APPENDIX TO [BFN]: MORITA EQUIVALENCE
FOR CONVOLUTION CATEGORIES

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ABSTRACT. In this brief postscript to [BFN], we describe a Morita equivalence for derived, categoried matrix algebras implied by theory [G2, P, G1] developed since the appearance of [BFN]. We work in the setting of perfect stacks X and their stable ∞-categories $\mathrm{Q}(X)$ of quasicoherent sheaves. Perfect stacks include all varieties and common stacks in characteristic zero, and their stable ∞-categories of sheaves are well behaved refinements of their quasicoherent derived categories, satisfying natural analogues of common properties of function spaces.

To a morphism of perfect stacks $\pi : X \to Y$, we associate the categorified matrix algebra $\mathrm{Q}(X \times_Y X)$ of sheaves on the derived fiber product equipped with its monoidal convolution product. We show that for π faithfully flat (as a corollary of the 1-affineness theorem of Gaisgory [G2]) or for π proper and surjective and X, Y smooth (as an application of proper descent [P, G1]), there is a Morita equivalence between $\mathrm{Q}(X \times_Y X)$ and $\mathrm{Q}(Y)$, that is, an equivalence of their ∞-categories of stable module ∞-categories. In particular, this immediately implies an identification of their Drinfeld centers (as previously established in [BFN]), and more generally, an identification of their associated topological field theories. Another consequence is that for an affine algebraic group G in characteristic zero and an algebraic subgroup K, passing to K-invariants induces an equivalence from stable ∞-categories with algebraic G-action to modules for the Hecke category $\mathrm{Q}(K \backslash G / K)$.

1. INTRODUCTION

In this postscript to [BFN], we establish a derived, categorified analogue of the following well-known observation. Let $X \to Y$ denote a surjective map of finite sets, and consider the vector spaces $\mathbb{C}[X], \mathbb{C}[Y]$ of functions. Pullback of functions realizes $\mathbb{C}[X]$ as a module over $\mathbb{C}[Y]$ equipped with its usual commutative pointwise multiplication.

Consider the vector space $\mathbb{C}[X \times_Y X]$ of functions on the fiber product. Concretely, it forms the matrix algebra of block diagonal matrices with indices labelled by X and blocks labelled by Y. It naturally acts on $\mathbb{C}[X]$ by matrix multiplication, and $\mathbb{C}[Y]$ is the subalgebra of block scalar matrices. The $\mathbb{C}[Y]$–$\mathbb{C}[X \times_Y X]$ bimodule $\mathbb{C}[X]$ provides a Morita equivalence identifying the categories of $\mathbb{C}[Y]$-modules and $\mathbb{C}[X \times_Y X]$-modules.

We will establish an analogue of this Morita equivalence in which finite sets are replaced by perfect stacks (for example, varieties or common stacks in characteristic zero) and vector spaces of functions by stable ∞-categories of quasicoherent sheaves.

1.1. **Statements of Morita equivalence.** We will continue in the general setting of [BFN], and only comment here specifically where we interface with it. In particular, we will work with perfect stacks X as introduced in [BFN] where the stable ∞-category $\mathrm{Q}(X)$ of quasicoherent sheaves is "generated by finite objects" (see also the discussion in [L5, Section 8]). More precisely, a derived stack X with affine diagonal is said to be perfect if $\mathrm{Q}(X)$ is the inductive limit

$$\mathrm{Q}(X) \simeq \operatorname{Ind} \operatorname{Perf}(X)$$

of the full ∞-subcategory $\operatorname{Perf}(X)$ of perfect complexes (objects locally equivalent to finite complexes of finite vector bundles), or equivalently, $\mathrm{Q}(X)$ is compactly generated (there is no right orthogonal to the compact objects), and compact objects and dualizable objects coincide.

The class of perfect stacks is very broad, and includes all (quasi-compact and separated) schemes and common stacks in characteristic zero (in particular, the quotient X/G of a quasi-projective

derived scheme X by a linear action of an affine group G). It is also closed under fiber products, passing to total spaces of quasiprojective morphisms, and quotients by finite group schemes in characteristic zero.

The main technical result of [BFN] is the following theorem.

Theorem 1.1 ([BFN])**.** *Let $\pi : X \to Y$ denote a morphism between perfect stacks. Then $Q(X)$ is canonically self-dual as a $Q(Y)$-module, and there are canonical equivalences of $Q(X)$-bimodules*

$$Q(X) \otimes_{Q(Y)} Q(X) \xrightarrow{\ \sim\ } Q(X \times_Y X) \xrightarrow{\ \sim\ } \mathrm{Hom}_{Q(Y)}(Q(X), Q(X))$$

Moreover, the convolution monoidal structure on the middle term is identified with the composition monoidal structure of the third term.

Remark 1.2. For perfect stacks X_1, X_2 and a morphism $f : X_1 \to X_2$ over the perfect stack Y, the canonical self-duality of $Q(X_1), Q(X_2)$ over $Q(Y)$ identifies the transpose of the pullback $f^* : Q(X_2) \to Q(X_1)$ with the pushforward $f_* : Q(X_1) \to Q(X_2)$ and vice versa. Thus not only is the natural algebra structure on $Q(X \times_Y X)$ compatible with that on $\mathrm{Hom}_{Q(Y)}(Q(X), Q(X))$, but also the natural coalgebra structure on $Q(X \times_Y X)$ is compatible with that on $Q(X) \otimes_{Q(Y)} Q(X)$.

We will focus here on the module theory of the convolution algebra $Q(X \times_Y X)$, and its description in terms of the commutative algebra $Q(Y)$. In general, when we speak about modules for an algebra object $\mathcal{A}$ in a symmetric monoidal ∞-category $\mathcal{C}$, we mean $\mathcal{A}$-module objects in $\mathcal{C}$ and denote their ∞-category by $\mathrm{Mod}(\mathcal{A})$. In particular, we regard the monoidal ∞-categories $Q(Y)$, $Q(X \times_Y X)$ as algebra objects in the symmetric monoidal ∞-category St of stable presentable ∞-categories.

In [BFN], we constructed (under the hypothesis that descent holds for π) an equivalence of Drinfeld centers (or Hochschild cohomology categories)

$$\mathcal{Z}(Q(Y)) \simeq Q(\mathcal{L}Y) \simeq \mathcal{Z}(Q(X \times_Y X)).$$

where $\mathcal{L}Y = \mathrm{Map}(S^1, Y)$ is the derived loop space of Y. Under appropriate assumptions, we will upgrade this here to a Morita equivalence between the two algebras.

Theorem 1.3. *Let $\pi : X \to Y$ denote a morphism of perfect stacks. If either*

(1) *π is faithfully flat, or*
(2) *X, Y are smooth and π is proper and surjective,*

then the $Q(Y)$– $Q(X \times_Y X)$-bimodule $Q(X)$ defines an equivalence

$$Q(X) \otimes_{Q(Y)} (-) : \mathrm{Mod}(Q(Y)) \xrightarrow{\ \sim\ } \mathrm{Mod}(Q(X \times_Y X))$$

of ∞-categories of stable presentable module categories.

The proof is a straightforward application of Lurie's Barr-Beck Theorem. An evident necessary condition is that the functor $Q(X) \otimes_{Q(Y)} (-)$ is conservative. The stated assumptions provide common geometric settings where this can be seen by appealing to established theory.

First, the flat Morita equivalence is a categorified analogue of faithfully flat descent for quasicoherent sheaves (due to Lurie [L4, L5] in the derived setting). Here the key input is the theorem of Gaitsgory [G2] that many geometric stacks X (including perfect stacks) are *1-affine* in the sense that localization and global sections define inverse equivalences between $Q(X)$-modules and quasicoherent stacks of categories over X.

Second, the strange hybrid setting for the proper Morita equivalence results from combining two inputs. On the one hand, there is a general descent theorem for proper morphisms due to Preygel and Gaitsgory (see [P, Proposition A.2.8] and [G1, 7.2.2]) which applies not to $Q(X)$ but to its "dual counterpart" the ∞-category $Q^!(X) = \mathrm{Ind}(\mathrm{Coh}(X))$ of *ind-coherent sheaves*. Just as quasicoherent sheaves are a categorified analogue of functions or cohomology, ind-coherent sheaves are a categorified analogue of measures or homology (many of their beautiful properties are developed in [G1]). However, the analogue of Theorem 1.1 *does not hold* for ind-coherent sheaves even in the

restricted setting of Theorem 1.3(2) (though it holds in the "absolute" case when $Y = pt$), and proper Morita equivalence for $Q^!$ also fails.

On the other hand, quasicoherent and ind-coherent sheaves coincide if and only if the geometric stack is smooth. Thus our assumption that X, Y are smooth implies that we have proper descent for quasicoherent sheaves. Moreover, while usual descent data is given by a coalgebra (or comonad), proper descent uses an opposite adjunction and hence is described by an algebra (monad). Thus we can deduce proper Morita equivalence from general facts about modules over algebras.

1.2. **Applications.** First, we mention some relations to topological field theory. Recall [L3] the Morita $(\infty, 2)$-category $2Alg$ of algebra objects in stable presentable ∞-categories St with 1-morphisms given by bimodules, and 2-morphisms given by natural transformations.

Corollary 1.4. *Let $\pi : X \to Y$ satisfy either hypotheses (1) or (2) of Theorem 1.3, and $\mathcal{H} = Q(X \times_Y X)$ denote the resulting convolution category.*

    (1) *As an object of $2Alg_k$, the convolution category $\mathcal{H}$ is equivalent to $Q(Y)$, and hence it is 2-dualizable and admits a canonical $O(2)$-fixed structure.*

    (2) *The two-dimensional unoriented topological field theory corresponding to $\mathcal{H}$ is equivalent to that corresponding to $Q(Y)$.*

    (3) *The Hochschild homology and cohomology of $\mathcal{H}$ are canonically identified as*

$$HH_*(\mathcal{H}) \simeq HH^*(\mathcal{H}) \simeq Q(\mathcal{L}Y).$$

Proof. Theorem 1.3 provides an equivalence between $\mathcal{H}$ and $Q(Y)$ as objects of $2Alg$, so the remaining assertions follow from the corresponding assertions for $Q(Y)$ itself. The 2-dualizability of $Q(Y)$ follows from the commutativity, or can be seen from the more general criterion for $SO(2)$-fixed monoidal categories of [BN]. In particular, the canonical $SO(2)$-invariant trace is given by the right adjoint to the unit morphism $\mathcal{F} \mapsto \mathrm{Hom}(1, \mathcal{F})$. $\square$

Now let G be an affine algebraic group, and $X = pt$ a point, and $Y = BG$ the classifying stack of G, so that then $G = pt \times_{BG} pt$.

Define the *quasicoherent group algebra* of G to be the ∞-category $Q(G)$ equipped with its monoidal structure coming from convolution along the group multiplication, or equivalently, the natural convolution product on $Q(pt \times_{BG} pt)$. Define a *quasicoherent G-category* to a $Q(G)$-module category.

As a specific instance of Gaitsgory's general results [G2], the classifying stack BG is *1-affine*: localization and global sections define inverse equivalences between $Q(BG)$-modules and quasicoherent stacks of categories over BG. As a consequence, taking $Q(G)$-invariants gives an equivalence from quasicoherent G-categories to $Q(BG)$-modules. This is a categorified analogue of the Koszul duality theorem of Goresky-Kottwitz-MacPherson [GKM] that provides a derived Morita equivalence (under certain auxiliary conditions) between the "topological group algebra" $C_*(G)$ and equivariant cochain complex $C^*(BG)$. Note this is in marked distinction with ordinary representation theory of a finite or reductive group, where passage to G-invariants is essentially never an equivalence.

The flat Morita equivalence (itself an application of [G2]) gives the following generalization. For an algebraic subgroup $K \subset G$, consider the double coset stack $K \backslash G / K = BK \times_{BG} BK$ and the resulting Hecke algebra $Q(K \backslash G / K)$. By Theorem 1.3, such algebras are all Morita equivalent to $Q(BG)$, and hence to each other. Tracing through the constructions, the Morita functor from quasicoherent G-categories to Hecke modules is given by passage to K-equivariant objects.

Corollary 1.5. *For affine algebraic groups $K \subset G$, there is a canonical equivalence*

$$(-)^K : \mathrm{Mod}(Q(G)) \xrightarrow{\;\sim\;} \mathrm{Mod}(Q(K \backslash G / K))$$

which sends a quasicoherent G-category $\mathcal{M}$ to the ∞-category of K-equivariant objects

$$\mathcal{M}^K \simeq \mathrm{Hom}_{Q(G)}(Q(G/K), \mathcal{M})$$

The special setting (a proper map between smooth stacks) of the proper Morita equivalence has an important instance: the Grothendieck-Springer simultaneous resolution

$$\mu : \widetilde{G}/G \longrightarrow G/G$$

of the adjoint quotient of a reductive group. The corresponding fiber product is the adjoint quotient of the Steinberg variety

$$\mathcal{S}t/G = \widetilde{G}/G \times_{G/G} \widetilde{G}/G \simeq (\widetilde{G} \times_G \widetilde{G})/G$$

Corollary 1.6. *There is a canonical equivalence*

$$\mathrm{Mod}(\mathrm{Q}(\mathcal{S}t/G)) \simeq \mathrm{Mod}(\mathrm{Q}(G/G))$$

of objects of $2Alg$ *The corresponding unoriented two-dimensional topological field theory valued in* $2Alg$ *assigns to the circle the stable* ∞*-category*

$$\mathrm{Q}(\mathcal{L}(G/G)) = \mathrm{Q}(BG^{T^2}) = \mathrm{Q}(\mathcal{L}oc_G(T^2))$$

of quasicoherent sheaves on the derived stack of G-local systems on the two-torus T^2.

Remark 1.7. A primary motivation for the consideration of $\mathrm{Q}(\mathcal{S}t/G)$ comes from its close relation to the affine Hecke category $\mathcal{H}^{aff}$ refining the affine Hecke algebra. Namely, a theorem of Bezrukavnikov [Be] describes $\mathcal{H}^{aff}$ (or its ind-complete version) in terms of G-equivariant coherent (or ind-coherent) sheaves on the Steinberg stack $\mathcal{S}t/G$.

From this perspective, the study of $\mathcal{H}^{aff}$ is subtle in part due to the failure of Morita equivalence: in fact, the natural action of $\mathcal{H}^{aff}$ on the standard module $\mathrm{Q}(\widetilde{G}/G)$ factors through the endofunctors $\mathrm{Q}(\mathcal{S}t/G)$. The Morita equivalence of the above corollary can be viewed as an incomplete categorification of the Kazhdan-Lusztig description of modules over the affine Hecke algebra. It successfully describes $\mathcal{H}^{aff}$-modules accessible via the standard module $\mathrm{Q}(\widetilde{G}/G)$.

A central problem going forward is to extend this picture to the full affine Hecke category $\mathcal{H}^{aff}$. For example, one might expect the Hochschild homology of $\mathcal{H}^{aff}$ to involve those ind-coherent sheaves with nilpotent singular support on the derived stack of G-local systems on the two-torus T^2 (as appears in the refined geometric Langlands conjecture of [AG]).

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2. Morita equivalence via Barr-Beck Theorem

Let $\pi : X \to Y$ be a morphism of perfect stacks.

We have the standard adjunction

$$\pi^* : \mathrm{Q}(Y) \longleftrightarrow \mathrm{Q}(X) : \pi_*$$

The pullback π^* is monoidal, and $\mathrm{Q}(X)$ is canonically self-dual as a $\mathrm{Q}(Y)$-module (as stated in Theorem 1.1 and proved in [BFN]).

Consider the induced adjunction

$$\Pi^* : \mathrm{Mod}(\mathrm{Q}(Y)) \longleftrightarrow \mathrm{Mod}(\mathrm{Q}(X)) : \Pi_*$$

where the pullback Π^* is given by

$$\Pi^*(-) = \mathrm{Q}(X) \otimes_{\mathrm{Q}(Y)} (-)$$

and the pushforward Π_* is the forgetful functor given by restricting $\mathrm{Q}(X)$-modules to $\mathrm{Q}(Y)$-modules along π^*.

By construction, the pullback Π^* preserves colimits, and since $\mathrm{Q}(X)$ is dualizable over $\mathrm{Q}(Y)$, it also preserves limits.

Now consider the convolution algebra

$$Q(X \times_Y X) \simeq \mathrm{End}_{Q(Y)}(Q(X))$$

Proposition 2.1. *Assume Π^* is conservative.*
Then the functor

$$Q(X) \otimes_{Q(Y)} (-) : \mathrm{Mod}(Q(Y)) \longrightarrow \mathrm{Mod}(Q(X \times_Y X))$$

is an equivalence.

Proof. Applying the comonadic form of Lurie's Barr-Beck Theorem to the adjunction

$$\Pi^* : \mathrm{Mod}(Q(Y)) \longleftrightarrow \mathrm{Mod}(Q(X)) : \Pi_*$$

we have an equivalence

$$Q(X) \otimes_{Q(Y)} (-) : \mathrm{Mod}(Q(Y)) \overset{\sim}{\longrightarrow} \mathrm{Comod}_{T^\vee}(\mathrm{Mod}(Q(X)))$$

where $T^\vee$ is the comonad of the adjunction with underlying endofunctor $\Pi^* \Pi_*$.
To calculate $T^\vee$, observe that it is represented by the coalgebra object

$$T^\vee(Q(X)) \simeq Q(X) \otimes_{Q(Y)} Q(X)$$

inside of $Q(X)$-bimodules. The fact that $Q(X)$ is dualizable over $Q(Y)$ implies that

$$Q(X) \otimes_{Q(Y)} Q(X) \simeq Q(X \times_Y X)$$

is dualizable as a $Q(X)$-bimodule. Moreover, the canonical self-duality of $Q(X)$ over $Q(Y)$ provides a canonical identification of the $Q(X)$-bimodule dual of this coalgebra with the algebra object

$$\mathrm{End}_{Q(Y)} Q(X) \simeq Q(X \times_Y X)$$

$\square$

Now we present two geometric contexts where the conservativity assumed in Proposition 2.1 can be readily verified.

2.1. **Flat Morita equivalence for perfect stacks.**

Proposition 2.2. *Suppose $\pi : X \to Y$ is faithfully flat. Then Π^* is conservative.*

Proof. Observe that Π^* factors as a composition of the localization functor from $Q(Y)$-modules to quasicoherent stacks on Y, pullback of quasicoherent stacks along π, and the global sections functor from quasicoherent stacks on X to $Q(X)$-modules.
By the 1-affineness of X and Y established in [G2], the first and third functors are equivalences. Finally, the pullback of quasicoherent stacks along π is conservative since the map π has a section locally in the flat topology whose pullback provides a left inverse to pullback along π. $\square$

2.2. **Proper Morita equivalence for smooth perfect stacks.** Now assume $\pi : X \to Y$ is a proper surjective morphism of perfect stacks.
We have the adjunction

$$\pi_* : Q^!(X) \longleftrightarrow Q^!(Y) : \pi^!$$

Consider the induced monad T with underlying endofunctor $\pi^! \pi_*$. By the proper descent of [P, G1], we have an equivalence

$$Q^!(Y) \simeq \mathrm{Mod}_T(Q^!(X))$$

Now assume in addition that X and Y are smooth so that we have equivalences

$$Q(X) \simeq Q^!(X) \qquad Q(Y) \simeq Q^!(Y)$$

Thus we have an equivalence

$$Q(Y) \simeq \mathrm{Mod}_T(Q(X))$$

where the monad T is represented by an algebra object in $\mathrm{End}_{Q(Y)}(Q(X))$.

Proposition 2.3. *Suppose X and Y are smooth and $\pi : X \to Y$ is proper and surjective. Then Π^* is conservative.*

Proof. We appeal to Proposition 3.1 below. Namely, we can produce a left inverse to the functor Π^* by passing to T-module objects

$$\mathrm{Mod}_T(\Pi^*(\mathcal{M})) \simeq \mathrm{Mod}_T(Q(X)) \otimes_{Q(Y)} \mathcal{M} \simeq \mathcal{M}$$

$\square$

3. Algebras and Module Categories

In this section, we describe some useful relations between categories of modules over algebras in a monoidal category $\mathcal{C}$ and $\mathcal{C}$-module categories. The results form a straightforward modification of [BFN, Proposition 4.1] from symmetric monoidal to general monoidal ∞-categories $\mathcal{C}$. The proofs are included for the convenience of the reader.

Proposition 3.1. *Let $\mathcal{C}$ be a stable presentable monoidal ∞-category, and $A \in \mathcal{C}$ an associative algebra object. For any $\mathcal{C}$-module $\mathcal{M}$, there is a canonical equivalence of ∞-categories*

$$_A\mathcal{C} \otimes_{\mathcal{C}} \mathcal{M} \simeq {}_A\mathcal{M}.$$

Proof. We will prove that $_A\mathcal{C} \otimes_{\mathcal{C}} \mathcal{M}$ is equivalent to $_A\mathcal{M}$ by the natural evaluation functor. Consider the adjunction

$$\mathcal{C} \xrightarrow{\ \ F\ \ } {}_A\mathcal{C} \quad \xleftarrow{\ \ G\ \ }$$

where $F(-) = A \otimes -$ is the induction, and G is the forgetful functor.

The above adjunction induces an adjunction

$$\mathcal{M} \xrightarrow{\ F \otimes \mathrm{id}\ } {}_A\mathcal{C} \otimes_{\mathcal{C}} \mathcal{M} \xleftarrow{\ G \otimes \mathrm{id}\ } \longrightarrow \mathrm{Mod}_T(\mathcal{M})$$

and thus a functor to modules over the monad $T = (G \otimes \mathrm{id}) \circ (F \otimes \mathrm{id})$ acting on $\mathcal{M}$. The functor underlying T is given by tensoring with A, so we also have an equivalence $\mathrm{Mod}_T(\mathcal{M}) \simeq {}_A\mathcal{M}$.

By its universal characterization, the functor $G' = G \otimes \mathrm{id}$ is colimit preserving. We will now check that G' is conservative. It will follow that G' satisfies the monadic Barr-Beck conditions, and we obtain the desired equivalence $_A\mathcal{C} \otimes_{\mathcal{C}} \mathcal{M} \simeq {}_A\mathcal{M}$.

Observe that for any $\mathcal{D}$, the pullback

$$\mathrm{Fun}^{\mathrm{L}}(_A\mathcal{C} \otimes_{\mathcal{C}} \mathcal{M}, \mathcal{D}) \to \mathrm{Fun}^{\mathrm{L}}(\mathcal{M}, \mathcal{D})$$

induced by the induction $F : \mathcal{M} \to {}_A\mathcal{C} \otimes_{\mathcal{C}} \mathcal{M}$ is conservative. In other words, if a functor out of $_A\mathcal{C} \times \mathcal{M}$ (which preserves colimits in each variable) is trivial when restricted to $\mathcal{M}$, then it is necessarily trivial.

Consequently, switching to opposite categories, we have that the corresponding functor

$$\mathrm{Fun}^{\mathrm{R}}(\mathcal{D}, {}_A\mathcal{C} \otimes \mathcal{M}) \to \mathrm{Fun}^{\mathrm{R}}(\mathcal{D}, \mathcal{M})$$

induced by the forgetful functor $G' : {}_A\mathcal{C} \otimes_{\mathcal{C}} \mathcal{M} \to \mathcal{M}$ is conservative. Now we can apply [BFN, Lemma 4.2], which asserts that we can check conservativity of a colimit preserving right adjoint G on right adjoint functors from test categories $\mathcal{D}$ (in fact the lemma applies in a setting that is linear over some stable presentable symmetric monoidal ∞-category). This concludes the proof.

$\square$

Proposition 3.2. *Let $\mathcal{C}$ be a stable presentable monoidal ∞-category.*

(1) *For $A, A' \in \mathcal{C}$ associative algebra objects, there is a canonical equivalence of ∞-categories*

$$_A\mathcal{C}_{A'} \simeq {_A\mathcal{C}} \otimes_{\mathcal{C}} \mathcal{C}_{A'}.$$

(2) *The ∞-category of modules $_A\mathcal{C}$ is dualizable as a $\mathcal{C}$-module with dual given by the ∞-category of modules $\mathcal{C}_A$ over the opposite algebra. In particular for A, A' algebra objects we have an equivalence between functors and bimodules $Hom_{\mathcal{C}}(\mathcal{C}_A, \mathcal{C}_{A'}) \simeq {_A\mathcal{C}_{A'}}.$*

Proof. We apply Proposition 3.1 to the instance where $\mathcal{M}$ is the ∞-category of left modules over another associative algebra A' to conclude that there is a natural equivalence $_A\mathcal{C} \otimes_{\mathcal{C}} \mathcal{C}_{A'} \simeq {_A(\mathcal{C}_{A'})}$. We now have a chain of adjunctions

$$\mathcal{C} \underset{G'}{\overset{F'}{\rightleftarrows}} \mathcal{C}_{A'} \underset{G''}{\overset{F''}{\rightleftarrows}} {_A(\mathcal{C}_{A'})}$$

in which the composite $G' \circ G''$ is colimit preserving and conservative, and hence satisfies the monadic Barr-Beck conditions.

Furthermore, the above adjunction naturally extends to a diagram in which the cycle of left adjoints (denoted by bowed arrows), and hence also the cycle of right adjoints (denoted by straight arrows), commute

Here $F(-) = A_1 \otimes A_2 \otimes -$ is the induction, G is the forgetful functor, f is the natural functor factoring through $_A\mathcal{C} \otimes_{\mathcal{C}} \mathcal{C}_{A'}$, and g is its right adjoint. From this diagram, we obtain a morphism of monads

$$G'G''F''F' \longrightarrow G'G''gfF''F' \simeq GF.$$

Now the underlying functors of the monads $GF(-)$ and $G'G''F''F'(-)$ are both equivalent to the tensor $A \otimes (-) \otimes A'$, so the above morphism of monads is an equivalence. Thus we obtain the promised equivalence $_A\mathcal{C} \otimes_{\mathcal{C}} \mathcal{C}_{A'} \simeq {_A(\mathcal{C}_{A'})} \simeq {_A\mathcal{C}_{A'}}$.

Finally, we show that the ∞-category of left A-modules $_A\mathcal{C}$ is a dualizable $\mathcal{C}$-module by directly exhibiting the ∞-category of right A-modules $\mathcal{C}_A$ as its dual. The trace map is given by the two-sided bar construction

$$\tau : \mathcal{C}_A \otimes_{\mathcal{C}} {_A\mathcal{C}} \to \mathcal{C} \qquad M, N \mapsto M \otimes_A N$$

The unit map is given by the induction

$$u : \mathrm{Mod}_k \to {_A\mathcal{C}} \otimes_{\mathcal{C}} \mathcal{C}_A \simeq {_A\mathcal{C}_A} \qquad V \mapsto A \otimes V$$

where we regard $A \otimes c$ as an A-bimodule.

One can verify directly that the composition

$$\mathrm{Mod}_A(\mathcal{C}) \xrightarrow{\mathrm{id} \otimes u} \mathrm{Mod}_A(\mathcal{C}) \otimes_{\mathcal{C}} \mathrm{Mod}_{A^{\mathrm{op}}}(\mathcal{C}) \otimes_{\mathcal{C}} \mathrm{Mod}_A(\mathcal{C}) \xrightarrow{\tau \otimes \mathrm{id}} \mathrm{Mod}_A(\mathcal{C})$$

is equivalent to the identity. First, $(\mathrm{id} \otimes u)(M)$ is equivalent to $A \otimes M$ regarded as an $A \otimes A^{\mathrm{op}} \otimes A$-module, and second, $(\tau \otimes \mathrm{id})(A \otimes M)$ is equivalent to $A \otimes_A M \simeq M$. $\qquad \square$

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