

Proposition 8.16. Let $\Bbbk$ be a field. An $\mathrm{H}(\mathcal{L};\Bbbk)$-module is pure if and only if it is isomorphic to a direct sum of direct summands of modules of the form $\mathrm{CH}(\mathcal{L}^F;\Bbbk)[k]$, where $F \in \mathcal{L}$ and $k \in \mathbb{Z}$.

Proof. By Proposition 4.5(e), $\mathrm{CH}(\mathcal{L}^F;\Bbbk)[k]$ is pure, and since purity is clearly preserved under taking direct sums and direct summands, the "if" statement holds.

Considering $\mathrm{CH}(\mathcal{L}^F;\Bbbk)$ as an $\mathrm{H}(\mathcal{L};\Bbbk)$-module via the homomorphism $\mathrm{H}(\mathcal{L};\Bbbk) \to \mathrm{H}(\mathcal{L}^F;\Bbbk)$, we have $y_F\,\mathrm{CH}(\mathcal{L}^F;\Bbbk) \cong \Bbbk[-\operatorname{rk} F]$ and $y_G\,\mathrm{CH}(\mathcal{L}^F;\Bbbk) = 0$ for $G \nleq F$. It follows that $\mathrm{E}(\mathcal{L}^F;\Bbbk)$ is isomorphic to a direct summand of $\mathrm{CH}(\mathcal{L}^F;\Bbbk)$. Thus, by Theorem D, every indecomposable pure $\mathrm{H}(\mathcal{L};\Bbbk)$-module appears as a shift of a direct summand of $\mathrm{CH}(\mathcal{L}^F;\Bbbk)$ for some $F \in \mathcal{L}$. This proves the "only if" direction. □

Theorem D implies that any pure $\mathrm{H}(\mathcal{L};\Bbbk)$-module M can be decomposed as a direct sum

$$\mathrm{M} \cong \bigoplus_{F\in\mathcal{L}} W(\mathrm{M},F) \otimes \mathrm{E}(\mathcal{L}^F;\Bbbk) \tag{11}$$

for some finite-dimensional graded vector spaces $W(\mathrm{M},F)$. The next result, which proves Theorem F, shows how to compute these multiplicity spaces.

Proof of Theorem F. As with all of the results stated in the introduction, we are assuming that $\Bbbk$ is a field. We must show that $W(\mathrm{M},F)$ is isomorphic to the image of the costalk-stalk homomorphism $\beta_F(\mathrm{M})\colon \mathrm{M}_{[F]} \to \mathrm{M}_F$. This follows by applying $\operatorname{im}(\beta_G(-))$ to both sides of (11). Indeed, if $F \neq G$, then $\mathcal{Q}^F_{\mathcal{L},\Bbbk} \cong \Lambda\mathrm{E}(\mathcal{L}^F;\Bbbk)$ is taut at G. By Lemma 8.8, this implies that $\beta_G(\mathrm{E}(\mathcal{L}^F;\Bbbk)) = 0$, and therefore the F summand of the right-hand side vanishes. On the other hand, the maps

$$\Bbbk \cong \mathrm{E}(\mathcal{L}^G;\Bbbk)_{[G]} \longrightarrow y_G\mathrm{E}(\mathcal{L}^G;\Bbbk)[\operatorname{rk} G] \longrightarrow \mathrm{E}(\mathcal{L}^G;\Bbbk)_G \cong \Bbbk$$

are isomorphisms, therefore the G summand of the right-hand side is isomorphic to $W(\mathrm{M},G)$. Thus we obtain the equation $\operatorname{im}(\beta_G(\mathrm{M})) \cong W(\mathrm{M},G)$. □

Remark 8.17. Theorem F does not hold when $\Bbbk$ is a DVR. This is because it is possible for a pure module M to be taut at F even though $\beta_F(\mathrm{M})$ is not zero. The object in Example 8.10 is an example. Instead, the multiplicities in the decomposition of a pure $\mathrm{H}(\mathcal{L};\Bbbk)$-module are given by the graded dimensions of the images of the reduced maps $\overline{\beta_F(\mathrm{M})}$.

9. Properties of parity sheaves

The purpose of this section is to prove Theorems B and E.

9.1. **Top and bottom degrees.** In addition to giving a classification of indecomposable pure objects, Theorem 8.12 (or rather its proof) also gives an algorithm to construct them. Namely, to construct $\mathcal{Q}^F = \mathcal{Q}^F_{\mathcal{L},\Bbbk}$, start with $\mathcal{Q}^F(G) = 0$ for $G \nleq F$, $\mathcal{Q}^F(F) = \Bbbk$, and then use Proposition 8.9 to define $\mathcal{Q}^F(G)$ for $G < F$ in decreasing order of rank. The following proposition is an application of this algorithm.

Proposition 9.1. The indecomposable pure object $\mathcal{Q}^F = \mathcal{Q}^F_{\mathcal{L},\Bbbk}$ satisfies the following properties.

(a) For any $G \leqslant F$, we have $\mathcal{Q}^F(G)^i = 0$ if $i < 0$ or $i > \operatorname{rk} F - \operatorname{rk} G$.

(b) For any $G \leqslant F$, we have $\mathcal{Q}^F(G)^0 \cong \mathcal{Q}^F(G)^{\operatorname{rk} F - \operatorname{rk} G} \cong \Bbbk$.

(c) For any $G \leqslant H \leqslant F$, the restriction map

$$\varphi^H_G \colon \mathcal{Q}^F(G)^0 \longrightarrow \mathcal{Q}^F(H)^0$$

and the corestriction map

$$\psi^H_G \colon \mathcal{Q}^F(H)^{\operatorname{rk} F - \operatorname{rk} H} \longrightarrow \mathcal{Q}^F(G)^{\operatorname{rk} F - \operatorname{rk} G}$$

are both isomorphisms.

Proof. We will proceed by induction on the difference $\operatorname{rk} F - \operatorname{rk} G$. When $G = F$, the statements (a) and (b) hold because $\mathcal{Q}^F(F)$ is equal to $\Bbbk$ concentrated in degree 0. The only case of (c) to check is $G = H = F$, which holds trivially since φ^F_F and ψ^F_F are both the identity map.

Now suppose that we are given $G_0 < F$ such that the proposition holds for all G with $G_0 < G \leqslant F$. We apply these statements to the sections and cosections of $\mathcal{Q}^F$ on $\Sigma_{>G_0}$. Looking at sections

$$\Gamma_*(\Sigma_{>G_0}, \mathcal{Q}^F) = \varprojlim_{G_0 < G} \mathcal{Q}^F(G) = \varprojlim_{G_0 < G \leqslant F} \mathcal{Q}^F(G),$$

the fact that $\mathcal{Q}^F(G)^i$ is zero when $i < 0$ or $i > \operatorname{rk} F - \operatorname{rk} G$ implies that $\Gamma_*(\Sigma_{>G_0}, \mathcal{Q}^F)^i = 0$ for $i < 0$ and $i \geqslant \operatorname{rk} F - \operatorname{rk} G_0$. Furthermore, statement (c) implies that all of the maps in the limit between elements $G_0 < G \leqslant F$ are isomorphisms in degree 0, and so $\Gamma_*(\Sigma_{>G_0}, \mathcal{Q}^F)^0 \cong \Bbbk$ and the maps $\Gamma_*(\Sigma_{>G_0}, \mathcal{Q}^F)^0 \to \mathcal{Q}^F(G)^0$ are isomorphisms for all $G_0 < G \leqslant F$.

Dually, when we look at cosections

$$\Gamma_!(\Sigma_{>G_0}, \mathcal{Q}^F) = \varinjlim_{G_0 < G} \mathcal{Q}^F(G)[-\operatorname{rk} G] = \varinjlim_{G_0 < G \leqslant F} \mathcal{Q}^F(G)[-\operatorname{rk} G],$$

the statement (a) implies that $\mathcal{Q}^F(G)[-\operatorname{rk} G]$ vanishes in all degrees $i < \operatorname{rk} G$ and $i > \operatorname{rk} F$, so the limit vanishes for all degrees $i \leqslant \operatorname{rk} G_0$ and $i > \operatorname{rk} F$. Statement (c) implies that all of the maps in the limit between elements $G_0 < G \leqslant F$ are isomorphisms in degree $\operatorname{rk} F$, so we have $\Gamma_!(\Sigma_{>G_0}, \mathcal{Q}^F)^{\operatorname{rk} F} \cong \Bbbk$ and the maps $\mathcal{Q}^F(G)[-\operatorname{rk} G]^{\operatorname{rk} F} \to \Gamma_!(\Sigma_{>G_0}, \mathcal{Q}^F)^{\operatorname{rk} F}$ are isomorphisms for all $G_0 < G \leqslant F$.

We now consider the maps

$$\Gamma_!\big(\Sigma_{>G_0}, \mathcal{Q}^F\big)[\operatorname{rk} G_0] \xrightarrow{\Psi_{G_0}} \mathcal{Q}^F(G_0) \xrightarrow{\Phi_{G_0}} \Gamma_*\big(\Sigma_{>G_0}, \mathcal{Q}^F\big).$$

In degrees $i < 0$ and $i > \operatorname{rk} F - \operatorname{rk} G_0$, both the sections and cosections vanish, so $\mathcal{Q}^F(G_0)^i = 0$, giving the statement (a). In degree $i = 0$, the cosections vanish while the sections are $\Bbbk$, so we have $\mathcal{Q}^F(G_0)^0 \cong \Bbbk$. In degree $\operatorname{rk} F - \operatorname{rk} G_0$, the sections vanish while the cosections are $\Bbbk$, giving

$\mathcal{Q}^F(G_0)^{\operatorname{rk} F - \operatorname{rk} G_0} \cong \Bbbk$, so we have proved the statement (b) for G_0. It also follows that the statement (c) holds for G_0. □

Remark 9.2. Applying the functor Θ to $\mathcal{Q}^{\hat{1}}$, Proposition 9.1 immediately gives us the first two statements of Theorem B. We will tackle the third statement in Section 9.3.

9.2. **Next degrees in.** Assume in this section that $\Bbbk$ is a field of characteristic p (allowing the possibility that $p = 0$). Our next result gives a simple description of the spaces $\mathcal{Q}^F(G)$ in the next degrees "in from the outside", that is, in degrees 1 and $\operatorname{rk} F - \operatorname{rk} G - 1$. This gives an alternative proof of Proposition 1.9 that does not involve the embedding $\mathrm{E}(\mathcal{L}; \Bbbk)$ in the augmented Chow ring $\mathrm{CH}(\mathcal{L}; \Bbbk)$. A reader who is only interested in the proofs of Theorems B and E may skip this section.

To simplify notation, we will restrict ourselves to the case where F is the maximal element $\hat{1}$. Let $\mathcal{Q} = \mathcal{Q}^{\hat{1}}$. For an upper set $\Sigma \subseteq \mathcal{L}$, we let $\Sigma(p)$ denote the set of p-resonant elements in Σ, where p is the characteristic of $\Bbbk$.

Theorem 9.3. For any $G < \hat{1}$, there are isomorphisms

$$\mathcal{Q}(G)^1 \cong \Bbbk^{\mathcal{L}_G(p)} \cong \mathcal{Q}(G)^{\operatorname{crk} G - 1}$$

with the property that, for any $G < F < \hat{1}$, the maps

$$\psi_G^F \colon \mathcal{Q}(F)^{\operatorname{crk} F - 1} \longrightarrow \mathcal{Q}(G)^{\operatorname{crk} G - 1} \quad \text{and} \quad \varphi_G^F \colon \mathcal{Q}(G)^1 \longrightarrow \mathcal{Q}(F)^1$$

are given by the coordinate inclusion $\Bbbk^{\mathcal{L}_F(p)} \to \Bbbk^{\mathcal{L}_G(p)}$ and projection $\Bbbk^{\mathcal{L}_G(p)} \to \Bbbk^{\mathcal{L}_F(p)}$, respectively. If in addition F is a cover of G, so that $\operatorname{crk} F = \operatorname{crk} G - 1$, the map

$$\psi_G^F \colon \mathcal{Q}(F)^0 \longrightarrow \mathcal{Q}(G)^1$$

is the map $\Bbbk \to \Bbbk^{\mathcal{L}_G(p)}$ sending 1 to the characteristic function of $\mathcal{L}_G(p) \smallsetminus \mathcal{L}_{F \vee G}(p)$, and the map $\varphi_G^F \colon \mathcal{Q}(G)^{\operatorname{crk} G - 1} \to \mathcal{Q}(F)^{\operatorname{crk} F} \cong \Bbbk$ is its transpose.

Before proving Theorem 9.3, we formulate and prove a lemma. Given a map $a \colon \mathcal{L}^1 \to \Bbbk$, its **complementary Radon transform** is the map $C_a \colon \mathcal{L} \to \Bbbk$ given by

$$C_a(F) = \sum_{D \in \mathcal{L}^1 \text{ and } D \nleq F} a(D).$$

The following lemma is proved in [Kun79, Section 3] under the additional assumption that $p = 0$; we prove it below for arbitrary fields.

Lemma 9.4. The map $a \mapsto C_a|_{\mathcal{L}(p)}$ is injective. If $\hat{0}$ is p-resonant, then there exists an a such that, for any $F \in \mathcal{L}(p)$, we have $C_a(F) = \delta_{\hat{0}F}$.

Proof. First we prove injectivity. Suppose that C_a vanishes on $\mathcal{L}(p)$. If we can show that C_a vanishes on all of $\mathcal{L}$, it will follow that $a = 0$, since for any $D \in \mathcal{L}^1$ we have

$$a(D) = C_a(\hat{0}) - C_a(D).$$

We prove that $C_a(F) = 0$ for all $F \in \mathcal{L}$ by induction on the corank of F. When $F = \hat{1}$ the result is obvious, and when F has corank 1 it is p-resonant, so $C_a(F) = 0$ by assumption. Now take $F \in \mathcal{L}$ of corank at least 2, and assume that C_a vanishes on all elements of smaller corank. If F is p-resonant, then $C_a(F) = 0$ by assumption. Otherwise, there is an element $H > F$ with $\operatorname{rk} H = \operatorname{rk} F + 2$ such that the open interval (F, H) has cardinality m with $m - 1$ invertible in $\Bbbk$. An atom $D \in \mathcal{L}^1$ with $D \leqslant H$ and $D \nleqslant F$ lies below exactly one $G \in (F, H)$, and hence we have

$$\sum_{G \in (F,H)} C_a(G) = (m-1)C_a(F) + C_a(H).$$

Our inductive hypothesis says that the left sum and $C_a(H)$ both vanish, so $C_a(F) = 0$, as desired.

For the second statement, suppose that $\hat{0}$ is p-resonant. Let $a(D) = 1$ for all $D \in \mathcal{L}^1$, so that

$$C_a(F) = |\mathcal{L}^1| - |(\mathcal{L}^F)^1|.$$

Lemma 4.3 implies that, for any $F > \hat{0}$, $|(\mathcal{L}^F)^1|$ is congruent to 1 modulo p. Thus $C_a(F) = 1 - 1 = 0$ for all $F \neq \hat{0}$, while $C_a(\hat{0}) = 1 - 0 = 1$. □

Proof of Theorem 9.3. We will prove the statements about $\mathcal{Q}(G)^1$; the statements about $\mathcal{Q}(G)^{\operatorname{crk} G - 1}$ follow from duality. We proceed by induction on $\operatorname{crk} G$. When $\operatorname{crk} G = 1$, G is the only p-resonant element of $\mathcal{L}_G$, and Proposition 9.1 says that

$$\mathcal{Q}(G)^1 = \mathcal{Q}(G)^{\operatorname{crk} G} \cong \Bbbk \cong \Bbbk^{\mathcal{L}_G(p)}.$$

The statements about ψ_G^F and φ_G^F are trivial in this case.

Now suppose that $\operatorname{crk} G > 1$, and that the statements hold for all elements of smaller corank. We need to compute the degree 1 part of the map $\Xi_G \colon \Gamma_!(\mathcal{L}_{>G}, \mathcal{Q})[\operatorname{rk} G] \to \Gamma_*(\mathcal{L}_{>G}, \mathcal{Q})$. The source of this map is

$$\Gamma_!(\mathcal{L}_{>G}, \mathcal{Q})[\operatorname{rk} G]^1 = \Gamma_!(\mathcal{L}_{>G}, \mathcal{Q})^{\operatorname{rk} G + 1} = \varinjlim_{F > G} \mathcal{Q}(F)[-\operatorname{rk} F]^{\operatorname{rk} G + 1} = \varinjlim_{F > G} \mathcal{Q}(F)^{\operatorname{rk} G - \operatorname{rk} F + 1} \cong \Bbbk^{(\mathcal{L}_G)^1}.$$

The last isomorphism follows from Proposition 9.1 because, for $F > G$, the quantity $\operatorname{rk} G - \operatorname{rk} F + 1$ is negative unless F covers G, in which case it is zero. We represent elements of this space as functions $a \colon (\mathcal{L}_G)^1 \to \Bbbk$.

For any F' with $G < F' \leqslant F$, our inductive hypothesis tells us that the restriction map $\varphi_{F'}^F$ is a coordinate projection, thus the degree one part of the target of Ξ_G is

$$\Gamma_*(\mathcal{L}_{>G}, \mathcal{Q})^1 = \varprojlim_{F > G} \Bbbk^{\mathcal{L}_F(p)} = \Bbbk^{\Sigma_{>G}(p)}.$$

If we represent elements of this space as functions $\Sigma_{>G}(p) \to \Bbbk$, we claim that

$$\Xi_G(a) = C_a|_{\Sigma_{>G}(p)},$$

where the complementary Radon transform C_a is computed in the lattice $\mathcal{L}_G = \Sigma_{\geqslant G}$.

To see this, let a_F be the basis element in $\Bbbk^{(\mathcal{L}_G)^1}$ corresponding to a cover F of G. It is the image of $1 \in \mathcal{Q}(F)^0$ in $\Gamma_!(\mathcal{L}_{>G}, \mathcal{Q})[\operatorname{rk} G]^1$. Then $\Xi_G(a_F)$ is determined by its image in $\mathcal{Q}(H)^1$ over all $H > G$. If $F \leqslant H$, then the image is zero, since $F \wedge H = F \neq G$. This agrees with the claim, since C_{a_F} is zero on $\mathcal{L}_H$ when $H \geqslant F$. On the other hand, if $F \nleqslant H$, then $F \vee H$ covers H, and the image of $\Xi_G(a_F)$ in $\mathcal{Q}(H)^1$ is the image of 1 under the map

$$\Bbbk = \mathcal{Q}(F)^0 \xrightarrow{\varphi_F^{F\vee H}} \mathcal{Q}(F \vee H)^0 \xrightarrow{\psi_H^{F\vee H}} \mathcal{Q}(H)^1 \cong \Bbbk^{\mathcal{L}_H(p)}.$$

By the inductive assumption, $\varphi_F^{F\vee H}$ is the identity and $\psi_H^{F\vee H}$ sends 1 to the characteristic function of $\mathcal{L}_H(p) \smallsetminus \mathcal{L}_{F\vee H}(p)$, which is the restriction of C_{a_F} to $\mathcal{L}_H(p)$. The claim follows.

Now consider the diagram

$$\begin{tikzcd}
\Bbbk^{(\mathcal{L}_G)^1} \arrow[rr, "\Xi_G"] \arrow[dr, "\Psi_G"'] & & \Bbbk^{\Sigma_{>G}(p)} \\
& \Bbbk^{\mathcal{L}_G(p)} \arrow[ur, "\Phi_G"'] &
\end{tikzcd}$$

where Ψ_G is the map $a \mapsto C_a|_{\mathcal{L}_G(p)}$ and Φ_G is the restriction to $\Sigma_{>G}(p)$. The triangle commutes, so this diagram describes the degree 1 part of an extension of $\mathcal{Q}|_{\Sigma_{>G}}$ to $\Sigma_{\geqslant G} = \mathcal{L}_G$. This extension is pure, because Φ_G is surjective and Ψ_G is injective, by Lemma 9.4. Finally, we must check that the extension is taut at G. If G is not p-resonant, then Φ_G is an isomorphism, and tautness is automatic. If G is p-resonant, then the kernel of Φ_G is spanned by the characteristic function of G, which is in the image of Ψ_G by the second part of Lemma 9.4. □

9.3. **Künneth isomorphism.** In this section we prove the last part of Theorem B, which is a Künneth isomorphism for parity modules. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be geometric lattices. Then $\mathcal{L}_1 \times \mathcal{L}_2$ is a geometric lattice, with rank function $\operatorname{rk}(F_1, F_2) = \operatorname{rk} F_1 + \operatorname{rk} F_2$. We have an isomorphism of rings

$$\mathrm{H}(\mathcal{L}_1; \Bbbk) \otimes \mathrm{H}(\mathcal{L}_2; \Bbbk) \cong \mathrm{H}(\mathcal{L}_1 \times \mathcal{L}_2; \Bbbk), \quad y_{F_1} \otimes y_{F_2} \longmapsto y_{(F_1,F_2)}.$$

Lemma 9.5. Let M_i be a graded $\mathrm{H}(\mathcal{L}_i)$-module for $i = 1, 2$. If M_1 and M_2 are both pure, then so is their external tensor product $\mathrm{M}_1 \otimes \mathrm{M}_2$.

Proof. Let $\mathrm{M} = \mathrm{M}_1 \otimes M_2$. To show that M is flabby and coflabby, we will use the criterion of Lemma 7.9. Let $\mathcal{V} = \Lambda\mathrm{M}$, and take $(F_1, F_2) \in \mathcal{L}_\times \mathcal{L}_2$. We wish to show that the corestriction

$$\Gamma_!(\Sigma_{>(F_1,F_2)}, \mathcal{V}) \longrightarrow \Gamma_!(\Sigma_{\geqslant(F_1,F_2)}, \mathcal{V}) = y_{(F_1,F_2)}\mathrm{M} = y_{F_1}\mathrm{M}_1 \otimes y_{F_2}\mathrm{M}_2$$

is injective and the restriction

$$(y_{F_1}\mathrm{M}_1 \otimes y_{F_2}\mathrm{M}_2)\,[\operatorname{rk} F_1 + \operatorname{rk} F_2] = \Gamma_*(\Sigma_{\geqslant(F_1,F_2)}, \mathcal{V}) \longrightarrow \Gamma_*(\Sigma_{>(F_1,F_2)}, \mathcal{V})$$

is surjective.

Using the cover

$$\Sigma_{>(F_1,F_2)} = (\Sigma_{>F_1} \times \Sigma_{\geqslant F_2}) \cup (\Sigma_{\geqslant F_1} \times \Sigma_{>F_2})$$

and the fact that $\Gamma_!(-, \mathcal{V})$ is a cosheaf gives us an exact sequence

$$\Gamma_!(\Sigma_{>F_1} \times \Sigma_{>F_2}, \mathcal{V}) \longrightarrow \Gamma_!(\Sigma_{>F_1} \times \Sigma_{\geqslant F_2}, \mathcal{V}) \oplus \Gamma_!(\Sigma_{\geqslant F_1} \times \Sigma_{>F_2}, \mathcal{V}) \longrightarrow \Gamma_!(\Sigma_{>(F_1,F_2)}, \mathcal{V}) \longrightarrow 0.$$

Since the subset $\Sigma_{>F_1} \times \{F_2\} \subseteq \Sigma_{>F_1} \times \Sigma_{\geqslant F_2}$ is cofinal, we have

$$\Gamma_!(\Sigma_{>F_1} \times \Sigma_{\geqslant F_2}, \mathcal{V}) = \varinjlim_{\substack{G_1 > F_1 \\ G_2 \geqslant F_2}} y_{G_1}\mathrm{M}_1 \otimes y_{G_2}\mathrm{M}_2 \cong \varinjlim_{G_1 > F_1} y_{G_1}\mathrm{M}_1 \otimes y_{F_2}\mathrm{M}_2 \cong \left(\varinjlim_{G_1 > F_1} y_{G_1}\mathrm{M}_1\right) \otimes y_{F_2}\mathrm{M}_2,$$

where the last isomorphism holds because $y_{F_2}\mathrm{M}_2$ is a free $\Bbbk$-module. Since M_1 is coflabby and locally free, the map from this space of cosections to $\mathrm{M}_1 \otimes \mathrm{M}_2$ is a split injection, with image

$$\left(\operatorname{span}_{\Sigma_{>F_1}} \mathrm{M}_1\right) \otimes y_{F_2}\mathrm{M}_2 = \operatorname{span}_{\Sigma_{>F_1} \times \Sigma_{\geqslant F_2}} \mathrm{M}.$$

Similarly, $\Gamma_!(\Sigma_{\geqslant F_1} \times \Sigma_{>F_2}, \mathcal{V})$ maps by a split injection into $\mathrm{M}_1 \otimes \mathrm{M}_2$ with image

$$y_{F_1}\mathrm{M}_1 \otimes \left(\operatorname{span}_{\Sigma_{>F_2}} \mathrm{M}_2\right) = \operatorname{span}_{\Sigma_{\geqslant F_1} \times \Sigma_{>F_2}} \mathrm{M}.$$

The intersection of these images is

$$\left(\operatorname{span}_{\Sigma_{>F_1}} \mathrm{M}_1\right) \otimes \left(\operatorname{span}_{\Sigma_{>F_2}} \mathrm{M}_2\right) = \operatorname{span}_{\Sigma_{>F_1} \times \Sigma_{>F_2}} \mathrm{M},$$

which is the image of $\Gamma_!(\Sigma_{>F_1} \times \Sigma_{>F_2}, \mathcal{V})$ in $\mathrm{M}_1 \otimes \mathrm{M}_2$. It follows that the map

$$\Gamma_!(\Sigma_{>(F_1,F_2)}, \mathcal{V}) \to \Gamma_!(\Sigma_{\geqslant(F_1,F_2)}, \mathcal{V}) = y_{F_1}\mathrm{M}_1 \otimes y_{F_2}\mathrm{M}_2$$

is a split injection, which proves that M is coflabby and locally free.

The proof that M is flabby is similar, using the sheaf property of $\Gamma_*(-, \mathcal{V})$. □

Proof of Theorem B. Parts (a) and (b) follow from Proposition 9.1, so it will suffice to prove part (c). We need to show that $\mathrm{E}(\mathcal{L}_1) \otimes \mathrm{E}(\mathcal{L}_2)$ satisfies the conditions of Theorem A. Indecomposability is clear, and purity follows from Lemma 9.5. Finally, we have

$$y_{(\hat{1},\hat{1})}\big(\mathrm{E}(\mathcal{L}_1) \otimes \mathrm{E}(\mathcal{L}_2)\big) = y_{\hat{1}}\mathrm{E}(\mathcal{L}_1) \otimes y_{\hat{1}}\mathrm{E}(\mathcal{L}_2) \cong \Bbbk[-\operatorname{rk}\mathcal{L}_1] \otimes \Bbbk[-\operatorname{rk}\mathcal{L}_2] \cong \Bbbk[-\operatorname{rk}(\mathcal{L}_1 \times \mathcal{L}_2)]. \quad \square$$

9.4. **Purity and base change.** In this section, we consider how pure modules behave when the coefficient ring $\Bbbk$ changes. Our ultimate goal is to prove Theorem E, along with the statement that parity modules behave nicely under field extensions.

Suppose that $\Bbbk \to \Bbbk'$ is a ring homomorphism, where $\Bbbk$ is either a field or a DVR and $\Bbbk'$ is a field. We will mainly be interested in the cases (1) $\Bbbk'$ is an extension field of $\Bbbk$, (2) $\Bbbk$ is a DVR and $\Bbbk' = \overline{\Bbbk}$ is its residue field, and (3) $\Bbbk$ is a DVR and $\Bbbk' = \mathbb{K}$ is its fraction field. Given an upper set Σ, we have a functor $- \otimes_{\Bbbk} \Bbbk' \colon \mathcal{C}(\Sigma; \Bbbk) \to \mathcal{C}(\Sigma; \Bbbk')$.

Theorem 9.6. Let $\mathcal{V}$ be an object of $\mathcal{C}(\Sigma; \Bbbk)$.

(a) For any $\Sigma' \subseteq \Sigma$, the natural map $\Gamma_!(\Sigma', \mathcal{V}) \otimes_{\Bbbk} \Bbbk' \to \Gamma_!(\Sigma', \mathcal{V} \otimes_{\Bbbk} \Bbbk')$ is an isomorphism.

(b) If $\mathcal{V}$ is pure, then the natural map $\Gamma_*(\Sigma', \mathcal{V}) \otimes_{\Bbbk} \Bbbk' \to \Gamma_*(\Sigma', \mathcal{V} \otimes_{\Bbbk} \Bbbk')$ is an isomorphism.

(c) If $\mathcal{V}$ is pure, then so is $\mathcal{V} \otimes_{\Bbbk} \Bbbk'$.

Proof. Tensoring with $\Bbbk'$ is right exact, which means that it commutes with finite colimits, and in particular it commutes with taking cosections. This proves (a).

The proof of (b) is by induction on $|\Sigma'|$. When $|\Sigma'| = 1$, or more generally when $\Sigma' = \Sigma_{\geqslant F}$ for an element F, the statement is true by definition. So assume $|\Sigma'| > 1$, Σ' has more than one minimal element, and the statement is true for all smaller upper sets. Let F be a minimal element of Σ' and let $\Sigma'' = \Sigma' \backslash \{F\}$.

Since $\mathcal{V}$ is flabby, Lemma 7.8(a) gives us an exact sequence

$$0 \longrightarrow \mathcal{V}_{[F]} \longrightarrow \Gamma_*(\Sigma', \mathcal{V}) \longrightarrow \Gamma_*(\Sigma'', \mathcal{V}) \longrightarrow 0.$$

Proposition 7.13(d) implies that $\Gamma_*(\Sigma'', \mathcal{V})$ is free, thus this exact sequence splits. It follows that tensoring with $\Bbbk'$ preserves exactness. Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{V}_{[F]} \otimes_{\Bbbk} \Bbbk' \arrow[r] \arrow[d] & \Gamma_*(\Sigma', \mathcal{V}) \otimes_{\Bbbk} \Bbbk' \arrow[r] \arrow[d] & \Gamma_*(\Sigma'', \mathcal{V}) \otimes_{\Bbbk} \Bbbk' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\mathcal{V} \otimes_{\Bbbk} \Bbbk')_{[F]} \arrow[r] & \Gamma_*(\Sigma', \mathcal{V} \otimes_{\Bbbk} \Bbbk') \arrow[r] & \Gamma_*(\Sigma'', \mathcal{V} \otimes_{\Bbbk} \Bbbk') \arrow[r] & 0
\end{tikzcd}$$

We have already argued that the top row is exact. Left-exactness of the bottom row follows from Lemma 7.8. The right vertical map is an isomorphism by our induction hypothesis, which implies that the bottom row is also right exact.

We want to prove the middle vertical map is an isomorphism, and by the Five Lemma, it is sufficient to show that the left vertical map is an isomorphism. To do this, we consider the same diagram with Σ' replaced by $\Sigma_{\geqslant F}$ and Σ'' replaced by $\Sigma_{>F}$. Now the second and third vertical maps are isomorphisms by the induction hypothesis, which shows that $\mathcal{V}_{[F]} \otimes_{\Bbbk} \Bbbk' \to (\mathcal{V} \otimes_{\Bbbk} \Bbbk')_{[F]}$ is an isomorphism. This completes the proof of (b).

Finally, we prove (c). If $\mathcal{V}$ is pure, then its restriction homomorphisms are surjections, and (b) implies that the restriction homomorphisms of $\mathcal{V} \otimes_{\Bbbk} \Bbbk'$ are surjections. Thus $\mathcal{V} \otimes_{\Bbbk} \Bbbk'$ is flabby. Similarly, the corestrictions of $\mathcal{V}$ are split injections by Proposition 7.13, so (a) implies that $\mathcal{V} \otimes_{\Bbbk} \Bbbk'$ is coflabby. Since we are assuming $\Bbbk'$ is a field, local freeness is automatic. Thus $\mathcal{V} \otimes_{\Bbbk} \Bbbk'$ is pure. □

Proposition 9.7. If $\Bbbk \to \Bbbk'$ is a field extension, or if $\Bbbk$ is a DVR and $\Bbbk' = \overline{\Bbbk}$ is its residue field, then there is an isomorphism

$$\mathrm{E}(\mathcal{L}; \Bbbk') \cong \mathrm{E}(\mathcal{L}; \Bbbk) \otimes_{\Bbbk} \Bbbk'.$$

Proof. By Theorem 9.6, $\mathrm{E}(\mathcal{L}; \Bbbk) \otimes_{\Bbbk} \Bbbk'$ is pure, and it is clearly not annihilated by $y_{\hat{1}}$, so it only remains to show that it is indecomposable. By Theorem 8.12, it is enough to show that it is taut at all elements $F \neq \hat{1}$.

Consider first the case of a field extension. Let $\mathcal{V} = \Lambda\mathrm{E}(\mathcal{L}; \Bbbk)$, and take an element $F \neq \hat{1}$. Since $\Bbbk$ is a field, $\mathcal{V}$ is taut at F if and only if $\ker \Phi_F \subseteq \operatorname{im} \Psi_F$. Since tensoring with $\Bbbk'$ is exact, we have

$$\ker(\Phi_F \otimes_{\Bbbk} \Bbbk') = (\ker \Phi_F) \otimes_{\Bbbk} \Bbbk' \subseteq (\operatorname{im} \Psi_F) \otimes_{\Bbbk} \Bbbk' = \operatorname{im}(\Psi_F \otimes_{\Bbbk} \Bbbk').$$

By Theorem 9.6, $\Phi_F \otimes_{\Bbbk} \Bbbk'$ and $\Psi_F \otimes_{\Bbbk} \Bbbk'$ are the Φ and Ψ maps for $\mathcal{V} \otimes_{\Bbbk} \Bbbk'$. So we have shown that $\mathcal{V} \otimes_{\Bbbk} \Bbbk'$ is taut at F, as desired.

The case $\Bbbk \mapsto \overline{\Bbbk}$ is similar. Since $\mathcal{V} = \Lambda\mathrm{E}(\mathcal{L}; \Bbbk)$ is taut at F, we have $\ker \overline{\Phi_F} \subseteq \operatorname{im} \overline{\Psi_F}$. But Theorem 9.6 implies that $\overline{\Phi_F}$ and $\overline{\Psi_F}$ are the Φ and Ψ maps for $\bar{\mathcal{V}} = \mathcal{V} \otimes_{\Bbbk} \overline{\Bbbk}$. Thus we see that $\mathrm{E}(\mathcal{L}; \Bbbk) \otimes_{\Bbbk} \Bbbk'$ is taut at F for all $F \neq \hat{1}$, so it is indecomposable. □

The field extension case of Proposition 9.7 has the following immediate corollary.

Corollary 9.8. Let $\Bbbk$ be a field of characteristic p. We have

$${}_pZ_{\mathcal{L}}(t) = \sum_{i \geqslant 0} t^i \dim \mathrm{E}^i(\mathcal{L}; \Bbbk) \quad \text{and} \quad {}_pP_{\mathcal{L}}(t) \coloneqq \sum_{i \geqslant 0} t^i \dim \underline{\mathrm{E}}^i(\mathcal{L}; \Bbbk).$$

Proof of Theorem E. We use base change by the homomorphisms $\mathbb{F}_p \leftarrow \mathbb{Z}_p \to \mathbb{Q}_p$. Proposition 9.7 tells us that

$${}_pZ_{\mathcal{L}}(t) = \sum_{i \geqslant 0} t^i \dim_{\mathbb{F}_p} \mathrm{E}^i(\mathcal{L}; \mathbb{F}_p) = \sum_{i \geqslant 0} t^i \operatorname{rk}_{\mathbb{Z}_p} \mathrm{E}^i(\mathcal{L}; \mathbb{Z}_p) = \sum_{i \geqslant 0} t^i \dim_{\mathbb{Q}_p} \mathrm{E}^i(\mathcal{L}; \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Since $\mathrm{E}(\mathcal{L}; \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is pure and is not annihilated by $y_{\hat{1}}$, it has $\mathrm{E}(\mathcal{L}; \mathbb{Q}_p)$ as a direct summand. The first inequality follows. The proof of the inequality on stalks is similar. □

10. Dimension of the space of homomorphisms

In this section we assume that $\Bbbk$ is a field. It is possible to extend these results to the case where $\Bbbk$ is a DVR, but we do not need it.

Let Σ be an upper set, and consider objects $\mathcal{V}, \mathcal{W}$ in $\mathcal{C}(\Sigma;\Bbbk)$. Let $\hom_{\mathcal{C}(\Sigma;\Bbbk)}(\mathcal{V},\mathcal{W})$ be the **graded hom space** whose i^{th} graded piece is $\mathrm{Hom}_{\mathcal{C}(\Sigma;\Bbbk)}(\mathcal{V},\mathcal{W}[i])$. The graded hom space between two graded $\Bbbk$-vector spaces is defined in the same way.

Theorem 10.1. If $\mathcal{V}$ and $\mathcal{W}$ are both pure, then there is a non-canonical isomorphism of graded vector spaces

$$\hom_{\mathcal{C}(\Sigma;\Bbbk)}(\mathcal{V},\mathcal{W}) \cong \bigoplus_{F\in\Sigma} \hom(\mathcal{V}_F, \mathcal{W}_{[F]}).$$

Proof. We proceed by induction on $|\Sigma|$. If $|\Sigma| = 1$, then $\Sigma = \{\hat{1}\}$ and the category $\mathcal{C}(\Sigma;\Bbbk)$ is equivalent to the category of graded $\Bbbk$-vector spaces, so the result is trivial.

Suppose that $|\Sigma| > 1$, and let F be a minimal element in Σ, so that $\Sigma' = \Sigma\backslash\{F\}$ is an upper set. Then we have an exact sequence of graded vector spaces

$$0 \longrightarrow \hom(\mathcal{V}_F, \mathcal{W}_{[F]}) \longrightarrow \hom_{\mathcal{C}(\Sigma;\Bbbk)}(\mathcal{V},\mathcal{W}) \longrightarrow \hom_{\mathcal{C}(\Sigma';\Bbbk)}(\mathcal{V}|_{\Sigma'}, \mathcal{W}|_{\Sigma'}) \longrightarrow 0.$$

The second map is given by restriction, and it is surjective by Proposition 8.6. The first map is given by sending $\alpha\colon \operatorname{coker}\Psi_{\mathcal{V},F} \to \ker\Phi_{\mathcal{W},F}$ to a map that restricts to zero on Σ', and whose component at F is the composition

$$\mathcal{V}(F) \twoheadrightarrow \mathcal{V}_F \xrightarrow{\alpha} \mathcal{W}_{[F]} \hookrightarrow \mathcal{W}(F).$$

Applying the inductive hypothesis to morphisms $\mathcal{V}|_{\Sigma'} \to \mathcal{W}|_{\Sigma'}$ gives the result. □

Proposition 6.3, Lemma 7.6, and Theorem 10.1 together imply the following corollary.

Corollary 10.2. If M and M' are pure graded $\mathrm{H}(\mathcal{L};\Bbbk)$-modules, then there is a non-canonical isomorphism of graded vector spaces

$$\hom_{\mathrm{H}(\mathcal{L};\Bbbk)}(\mathrm{M},\mathrm{M}') \cong \bigoplus_{F\in\mathcal{L}} \hom(\mathrm{M}_F, \mathrm{M}'_{[F]}).$$

Taking both M and M' to be the parity module $\mathrm{E}(\mathcal{L};\Bbbk)$, we obtain the following result.

Corollary 10.3. We have $\mathrm{End}_{\mathrm{H}(\mathcal{L};\Bbbk)}\mathrm{E}(\mathcal{L};\Bbbk) \cong \Bbbk$ if and only if, for every $F \neq \hat{1}$ and every degree k, either the stalk $\mathrm{E}(\mathcal{L};\Bbbk)_F$ or the costalk $\mathrm{E}(\mathcal{L};\Bbbk)_{[F]}$ vanishes in degree k.

In [BHM+, Proposition 6.4], we showed that the only endomorphisms of $\mathrm{IH}(\mathcal{L}) = \mathrm{E}(\mathcal{L};\mathbb{Q})$ are scalar multiplication. The proof relied on [BHM+, Proposition 6.3], which states that the stalk $\mathrm{IH}(\mathcal{L})_F$ vanishes in degrees greater than or equal to $(\operatorname{crk} F)/2$ and the costalk $\mathrm{IH}(\mathcal{L})_{[F]}$ vanishes in

degrees less than or equal to $(\operatorname{crk} F)/2$. Although the argument there was expressed in somewhat different language, it follows the same outline given here.

Example 10.4. Let $\mathcal{L} = U_{2,n}$ be the lattice of flats for the uniform matroid of rank 2 on n elements, and let $\Bbbk$ be a field of characteristic p. If $n \equiv 1$ modulo p, then $\mathrm{E}(\mathcal{L};\Bbbk) \cong \mathrm{CH}(\mathcal{L};\Bbbk)$ by Theorem 1.12. The degree 1 parts of the stalk $\mathrm{CH}(\mathcal{L};\Bbbk)_{\hat{0}}$ and the costalk $\mathrm{CH}(\mathcal{L};\Bbbk)_{[\hat{0}]}$ are both isomorphic to $\Bbbk$, and we have $\operatorname{End}_{\mathrm{H}(\mathcal{L};\Bbbk)} \mathrm{E}(\mathcal{L};\Bbbk) \cong \Bbbk[\varepsilon]/\langle\varepsilon^2\rangle$.

11. Relation with topology

In this section, we assume that $\mathcal{L}$ is the lattice of flats of a matroid, and that this matroid is realizable by a linear subspace $V \subseteq \mathbb{C}^{\mathcal{L}^1}$, and we write $Y = Y_V \coloneqq \overline{V} \subseteq (\mathbb{P}^1)^{\mathcal{L}^1}$ to denote the associated arrangement Schubert variety. Given $p \in Y$ and $G \in \mathcal{L}^1$, we will write $p_G \in \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ to denote the corresponding coordinate. For any $F \in \mathcal{L}$, let

$$S_F \coloneqq \{p \in Y \mid p_G = \infty \iff G \nleq F\}.$$

By [PXY18, Lemma 7.6], we have $S_F \cong \mathbb{C}^{\operatorname{rk} F}$ and the stratification

$$Y = \bigsqcup_{F \in \mathcal{L}} S_F.$$

The strata $\mathcal{S} = \{S_F \mid F \in \mathcal{L}\}$ are the orbits of the action of the additive group $(V, +)$ on Y. This implies that they form an algebraic stratification in the sense of Chriss–Ginzburg [CG97, Definition 3.2.23] and a good stratification in the sense of Achar [Ach21, Definition 2.3.20]. It follows that for any inclusion f of locally closed unions of strata, the functors $f_*, f^*, f_!, f^!$ all preserve $\mathcal{S}$-constructibility.

Let $\Bbbk$ be either a field of arbitrary characteristic or a complete DVR of characteristic 0. Let $D^b_{\mathcal{S}}(Y;\Bbbk)$ denote the derived category of sheaves of $\Bbbk$-modules on Y that are constructible with respect to $\mathcal{S}$. For each element $F \in \mathcal{L}$, let $i_F \colon S_F \to Y$ denote the inclusion. An object $\mathcal{F}$ in $D^b_{\mathcal{S}}(Y;\Bbbk)$ is called **$*$-even** (resp. **!-even**) if, for every $F \in \mathcal{L}$, the cohomology sheaves $\mathcal{H}^k(i_F^*\mathcal{F})$ (resp. $\mathcal{H}^k(i_F^!\mathcal{F})$) vanish for all odd $k \in \mathbb{Z}$ and are free $\Bbbk$-local systems for all even $k \in \mathbb{Z}$. It is called **even** if it is both $*$-even and !-even. Juteau, Mautner, and Williamson defined a class of objects in $D^b_{\mathcal{S}}(Y;\Bbbk)$ called **parity sheaves**, described by the following theorem [JMW14].

Theorem 11.1. There exists a collection $\{\mathcal{E}(F;\Bbbk) \mid F \in \mathcal{L}\}$ of objects of $D^b_{\mathcal{S}}(Y;\Bbbk)$ satisfying the following properties:

(1) $\mathcal{E}(F;\Bbbk)$ is even and indecomposable.

(2) The support of $\mathcal{E}(F;\Bbbk)$ is equal to $\overline{S_F}$.

(3) The pullback $i_F^*\mathcal{E}(F;\Bbbk)$ to S_F is a rank one constant $\Bbbk$-local system in degree zero.

This collection is unique up to isomorphism. Every even object in $D^b_{\mathcal{S}}(Y;\Bbbk)$ is isomorphic to a direct sum of objects of the form $\mathcal{E}(F;\Bbbk)[2k]$ for some $k \in \mathbb{Z}$.

Proof. The uniqueness and support condition are [JMW14, Theorem 2.12]. Existence of the complexes $\mathcal{E}(F;\Bbbk)$ follows from [JMW14, Corollary 2.35], since the closures of strata $\overline{S_F}$, which are smaller arrangement Schubert varieties, have $(V,+)$-equivariant resolutions whose fibers have no odd cohomology. □

For any object $\mathcal{F}$ in $D^b_{\mathcal{S}}(Y;\Bbbk)$, the hypercohomology $\mathbb{H}^\bullet(\mathcal{F})$ is naturally a graded module over $\mathrm{H}^\bullet(Y;\Bbbk)$. The next result, which is the main result of this section, identifies the hypercohomology of the parity sheaves on Y with the parity modules of $\mathcal{L}$. The special case where $F = \hat{1}$ and $\Bbbk$ is a field gives us Theorem C from the introduction.

Theorem 11.2. For any $F \in \mathcal{L}$, there is a degree-doubling isomorphism

$$\mathrm{E}(\mathcal{L}^F;\Bbbk) \cong \mathbb{H}^\bullet(\mathcal{E}(F;\Bbbk))$$

of modules over $\mathrm{H}(\mathcal{L};\Bbbk) \cong \mathrm{H}^\bullet(Y;\Bbbk)$.

The proof of this theorem will proceed in several stages. We begin by associating a sheaf and a cosheaf to an even object in $D^b_{\mathcal{S}}(Y;\Bbbk)$. For an upper set $\Sigma \subseteq \mathcal{L}$, let

$$U_\Sigma \coloneqq \bigcup_{F\in\Sigma} S_F$$

be the corresponding open union of strata, and let $j_\Sigma \colon U_\Sigma \to Y$ be the inclusion. Let $\mathcal{F} \in D^b_{\mathcal{S}}(Y;\Bbbk)$ be even.

Proposition 11.3. The assignment

$$\Sigma \longmapsto \mathbb{H}^\bullet(j_\Sigma^*\mathcal{F})$$

is a flabby sheaf on $\mathcal{L}$, and the assignment

$$\Sigma \longmapsto \mathbb{H}^\bullet_c(j_\Sigma^*\mathcal{F})$$

is a coflabby cosheaf on $\mathcal{L}$. For any upper set Σ, both of these graded $\Bbbk$-modules vanish in odd degrees.

Proof. We first prove the parity vanishing statements, using induction on the cardinality of Σ. The statements are trivial when Σ is empty. Assume it is nonempty, and choose a minimal element $F \in \Sigma$, so that $\Sigma' \coloneqq \Sigma\backslash\{F\}$ is an upper set. By induction, both $\mathbb{H}^\bullet(j_{\Sigma'}^*\mathcal{F})$ and $\mathbb{H}^\bullet_c(j_{\Sigma'}^*\mathcal{F})$ vanish in odd degrees. Applying hypercohomology to the distinguished triangles of [Ach21, Theorem 1.2.10] gives exact sequences

$$\cdots \longrightarrow \mathbb{H}^\bullet(i_F^!\mathcal{F}) \longrightarrow \mathbb{H}^\bullet(j_\Sigma^*\mathcal{F}) \longrightarrow \mathbb{H}^\bullet(j_{\Sigma'}^*\mathcal{F}) \xrightarrow{[1]} \cdots \tag{12}$$

and

$$\cdots \longrightarrow \mathbb{H}^\bullet_c(j^*_{\Sigma'}\mathcal{F}) \longrightarrow \mathbb{H}^\bullet_c(j^*_{\Sigma}\mathcal{F}) \longrightarrow \mathbb{H}^\bullet_c(i^*_F\mathcal{F}) \xrightarrow{[1]} \cdots . \tag{13}$$

Since $i^*_F\mathcal{F}$ and $i^!_F\mathcal{F}$ are both isomorphic to direct sums of constant free $\Bbbk$-local systems with only even shifts, it follows that $\mathbb{H}^\bullet(i^!_F\mathcal{F})$ and $\mathbb{H}^\bullet_c(i^*_F\mathcal{F})$ both vanish in odd degrees and are free in even degrees. Since the outer terms of the exact sequences vanish in odd degrees, the same is true of the middle terms, completing the induction.

This argument also shows that the connecting homomorphisms in these exact sequences vanish, so $\mathbb{H}^\bullet(j^*_\Sigma\mathcal{F}) \to \mathbb{H}^\bullet(j^*_{\Sigma'}\mathcal{F})$ is surjective and $\mathbb{H}^\bullet_c(j^*_{\Sigma'}\mathcal{F}) \to \mathbb{H}^\bullet_c(j^*_\Sigma\mathcal{F})$ is split injective for any inclusion $\Sigma' \subseteq \Sigma$ of upper sets with $|\Sigma'| = |\Sigma| - 1$. By induction, the same holds for any inclusion of upper sets, which proves flabbiness and coflabbiness.

Finally, the parity vanishing proved above implies that, for any upper sets $\Sigma_1, \Sigma_2 \subseteq \mathcal{L}$, the Mayer–Vietoris sequences

$$\cdots \longrightarrow \mathbb{H}^\bullet(j^*_{\Sigma_1\cup\Sigma_2}\mathcal{F}) \longrightarrow \mathbb{H}^\bullet(j^*_{\Sigma_1}\mathcal{F}) \oplus \mathbb{H}^\bullet(j^*_{\Sigma_2}\mathcal{F}) \longrightarrow \mathbb{H}^\bullet(j^*_{\Sigma_1\cap\Sigma_2}\mathcal{F}) \xrightarrow{[1]} \cdots$$

and

$$\cdots \longrightarrow \mathbb{H}^\bullet_c(j^*_{\Sigma_1\cap\Sigma_2}\mathcal{F}) \longrightarrow \mathbb{H}^\bullet_c(j^*_{\Sigma_1}\mathcal{F}) \oplus \mathbb{H}^\bullet_c(j^*_{\Sigma_2}\mathcal{F}) \longrightarrow \mathbb{H}^\bullet_c(j^*_{\Sigma_1\cup\Sigma_2}\mathcal{F}) \xrightarrow{[1]} \cdots$$

split into short exact sequences. The first exact sequence says that $\Sigma \mapsto \mathbb{H}^\bullet(j^*_\Sigma\mathcal{F})$ is a sheaf, and the second says that $\Sigma \mapsto \mathbb{H}^\bullet_c(j^*_\Sigma\mathcal{F})$ is a cosheaf. □

Proposition 11.3 holds for even objects on any variety stratified by affine spaces; for example, it also holds for the Schubert stratification of a flag variety. But the exploded modules introduced in Section 6.1 have additional structure, namely an identification (up to shift) of the sections of the sheaf with the cosections of the cosheaf on principal open sets. In order to prove Theorem 11.2, we need this additional structure. It is easy to see that the sheaf and cosheaf of Proposition 11.3 agree on the maximal open set $U_\mathcal{L} = Y$, since Y is compact and therefore

$$\mathbb{H}^\bullet(\mathcal{F}) = \mathbb{H}^\bullet_c(\mathcal{F})$$

for any $\mathcal{F} \in D^b_\mathcal{S}(Y;\Bbbk)$. But to relate the sheaf and cosheaf beyond this extreme case we need the following proposition, which is particular to the geometry of the arrangement Schubert variety Y.

For all $F \in \mathcal{L}$, let $U_F = U_{\Sigma_{\geqslant F}}$ and $j_F = j_{\Sigma_{\geqslant F}}$.

Proposition 11.4. For any $\mathcal{F} \in D^b_\mathcal{S}(Y;\Bbbk)$ and any $F \in \mathcal{L}$, there is a natural isomorphism

$$\mathbb{H}^\bullet(j^*_F\mathcal{F}) \xrightarrow{\sim} \mathbb{H}^\bullet_c(j^*_F\mathcal{F})[2\operatorname{rk} F]$$

of functors from $D^b_\mathcal{S}(Y;\Bbbk)$ to graded $\Bbbk$-modules with the property that the composition

$$\mathbb{H}^\bullet(\mathcal{F}) \longrightarrow \mathbb{H}^\bullet(j^*_F\mathcal{F}) \longrightarrow \mathbb{H}^\bullet_c(j^*_F\mathcal{F})[2\operatorname{rk} F] \longrightarrow \mathbb{H}^\bullet_c(\mathcal{F})[2\operatorname{rk} F] = \mathbb{H}^\bullet(\mathcal{F})[2\operatorname{rk} F]$$

with the restriction and corestriction maps coincides with the action of $y_F \in \mathrm{H}(\mathcal{L};\Bbbk)$.

Before proving Proposition 11.4, we show how it implies Theorem 11.2. The proof will be based on the following result, which is an immediate consequence of Propositions 11.3 and 11.4.

Corollary 11.5. For any even object $\mathcal{F} \in D^b_{\mathcal{S}}(Y;\Bbbk)$ and any $F \in \mathcal{L}$, the kernel of the multiplication by y_F on $\mathbb{H}^\bullet(\mathcal{F})$ is equal to the kernel of restriction $\mathbb{H}^\bullet(\mathcal{F}) \to \mathbb{H}^\bullet(j_F^*\mathcal{F})$ and the image is equal to the image of the corestriction $\mathbb{H}^\bullet_c(j_F^*\mathcal{F}) \to \mathbb{H}^\bullet_c(\mathcal{F})$.

Remark 11.6. This corollary can be interpreted as saying that the exploded module $\mathcal{V} \coloneqq \Lambda\mathbb{H}^\bullet(\mathcal{F})$ is built from the spaces $\mathcal{V}(F) = \mathbb{H}^\bullet(j_F^*\mathcal{F}) \cong \mathbb{H}^\bullet_c(j_F^*\mathcal{F})[2\operatorname{rk} F]$, with the maps φ_F^G given by restriction and the maps ψ_F^G given by corestriction.

Proof of Theorem 11.2. We first show that, for any even object $\mathcal{F}$ in $D^b_{\mathcal{S}}(Y;\Bbbk)$, the $\mathrm{H}(\mathcal{L};\Bbbk)$-module $\mathrm{M} \coloneqq \mathbb{H}^\bullet(\mathcal{F})$ (with degrees halved) is pure. More precisely, following Definition 7.10, we show that the exploded module $\mathcal{V} \coloneqq \Lambda\mathrm{M}$ is flabby, coflabby, and locally free.

Let $\Sigma \subseteq \mathcal{L}$ be an upper set. We have isomorphisms

$$\Gamma_!(\Sigma, \mathcal{V}) \cong \varinjlim_{F\in\Sigma} \mathbb{H}^\bullet_c(j_F^*\mathcal{F}) \cong \mathbb{H}^\bullet_c(j_\Sigma^*\mathcal{F}),$$

where the first isomorphism follows from Corollary 11.5 and the second follows from the cosheaf property of Proposition 11.3. Proposition 11.3 then shows that $\mathcal{V}$ is coflabby.

In the same way, we have isomorphisms

$$\Gamma_*(\Sigma, \mathcal{V}) \cong \varprojlim_{F\in\Sigma} \mathbb{H}^\bullet(j_F^*\mathcal{F}) \cong \mathbb{H}^\bullet(j_\Sigma^*\mathcal{F}),$$

and we conclude by Proposition 11.3 that $\mathcal{V}$ is flabby.

Next, we show that the stalk and costalk of $\mathcal{V}$ at any $F \in \mathcal{L}$ are identified with their topological counterparts. For the stalk, we have

$$\begin{aligned}
\mathcal{V}_F &= \operatorname{coker} \Psi_{F,\mathcal{V}} \\
&\cong \operatorname{coker}\left(\mathbb{H}^\bullet_c(j^*_{\Sigma_{>F}}\mathcal{F})[2\operatorname{rk} F] \to \mathbb{H}^\bullet_c(j_F^*\mathcal{F})[2\operatorname{rk} F]\right) \\
&\cong \mathbb{H}^\bullet_c(i_F^*\mathcal{F})[2\operatorname{rk} F] \\
&\cong \mathbb{H}^\bullet(i_F^*\mathcal{F}),
\end{aligned}$$

where the second isomorphism follows from the exact sequence (13) (recall that the connecting homomorphisms of this sequence vanish). Since the cohomology sheaves of $i_F^*\mathcal{F}$ are constant free $\Bbbk$-local systems, $\mathbb{H}^\bullet(i_F^*\mathcal{F})$ is free, and so $\mathcal{V}$ is locally free. We have completed the proof that $\mathcal{V}$ is pure.

We have a dual isomorphism for the costalk:

$$\begin{aligned}\mathcal{V}_{[F]} &= \ker \Phi_{F,\mathcal{V}} \\ &\cong \ker \left(\mathbb{H}^\bullet(j_F^*\mathcal{F}) \to \mathbb{H}^\bullet(j_{\Sigma_{>F}}^*\mathcal{F})\right) \\ &\cong \mathbb{H}^\bullet(i_F^!\mathcal{F}),\end{aligned}$$

where the second isomorphism follows from the exact sequence (12). Combining these isomorphisms, we see that the costalk-stalk homomorphism $\beta_F(\mathrm{M})$, which is the composition

$$\mathcal{V}_{[F]} \longrightarrow \mathcal{V}(F) \longrightarrow \mathcal{V}_F,$$

is identified with the natural map

$$\mathbb{H}^\bullet(i_F^!\mathcal{F}) \longrightarrow \mathbb{H}^\bullet(i_F^*\mathcal{F}).$$

Now suppose that $\mathcal{F}$ is the parity sheaf $\mathcal{E}(F;\Bbbk)$. We wish to show that the exploded module $\mathcal{V} = \Lambda\mathbb{H}^\bullet(\mathcal{F})$ is isomorphic to $\mathcal{Q}^F_{\mathcal{L},\Bbbk}$ as defined in Theorem 8.12, since Lemma 8.14 then yields $\mathrm{M} \cong \Theta\mathcal{V} \cong \mathrm{E}(\mathcal{L}^F;\Bbbk)$. We know that $\mathcal{V}$ is pure, so we must verify the remaining conditions of Theorem 8.12.

For any $G \nleq F$ we have $\mathcal{V}(G) \cong \mathbb{H}^\bullet(j_G^*\mathcal{F}) = 0$, since $\mathcal{E}(F;\Bbbk)$ is supported on $\overline{S_F}$, and we also have $\mathcal{V}(F) \cong \mathbb{H}^\bullet(j_F^*\mathcal{F}) \cong \Bbbk$, since $\mathcal{E}(F;\Bbbk)$ restricts to a constant rank one local system on S_F.

It remains to show that $\mathcal{V}$ is taut at all $G \neq F$. This is trivial if $G \nleq F$ since $\mathcal{V}(G) = 0$, so suppose that $G < F$. The restriction $j_G^*\mathcal{F}$ is indecomposable by [JMW14, Proposition 2.11], and S_G is closed in U_G, so we can apply [JMW14, Lemma 2.21] together with the fact that S_G is contractible to deduce that the image of the natural homomorphism

$$\mathbb{H}^\bullet(i_G^!\mathcal{F}) \longrightarrow \mathbb{H}^\bullet(i_G^*\mathcal{F})$$

does not contain any non-trivial direct summands as $\Bbbk$-modules. Equivalently, the induced homomorphism of $\overline{\Bbbk}$-modules

$$\overline{\mathbb{H}^\bullet(i_G^!\mathcal{F})} \longrightarrow \overline{\mathbb{H}^\bullet(i_G^*\mathcal{F})}$$

is zero.[4] We have seen that this map is identified with $\overline{\beta_G(\mathrm{M})}$, so Lemma 8.8 shows that $\mathcal{V}$ is taut at G, as desired. □

Proof of Proposition 11.4. Let $B \subseteq \mathcal{L}^1$ be a basis for F. In other words, B is a minimal set of atoms whose join is F. We have $|B| = \operatorname{rk} F$.

Let $X = (\mathbb{P}^1)^B$, and let $p\colon Y \to X$ denote the inclusion of Y into $(\mathbb{P}^1)^{\mathcal{L}^1}$ followed by projection onto the factors indexed by B. This map is equivariant for the action of $(V,+)$, where the action

[4]Note that, if $\Bbbk$ is a field, then $\Bbbk = \overline{\Bbbk}$ and the two homomorphisms are the same.

of $v \in V$ on X forgets the coordinates not in B. If $W = \mathbb{C}^B \subseteq X$ is the locus of points where no coordinate is ∞, then we have $p^{-1}(W) = U_F$, the union of all S_G with $G \geqslant F$.

The class $y_F \in \mathrm{H}^{\mathrm{rk}\,F}(\mathcal{L};\Bbbk) \cong \mathrm{H}^{2\,\mathrm{rk}\,F}(Y;\Bbbk)$ is the product of y_b for $b \in B$. This implies that $y_F = p^*\alpha$, where $\alpha \in \mathrm{H}^{2\,\mathrm{rk}\,F}(X;\Bbbk)$ is the standard generator. We can describe the class α as a map of sheaves in the following way. Let i be the inclusion of $\{0\} \subseteq W$ into X. Under the natural isomorphism

$$\mathrm{H}^{2\,\mathrm{rk}\,F}(X;\Bbbk) \cong \mathrm{Hom}_{D^b(X;\Bbbk)}(\underline{\Bbbk}_X, \underline{\Bbbk}_X[2\,\mathrm{rk}\,F]),$$

the class α is given by the composition

$$\underline{\Bbbk}_X \longrightarrow i_* i^* \underline{\Bbbk}_X \xrightarrow{\cong} i_! i^! \underline{\Bbbk}_X[2\,\mathrm{rk}\,F] \longrightarrow \underline{\Bbbk}_X[2\,\mathrm{rk}\,F], \tag{14}$$

where the first and third maps are the adjunctions and the middle map is obtained using the standard orientation of X as a smooth complex variety. More precisely, if $a\colon X \to \{0\}$ is the projection, the orientation induces an isomorphism $a^!\underline{\Bbbk}_{\{0\}} \cong \underline{\Bbbk}_X[2\,\mathrm{rk}\,F]$, and applying $i^!$ gives an isomorphism

$$\underline{\Bbbk}_{\{0\}} \cong i^! a^! \underline{\Bbbk}_{\{0\}} \cong i^! \underline{\Bbbk}_X[2\,\mathrm{rk}\,F]$$

which, together with the fact that $i_* = i_!$, gives the required map.

We can obtain a map of sheaves representing the class $y_F = p^*\alpha$ by pulling back the map (14) by p. We have $p^*\underline{\Bbbk}_X = \underline{\Bbbk}_Y$, so this will give a map $\underline{\Bbbk}_Y \to \underline{\Bbbk}_Y[2\,\mathrm{rk}\,F]$. To see what the pullback of the middle sheaf $i_* i^* \underline{\Bbbk}_X$ is, consider $Y_F \coloneqq p^{-1}(0)$. It is the arrangement Schubert variety associated with the vector subspace $V_F \coloneqq V \cap p^{-1}(0)$, considered as a subspace of $\mathbb{C}^{\mathcal{L}^1 \setminus F}$. Let $f\colon Y_F \to Y$ denote the inclusion. Since $i_* = i_!$, proper base change implies that

$$p^* i_* i^* \underline{\Bbbk}_X \cong f_* f^* p^* \underline{\Bbbk}_X \cong f_* f^* \underline{\Bbbk}_Y.$$

Thus pulling back (14) by p expresses the cohomology class y_F as a composition

$$\underline{\Bbbk}_Y \longrightarrow f_* f^* \underline{\Bbbk}_Y \longrightarrow \underline{\Bbbk}_Y[2\,\mathrm{rk}\,F].$$

The action of y_F on the hypercohomology of $\mathcal{F} \in D^b_{\mathcal{S}}(Y;\Bbbk)$ is then given by tensoring this map with $\mathcal{F}$, using the natural isomorphism $\underline{\Bbbk}_Y \otimes \mathcal{F} \simeq \mathcal{F}$, and taking hypercohomology. The projection formula and compatibility of tensor with pullbacks give

$$\begin{aligned} f_* f^* \underline{\Bbbk}_Y \otimes \mathcal{F} &\cong f_! f^* \underline{\Bbbk}_Y \otimes \mathcal{F} \\ &\cong f_!(f^* \underline{\Bbbk}_Y \otimes f^* \mathcal{F}) \\ &\cong f_* f^*(\underline{\Bbbk}_Y \otimes \mathcal{F}) \\ &\cong f_* f^* \mathcal{F}. \end{aligned}$$

This gives the bottom row in the following diagram:

$$\begin{tikzcd}
 & (j_F)_* j_F^* \mathcal{F} \arrow[rr] \arrow[dr, "\zeta"] & & (j_F)_! j_F^* \mathcal{F}[2\operatorname{rk} F] \arrow[dr] & \\
\mathcal{F} \arrow[ur] \arrow[rr] & & f_* f^* \mathcal{F} \arrow[ur, "\eta"] \arrow[rr] & & \mathcal{F}[2\operatorname{rk} F]
\end{tikzcd}$$

The outer diagonal maps come from the adjunctions, together with the fact that $j_F^* = j_F^!$. To construct the inner diagonal maps ζ and η, apply $(j_F)_* j_F^*$ to the bottom left map, apply $(j_F)_! j_F^*$ to the bottom right map, and use the fact that the adjunction maps

$$(j_F)_! j_F^* f_* f^* \mathcal{F} \longrightarrow f_* f^* \mathcal{F} \longrightarrow (j_F)_* j_F^* f_* f^* \mathcal{F}$$

are isomorphisms, because the support of $f_* f^* \mathcal{F}$ is contained in Y_F, which is a closed subset of U_F.

Taking hypercohomology of the upper path in the diagram gives homomorphisms

$$\mathbb{H}^\bullet(\mathcal{F}) \longrightarrow \mathbb{H}^\bullet(j_F^* \mathcal{F}) \xrightarrow{\mathbb{H}^\bullet(\eta\zeta)} \mathbb{H}_c^\bullet(j_F^* \mathcal{F})[2\operatorname{rk} F] \longrightarrow \mathbb{H}_c^\bullet(\mathcal{F})[2\operatorname{rk} F] = \mathbb{H}^\bullet(\mathcal{F})[2\operatorname{rk} F]$$

factoring the action of y_F. What remains is to show that $\mathbb{H}^\bullet(\eta\zeta)$ is an isomorphism. In fact, we will show that $\mathbb{H}^\bullet(\eta)$ and $\mathbb{H}^\bullet(\zeta)$ are both isomorphisms. This will follow from understanding the local geometry of Y near Y_F. We have an isomorphism $(p, q)\colon U_F \xrightarrow{\sim} W \times Y_F$, where $q\colon U_F \to Y_F$ is defined as follows. Choose a complement $\widetilde{W}$ to V_F in V. The restriction $\tilde{p} = p|_{\widetilde{W}}\colon \widetilde{W} \to W$ is an isomorphism, and we let $q(x) = x - \tilde{p}^{-1}(p(x))$ for $x \in U_F$. The inverse map is given by $(w, x) \mapsto x + \tilde{p}^{-1}(w)$ for $w \in W$, $x \in Y_F$.

Using this, one can construct a stratification-preserving action of $\mathbb{C}^\times$ on U_F with fixed point set Y_F, and which contracts U_F onto Y_F, given by the formula[5]

$$(p(t \cdot x), q(t \cdot x)) = (tp(x), q(x)), \quad t \in \mathbb{C}^\times, x \in U_F.$$

If we let $g\colon Y_F \to U_F$ denote the inclusion, so $f = j_F \circ g$, then the contracting lemma [Ach21, Theorem 2.10.3] gives isomorphisms $q_* \simeq g^*$ and $q_! \simeq g^!$ of functors $D^b_{\mathcal{S}}(U_F; \Bbbk) \to D^b(Y_F; \Bbbk)$. In particular, for an object $\mathcal{G}$ of $D^b_{\mathcal{S}}(U_F; \Bbbk)$, we have natural isomorphisms

$$\mathbb{H}^\bullet(\mathcal{G}) \cong \mathbb{H}^\bullet(q_* \mathcal{G}) \cong \mathbb{H}^\bullet(g^* \mathcal{G}), \quad \mathbb{H}_c^\bullet(\mathcal{G}) \cong \mathbb{H}^\bullet(q_! \mathcal{G}) \cong \mathbb{H}^\bullet(g^! \mathcal{G}).$$

Since both the source and target of ζ are pushforwards of objects on U_F, to show that $\mathbb{H}^\bullet(\zeta)$ is an isomorphism it is enough to note that all of the maps in the left triangle become isomorphisms after applying $f^* = g^* j_F^*$. Similarly, the fact that $\mathbb{H}^\bullet(\eta)$ is an isomorphism follows by noting that the maps in the right triangle become isomorphisms after applying $f^! = g^! j_F^!$. $\square$

[5] Beware that this action does *not* extend to an action on Y.

REFERENCES

[Ach21] Pramod N. Achar, *Perverse sheaves and applications to representation theory*, Mathematical Surveys and Monographs, vol. 258, American Mathematical Society, Providence, RI, [2021] ©2021. 55, 56, 61

[AHK18] Karim Adiprasito, June Huh, and Eric Katz, *Hodge theory for combinatorial geometries*, Ann. of Math. (2) **188** (2018), no. 2, 381–452. 7

[AHL25] Omid Amini, June Huh, and Matt Larson, *A decomposition theorem for Lefschetz modules*, arXiv:2511.02026v2. 26

[Ati56] M. Atiyah, *On the Krull-Schmidt theorem with application to sheaves*, Bull. Soc. Math. France **84** (1956), 307–317. 8

[Bac75] Kenneth Bacławski, *Whitney numbers of geometric lattices*, Advances in Math. **16** (1975), 125–138. 30

[BBD82] A. A. Beĭlinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, vol. 100, Soc. Math. France, Paris, 1982, pp. 5–171. 1

[BHM$^+$] Tom Braden, June Huh, Jacob P. Matherne, Nicholas Proudfoot, and Botong Wang, *Singular Hodge theory for combinatorial geometries*, to appear in J. Amer. Math. Soc., arXiv:2010.06088. 1, 3, 4, 5, 6, 16, 21, 22, 26, 45, 54

[BHM$^+$22] ———, *A semi-small decomposition of the Chow ring of a matroid*, Adv. Math. **409** (2022), Paper No. 108646. 3, 7, 20, 21, 29

[BV20] Tom Braden and Artem Vysogorets, *Kazhdan-Lusztig polynomials of matroids under deletion*, Electron. J. Combin. **27** (2020), no. 1, Paper No. 1.17, 17. 5, 9

[CG97] Neil Chriss and Victor Ginzburg, *Representation theory and complex geometry*, Birkhäuser Boston, Inc., Boston, MA, 1997. 55

[CZ22] Dean Crnković and Tin Zrinski, *Constructing block designs with a prescribed automorphism group using genetic algorithm*, Journal of Combinatorial Designs **30** (2022), no. 7. 10

[DW74] Thomas A. Dowling and Richard M. Wilson, *The slimmest geometric lattices*, Trans. Amer. Math. Soc. **196** (1974), 203–215. 1

[EPW16] Ben Elias, Nicholas Proudfoot, and Max Wakefield, *The Kazhdan-Lusztig polynomial of a matroid*, Adv. Math. **299** (2016), 36–70. 1, 4, 19

[FMSV24] Luis Ferroni, Jacob P. Matherne, Matthew Stevens, and Lorenzo Vecchi, *Hilbert-Poincaré series of matroid Chow rings and intersection cohomology*, Adv. Math. **449** (2024), Paper No. 109733, 55. 6, 7

[HW17] June Huh and Botong Wang, *Enumeration of points, lines, planes, etc*, Acta Math. **218** (2017), no. 2, 297–317. 3

[JMW14] Daniel Juteau, Carl Mautner, and Geordie Williamson, *Parity sheaves*, J. Amer. Math. Soc. **27** (2014), no. 4, 1169–1212. 1, 3, 9, 55, 56, 59

[JW17] Lars Thorge Jensen and Geordie Williamson, *The p-canonical basis for Hecke algebras*, Categorification and higher representation theory, Contemp. Math., vol. 683, Amer. Math. Soc., Providence, RI, 2017, pp. 333–361. 5

[Kun79] Joseph P. S. Kung, *The Radon transforms of a combinatorial geometry. I*, J. Combin. Theory Ser. A **26** (1979), no. 2, 97–102. 48

[Oxl11] James Oxley, *Matroid theory*, second ed., Oxford Graduate Texts in Mathematics, vol. 21, Oxford University Press, Oxford, 2011. 2, 7

[Pro19] Nicholas Proudfoot, *Equivariant Kazhdan–Lusztig polynomials of q-niform matroids*, Algebraic Combinatorics **2** (2019), no. 4, 613–619. 7

[PXY18] Nicholas Proudfoot, Yuan Xu, and Ben Young, *The Z-polynomial of a matroid*, Electron. J. Combin. **25** (2018), no. 1, Paper 1.26, 21. 4, 55

[Sta92] Richard P. Stanley, *Subdivisions and local h-vectors*, J. Amer. Math. Soc. **5** (1992), no. 4, 805–851. 8

[Wel76] D. J. A. Welsh, *Matroid theory*, Academic Press [Harcourt Brace Jovanovich, Publishers], London-New York, 1976, L. M. S. Monographs, No. 8. 6, 7, 19

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF MASSACHUSETTS, AMHERST, MA.

Email address: braden@umass.edu

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NJ, AND KOREA INSTITUTE FOR ADVANCED STUDY, SEOUL, KOREA.

Email address: huh@princeton.edu

DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, RALEIGH, NC.

Email address: jpmather@ncsu.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OR.

Email address: njp@uoregon.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN-MADISON, MADISON, WI.

Email address: wang@math.wisc.edu