OVERVIEW

DAVID BEN-ZVI

Goal: develop basic tools of representation theory. and harmonic analysis in a categorified setting: groups acting on categories.

Why do we want to do that?

- Geometry. For G acting on X, we can assign a category $\mathrm{Vect}(X), \mathrm{Shv}(X)$, which carries a G-action.
- Algebra. For G acting on an algebra A, there is a G-action on A-mod.
- Physics. G-gauge theories, QFTs with G-symmetry. Then the categories of boundary conditions (defects, branes) carry a G-action.

GIven a category you car about, one can use symmetry to understand it better.

Historically, the first appearance of these kinds of G-categories are Lusztig's character sheaves: look like characters of groups acting on categories. These are vector bundles/sheaves on G (algebraic group), which are invariant under conjugation. This allowed Lusztig to construct characters of most finite simple groups geometrically. Many finite simple groups are groups of Lie type: $G(\mathbf{F}_q)$.

Geometric Langlands program: categorical analog of harmonic analysis on arithmetic locally-symmetric spaces: e.g. $SL_2\mathbf{Z}\backslash SL_2\mathbf{R}/SO_2$. Classical Langlands program is concerned with L^2-functions on this space. Categorical Langlands is concerned with D-modules on the moduli space of G-bundles on an algebraic curve.

Extended TFT: most field theories in ¿2 dimensions (all field theories in ¿3 dimensions) are gauge theories: involves principal bundles on spacetime.

Gauge theory as TFT:

- 2d: G-actions on vector spaces
- 3d: G-actions on categories
- 4d: G-actions on 2-categories

Geometirc Langlands is fits into the 4d story.

If G acts on $L^2(X)$, we have certain tools: spectral theory, Fourier transform, integral transforms.

We want to find an analog of these tools for G-actions on categories like $\mathrm{Shv}(X)$. Need functional analysis for categories. Need basic algebra for categories. These will be given by the theory of ∞-categories.

Homological algebra is a basic tool in modern algebra: studying operation in a linear setting (vector spaces, modules, abelian categories).

Now need a nonlinear version. We might consider rings, spaces, categories. This is what homotopical algebra is: a nonlinear version of homological algebra.