Equivariant incidence algebras and equivariant Kazhdan–Lusztig–Stanley theory

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Abstract. We establish a formalism for working with incidence algebras of posets with symmetries, and we develop equivariant Kazhdan–Lusztig–Stanley theory within this formalism. This gives a new way of thinking about the equivariant Kazhdan–Lusztig polynomial and equivariant $Z$-polynomial of a matroid.

1 Introduction

The incidence algebra of a locally finite poset was first introduced by Rota, and has proved to be a natural formalism for studying such notions as Möbius inversion [Rot64], generating functions [DRS72], and Kazhdan–Lusztig–Stanley polynomials [Sta92, Section 6].

A special class of Kazhdan–Lusztig–Stanley polynomials that have received a lot of attention recently is that of Kazhdan–Lusztig polynomials of matroids, where the relevant poset is the lattice of flats [EPW16, Pro18]. If a finite group $W$ acts on a matroid $M$ (and therefore on the lattice of flats), one can define the $W$-equivariant Kazhdan–Lusztig polynomial of $M$ [GPY17]. This is a polynomial whose coefficients are virtual representations of $W$, and has the property that taking dimensions recovers the ordinary Kazhdan–Lusztig polynomial of $M$. In the case of the uniform matroid of rank $d$ on $n$ elements, it is actually much easier to describe the $S_n$-equivariant Kazhdan–Lusztig polynomial, which admits a nice description in terms of partitions of $n$, than it is to describe the non-equivariant Kazhdan–Lusztig polynomial [GPY17, Theorem 3.1].

While the definition of Kazhdan–Lusztig–Stanley polynomials is greatly clarified by the language of incidence algebras, the definition of the equivariant Kazhdan–Lusztig polynomial of a matroid is completely ad hoc and not nearly as elegant. The purpose of this note is to define the equivariant incidence algebra of a poset with a finite group of symmetries, and to show that the basic constructions of Kazhdan–Lusztig–Stanley theory make sense in this more general setting. In the case of a matroid, we show that this approach recovers the same equivariant Kazhdan–Lusztig polynomials that were defined in [GPY17].

Acknowledgments: We thank Tom Braden for his feedback on a preliminary draft of this work.

2 The equivariant incidence algebra

Fix once and for all a field $k$. Let $P$ be a locally finite poset equipped with the action of a finite group $W$. We consider the category $C^W(P)$ whose objects consist of
• a $k$-vector space $V$

• a direct product decomposition $V = \prod_{x \leq y \in P} V_{xy}$, with each $V_{xy}$ finite dimensional

• an action of $W$ on $V$ compatible with the decomposition.

More concretely, for any $\sigma \in W$ and any $x \leq y \in P$, we have a linear map $\varphi_{xy}^\sigma : V_{xy} \to V_{\sigma(x)\sigma(y)}$, and we require that $\varphi_{xy}^\sigma \circ \varphi_{xy}^{\sigma'} = \varphi_{xy}^{\sigma' \sigma}$. Morphisms in $C^W(P)$ are defined to be linear maps that are compatible with both the decomposition and the action. This category admits a monoidal structure, with tensor product given by

$$(U \otimes V)_{xz} := \bigoplus_{x \leq y \leq z} U_{xy} \otimes V_{yz}.$$ 

Let $I^W(P)$ be the Grothendieck ring of $C^W(P)$; we call $I^W(P)$ the equivariant incidence algebra of $P$ with respect to the action of $W$.

**Example 2.1.** If $W$ is the trivial group, then $I^W(P)$ is isomorphic to the usual incidence algebra of $P$ with coefficients in $\mathbb{Z}$. That is, it is isomorphic as an abelian group to a direct product of copies of $\mathbb{Z}$, one for each interval in $P$, and multiplication is given by convolution.

**Remark 2.2.** If $W$ acts on $P$ and $\psi : W' \to W$ is a group homomorphism, then $\psi$ induces a functor $F_\psi : C^W(P) \to C^{W'}(P)$ and a ring homomorphism $R_\psi : I^W(P) \to I^{W'}(P)$.

We now give a second, more down to earth description of $I^W(P)$. Let $\text{VRep}(W)$ denote the ring of finite dimensional virtual representations of $W$ over the field $k$. A group homomorphism $\psi : W' \to W$ induces a ring homomorphism $\Lambda_\psi : \text{VRep}(W) \to \text{VRep}(W')$. For any $x \in P$, let $W_x \subset W$ be the stabilizer of $x$. We also define $W_{xy} := W_x \cap W_y$ and $W_{xyz} := W_x \cap W_y \cap W_z$. Note that, for any $x, y \in P$ and $\sigma \in W$, conjugation by $\sigma$ gives a group isomorphism

$$\psi_{xy}^\sigma : W_{xy} \to W_{\sigma(x)\sigma(y)},$$

which induces a ring isomorphism

$$\Lambda_{\psi_{xy}^\sigma} : \text{VRep}(W_{\sigma(x)\sigma(y)}) \to \text{VRep}(W_{xy}).$$

An element $f \in I^W(P)$ is uniquely determined by a collection

$$\{ f_{xy} \mid x \leq y \in P \},$$

where $f_{xy} \in \text{VRep}(W_{xy})$ and for any $\sigma \in W$ and $x \leq y \in P$, $f_{xy} = \Lambda_{\psi_{xy}^\sigma}(f_{\sigma(x)\sigma(y)})$. The unit $\delta \in I^W(P)$ is characterized by the property that $\delta_{xx}$ is the 1-dimensional trivial representation of $W_x$ for all $x \in P$ and $\delta_{xy} = 0$ for all $x < y \in P$. The following proposition describes the product structure on $I^W(P)$ in this representation.
Proposition 2.3. For any \( f, g \in I^W(P) \),

\[(fg)_{xz} := \sum_{x \leq y \leq z} \frac{|W_{xyz}|}{|W_{x}|} \text{Ind}^W_{W_{x}} \left( \left( \text{Res}^W_{W_{xy}} f_{xy} \right) \otimes \left( \text{Res}^W_{W_{yz}} g_{yz} \right) \right).\]

Remark 2.4. It may be surprising to see the fraction \( \frac{|W_{xyz}|}{|W_{xz}|} \) in the statement of Proposition 2.3, since VRep(\( W_{xy} \)) is not a vector space over the rational numbers. We could in fact replace the sum over \([x, z]\) with a sum over one representative of each \( W_{xz} \)-orbit in \([x, z]\) and then eliminate the factor of \( \frac{|W_{xyz}|}{|W_{xz}|} \). Including the fraction in the equation allows us to avoid choosing such representatives.

Remark 2.5. Proposition 2.3 could be taken as the definition of \( I^W(P) \). It is not so easy to prove associativity directly from this definition, though it can be done with the help of Mackey’s restriction formula (see for example [Bum13, Corollary 32.2]).

Remark 2.6. Suppose that \( \psi : W' \to W \) is a group homomorphism, and for any \( x, y \in P \), consider the induced group homomorphism \( \psi_{xy} : W_{xy} \to W_{xy} \). For any \( f \in I^W(P) \), we have, \( R_{\psi}(f)_{xy} = \Lambda_{\psi_{xy}}(f_{xy}) \). In particular, if \( W' \) is the trivial group, then \( R_{\psi}(f)_{xy} \) is equal to the dimension of the virtual representation \( f_{xy} \in \text{VRep}(W_{xy}) \).

Before proving Proposition 2.3, we state the following standard lemma in representation theory.

Lemma 2.7. Suppose that \( E = \bigoplus_{s \in S} E_s \) is a vector space that decomposes as a direct sum of pieces indexed by a finite set \( S \). Suppose that \( G \) acts linearly on \( E \) and acts by permutations on \( S \) such that, for all \( s \in S \) and \( \gamma \in G \), \( \gamma \cdot E_s = E_{\gamma s} \). For each \( x \in S \), let \( G_x \subset G \) denote the stabilizer of \( s \). Then there exists an isomorphism

\[ E \cong \bigoplus_{s \in S} \frac{|G_s|}{|G|} \text{Ind}^G_{G_s} \left( E_s \right) \]

of representations of \( G \).

Proof of Proposition 2.3. By linearity, it is sufficient to prove the proposition in the case where we have objects \( U \) and \( V \) of \( C^W(P) \) with \( f = [U] \) and \( g = [V] \). This means that, for all \( x \leq y \leq z \in P \), \( f_{xy} = [U_{xy}] \in \text{VRep}(W_{xy}) \), \( g_{yz} = [V_{yz}] \in \text{VRep}(W_{yz}) \), and

\[(fg)_{xz} = [(U \otimes V)_{xz}] = \left[ \bigoplus_{x \leq y \leq z} U_{xy} \otimes V_{yz} \right] \in \text{VRep}(W_{xz}).\]

The proposition then follows from Lemma 2.7 by taking \( E = (U \otimes V)_{xz} \), \( S = [x, z] \), and \( G = W_{xz} \).

Let \( R \) be a commutative ring. Given an element \( f \in I^W(P) \otimes R \) and a pair of elements \( x \leq y \in P \), we will write \( f_{xy} \) to denote the corresponding element of \( \text{VRep}(W_{xy}) \otimes R \).

\[1\] As in Remark 2.4, we may eliminate the fraction at the cost of choosing one representative of each \( W \)-orbit in \( S \).
Proposition 2.8. An element \( f \in I^W(P) \otimes R \) is (left or right) invertible if and only if \( f_{xx} \in \text{VRep}(W_x) \otimes R \) is invertible for all \( x \in P \). In this case, the left and right inverses are unique and they coincide.

Proof. By Proposition 2.3, an element \( g \) is a right inverse to \( f \) if and only if
\[
\sum_{x \leq y \leq z} \frac{|W_{xyz}|}{|W_{xz}|} \text{Ind}_{W_{xz}}^{W_{xyz}} ((\text{Res}_{W_{xy}}^{W_{xyz}} f_{xy}) \otimes (\text{Res}_{W_{xyz}}^{W_{xyz}} g_{yz})) = 0
\]
for all \( x < z \in P \). The second condition can be rewritten as
\[
\left( \text{Res}_{W_{xz}}^{W_{xx}} f_{xx} \right) \otimes g_{xz} = - \sum_{x < y \leq z} \frac{|W_{xyz}|}{|W_{xz}|} \text{Ind}_{W_{xz}}^{W_{xyz}} ((\text{Res}_{W_{xy}}^{W_{xyz}} f_{xy}) \otimes (\text{Res}_{W_{xyz}}^{W_{xyz}} g_{yz})).
\]

If \( f_{xx} \) is invertible in \( \text{VRep}(W_x) \otimes R \), then \( \text{Res}_{W_{xz}}^{W_{xx}} f_{xx} \) is invertible in \( \text{VRep}(W_{xz}) \otimes R \), and this equation has a unique solution for \( g \). Thus \( f \) has a right inverse if and only if \( f_{xx} \in \text{VRep}(W_x) \otimes R \) is invertible for all \( x \in P \). The argument for left inverses is identical, so it remains only to show that left and right inverses coincide.

Let \( g \) be right inverse to \( f \). Then \( g \) is also left inverse to some function, which we will denote \( h \). We then have
\[
f = f \delta = f(gh) = (fg)h = \delta h = h,
\]
so \( g \) is left inverse to \( f \), as well.

3 Equivariant Kazhdan–Lusztig–Stanley theory

In this section we take \( R \) to be the ring \( \mathbb{Z}[t] \) and for each \( f \in I^W(P) \otimes \mathbb{Z}[t] \) and \( x \leq y \in P \), we write \( f_{xy}(t) \) for the corresponding component of \( f \). One can regard \( f_{xy}(t) \) as a polynomial whose coefficients are virtual representations of \( W_{xy} \), or equivalently as a graded virtual representation of \( W_{xy} \). We assume that \( P \) is equipped with a \( W \)-invariant weak rank function in the sense of [Bre99, Section 2]. This is a collection of natural numbers \( \{r_{xy} \in \mathbb{N} \mid x \leq y \in P \} \) with the following properties:

- \( r_{xy} > 0 \) if \( x < y \)
- \( r_{xy} + r_{yz} = r_{xz} \) if \( x \leq y \leq z \)
- \( r_{xy} = r_{\sigma(x)\sigma(y)} \) if \( x \leq y \) and \( \sigma \in W \).

Following the notation of [Pro18, Section 2.1], we define
\[
\mathcal{R}^W(P) := \left\{ f \in I^W(P) \otimes \mathbb{Z}[t] \mid \text{deg} f_{xy}(t) \leq r_{xy} \text{ for all } x \leq y \right\}
\]

If the ring \( R \) has integer torsion, then we rewrite this equation without the fractions as described in Remark 2.4.
along with

\[ \mathcal{S}^W_{\frac{1}{2}}(P) := \left\{ f \in I^W(P) \otimes \mathbb{Z}[t] \mid \deg f_{xy}(t) < r_{xy}/2 \text{ for all } x < y \text{ and } f_{xx}(t) = \delta_{xx}(t) \text{ for all } x \right\}. \]

Note that \( \mathcal{S}^W(P) \) is a subalgebra of \( I^W(P) \), and we define an involution \( f \mapsto \bar{f} \) of \( \mathcal{S}^W(P) \) by putting \( \bar{f}_{xy}(t) := t^{r_{xy}} f_{xy}(t)^{-1} \). An element \( \kappa \in \mathcal{S}^W(P) \) is called a \textbf{P-kernel} if \( \kappa_{xx}(t) = \delta_{xx}(t) \) for all \( x \in P \) and \( \bar{\kappa} = \kappa^{-1} \).

**Theorem 3.1.** If \( \kappa \in \mathcal{S}^W(P) \) is a P-kernel, there exists a unique pair of functions \( f, g \in \mathcal{S}^W_{\frac{1}{2}}(P) \) such that \( \bar{f} = \kappa f \) and \( \bar{g} = g \kappa \).

**Proof.** We follow the proof in [Pro18, Theorem 2.2]. We will prove existence and uniqueness of \( f \); the proof for \( g \) is identical. Fix elements \( x < w \in P \). Suppose that \( f_{yw}(t) \) has been defined for all \( x < y \leq w \) and that the equation \( \bar{f} = \kappa f \) holds where defined. Let

\[
Q_{xw}(t) := \sum_{x < y \leq w} \frac{|W_{xyw}|}{|W_{xw}|} \text{Ind}_{W_{w_{zxy}}}^W \left( \left( \text{Res}_{W_{zxy}}^{W_{xy}} \kappa_{xy}(t) \right) \otimes \left( \text{Res}_{W_{zxy}}^{W_{yw}} f_{yw}(t) \right) \right) \in VRep(W_{xw}) \otimes \mathbb{Z}[t].
\]

The equation \( \bar{f} = \kappa f \) for the interval \( [x, w] \) translates to

\[
\bar{f}_{xw}(t) - f_{xw}(t) = Q_{xw}(t).
\]

It is clear that there is at most one polynomial \( f_{xw}(t) \) of degree strictly less than \( r_{xw}/2 \) satisfying this equation. The existence of such a polynomial is equivalent to the statement

\[
t^{r_{xw}} Q_{xw}(t^{-1}) = -Q_{xw}(t).
\]

To prove this, we observe that

\[
t^{r_{xw}} Q_{xw}(t^{-1}) = t^{r_{xw}} \sum_{x < y \leq w} \frac{|W_{xyw}|}{|W_{xw}|} \text{Ind}_{W_{w_{zxy}}}^W \left( \left( \text{Res}_{W_{zxy}}^{W_{xy}} t^{r_{xy}} \kappa_{xy}(t^{-1}) \right) \otimes \left( \text{Res}_{W_{zxy}}^{W_{yw}} t^{r_{yw}} f_{yw}(t^{-1}) \right) \right)
\]

\[
= \sum_{x < y \leq w} \frac{|W_{xyw}|}{|W_{xw}|} \text{Ind}_{W_{w_{zxy}}}^W \left( \left( \text{Res}_{W_{zxy}}^{W_{xy}} t^{r_{xy}} \kappa_{xy}(t^{-1}) \right) \otimes \left( \text{Res}_{W_{zxy}}^{W_{yw}} t^{r_{yw}} f_{yw}(t^{-1}) \right) \right)
\]

\[
= \sum_{x < y \leq w} \frac{|W_{xyw}|}{|W_{xw}|} \text{Ind}_{W_{w_{zxy}}}^W \left( \left( \text{Res}_{W_{zxy}}^{W_{xy}} \kappa_{xy}(t) \right) \otimes \left( \text{Res}_{W_{zxy}}^{W_{yw}} \bar{f}_{yw}(t) \right) \right)
\]

\[
= \sum_{x < y \leq w} \frac{|W_{xyw}|}{|W_{xw}|} \text{Ind}_{W_{w_{zxy}}}^W \left( \left( \text{Res}_{W_{zxy}}^{W_{xy}} \kappa_{xy}(t) \right) \otimes \left( \text{Res}_{W_{zxy}}^{W_{yw}} (\kappa f)_{yw}(t) \right) \right).
\]

This is formally equal to the expression for \( (\bar{\kappa}(\kappa f))_{xw} - (\kappa f)_{xw} \), which by associativity is equal to the expression for

\[
((\bar{\kappa}\kappa)f)_{xw} - (\kappa f)_{xw} = f_{xw} - (\kappa f)_{xw}.
\]
Thus we have
\[ t^{xw}Q_{xw}(t^{-1}) = -\sum_{x < y \leq w} \frac{|W_{xyw}|}{|W_{xw}|} \text{Ind}_{W_{xyw}}^{W_{xw}} \left( \left( \text{Res}_{W_{xyw}}^{W_{xy}} \kappa_{xy}(t) \right) \otimes \left( \text{Res}_{W_{xyw}}^{W_{yw}} f_{yw}(t) \right) \right) = -Q_{xw}(t). \]

Thus there is a unique choice of polynomial \( f_{xw}(t) \) consistent with the equation \( \bar{f} = \kappa f \) on the interval \( [x, w] \).

We will refer to the element \( f \in \mathcal{I}^W(P) \) from Theorem 3.1 is the \textbf{right equivariant KLS-function} associated with \( \kappa \), and to \( g \) as the \textbf{left equivariant KLS-function} associated with \( \kappa \). For any \( x \leq y \), we will refer to the graded virtual representations \( f_{xy}(t) \) and \( g_{xy}(t) \) as (right or left) \textbf{equivariant KLS-polynomials}. When \( W \) is the trivial group, these definitions specialize to the ones in [Pro18, Section 2].

**Example 3.2.** Let \( \zeta \in \mathcal{I}^W(P) \) be the element defined by letting \( \zeta_{xy}(t) \) be the trivial representation of \( W_{xy} \) in degree zero for all \( x \leq y \), and let \( \chi := \zeta^{-1} \bar{\zeta} \). The function \( \chi \) is called the \textbf{equivariant characteristic function} of \( P \) with respect to the action of \( W \). We have \( \chi^{-1} = \bar{\zeta}^{-1} \bar{\zeta} = \bar{\chi} \), so \( \chi \) is a \( P \)-kernel. Since \( \bar{\zeta} = \zeta \chi \), \( \zeta \) is equal to the left KLS-function associated with \( \chi \). However, the right KLS-function \( f \) associated with \( \chi \) is much more interesting! See Propositions 4.1 and 4.3 for a special case of this construction.

We next introduce the equivariant analogue of the material in [Pro18, Section 2.3]. If \( \kappa \) is a \( P \)-kernel with right and left KLS-functions \( f \) and \( g \), we define \( Z := g \kappa f \in \mathcal{I}^W(P) \), which we call the \textbf{equivariant Z-function} associated with \( \kappa \). For any \( x \leq y \), we will refer to the graded virtual representation \( Z_{xy}(t) \) as an \textbf{equivariant Z-polynomial}.

**Proposition 3.3.** We have \( \bar{Z} = Z \).

**Proof.** Since \( \bar{g} = g \kappa \), we have \( Z = g \kappa f = \bar{g} f \). Since \( \bar{f} = \kappa f \), we have \( Z = g \kappa f = g \bar{f} \). Thus \( \bar{Z} = \bar{g} \bar{f} = \bar{g} \bar{f} = g \bar{f} = Z \). \( \square \)

**Remark 3.4.** Suppose that \( \kappa \in I^W(P) \) is a \( P \)-kernel and \( f, g, Z \in I^W(P) \) are the associated equivariant KLS-functions and equivariant Z-function. It is immediate from the definitions that, if \( \psi : W' \to W \) is a group homomorphism, then \( R_\psi(f), R_\psi(g), R_\psi(Z) \in I^{W'}(P) \) are the equivariant KLS-functions and equivariant Z-function associated with the \( P \)-kernel \( R_\psi(\kappa) \in I^{W'}(P) \). In particular, if we take \( W' \) to be the trivial group, then Remark 2.6 tells us that the ordinary KLS-polynomials and Z-polynomials are recovered from the equivariant KLS-polynomials and Z-polynomials by sending virtual representations to their dimensions.

### 4 Matroids

Let \( M \) be a matroid, let \( L \) be the lattice of flats of \( M \) equipped with the usual weak rank function, and let \( W \) be a finite group acting on \( L \). Let \( OS_M^W(t) \) be the Orlik–Solomon algebra of \( M \) [OS80],
regarded as a graded representation of $W$. Following [G PY17 Section 2], we define

$$H^W_M(t) := t^{rk} M \cdot OS^W_M(-t^{-1}) \in VRep(W) \otimes \mathbb{Z}[t].$$

If $W$ is trivial, then $H^W_M(t) \in \mathbb{Z}[t]$ is equal to the characteristic polynomial of $M$. For any $F \leq G \subseteq L$, let $M_{FG}$ be the minor of $M$ with lattice of flats $[F, G]$ obtained by deleting the complement of $G$ and contracting $F$; this matroid inherits an action of the stabilizer group $W_{FG} \subset W$. Define $H \in \mathcal{A}^W(L)$ by putting $H_{FG}(t) = H^W_{M_{FG}}(t)$ for all $F \leq G$.

**Proposition 4.1.** The function $H$ is equal to the equivariant characteristic function of $L$.

*Proof.* It is proved in [G PY17 Lemma 2.5] that $\zeta H = \tilde{\zeta}$. Multiplying on the left by $\zeta^{-1}$, we have $H = \zeta^{-1} \tilde{\zeta}$, which is the definition of the equivariant characteristic function of $L$. 

**Remark 4.2.** The proof of [G PY17 Lemma 2.5] is surprisingly difficult. Consequently, Proposition 4.1 is a deep fact about Orlik–Solomon algebras, not just a formal consequence of the definitions.

The **equivariant Kazhdan–Lusztig polynomial** $P^W_M(t) \in VRep(W) \otimes \mathbb{Z}[t]$ was introduced in [GPY17 Section 2.2]. Define $P \in \mathcal{A}^W(L)$ by putting $P_{FG}(t) = P^W_{M_{FG}}(t)$ for all $F \leq G$. The defining recursion for $P^W_M(t)$ in [GPY17 Theorem 2.8] translates to the formula $\hat{P} = H P$, which immediately implies the following proposition.

**Proposition 4.3.** The function $P$ is the right equivariant KLS-function associated with $H$.

The **equivariant $Z$-polynomial** $Z^W_M(t) \in VRep(W) \otimes \mathbb{Z}[t]$ was introduced in [PXY18 Section 6]. Define $Z \in \mathcal{A}^W(L)$ by putting $Z_{FG}(t) = Z^W_{M_{FG}}(t)$ for all $F \leq G$. The defining recursion for $Z^W_M(t)$ in [PXY18 Section 6] translates to the formula $Z = \tilde{\zeta} P$.

**Proposition 4.4.** The function $Z$ is the $Z$-function associated with $H$.

*Proof.* Example 3.2 tells us that the right KLS-function associated with $H$ is $\zeta$ and Proposition 4.3 tells us that the left KLS-function associated with $H$ is $P$, thus the $Z$-function is equal $\zeta HP = \tilde{\zeta} P = Z$. 

The following corollary was asserted without proof in [PXY18 Section 6], and follows immediately from Propositions 3.3 and 4.4.

**Corollary 4.5.** The polynomial $Z^W_M(t)$ is palindromic. That is, $t^{rk}MZ^W_M(t^{-1}) = Z^W_M(t)$.

When $W$ is the trivial group, Gao and Xie define polynomials $Q_M(t)$ and $\hat{Q}_M(t) = (-1)^{rk} M Q_M(t)$ with the property that $(P^{-1})_{FG}(t) = \hat{Q}_{M_{FG}}(t)$ [GX20]. If $\hat{0}$ and $\hat{1}$ are the minimal and maximal flats of $M$, this is equivalent to the statement that $Q_M(t) = (-1)^{rk} M (P^{-1})_{\hat{0} \hat{1}}(t)$. The polynomial $Q_M(t)$ is called the **inverse Kazhdan–Lusztig polynomial** of $M$ [P] Using the machinery of

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3 The difficult part appears in the proof of Lemma 2.4, which is then used to prove Lemma 2.5.
4 The reason for bestowing this name on $Q_M(t)$ rather than $\hat{Q}_M(t)$ is that $Q_M(t)$ has non-negative coefficients; this was conjectured in [GX20 Conjecture 4.1] and proved in [BHM].
this paper, we may extend their definition to the equivariant setting by defining the **equivariant inverse Kazhdan–Lusztig polynomial**

\[ Q^W_M(t) := (-1)^{rk M} (P^{-1})_{\hat{0}\hat{1}} (t). \]

If we then define \( \hat{Q} \in J_{1/2}^W(L) \) by putting \( \hat{Q}_{FG}(t) = (-1)^{r_{FG}} Q^W_{M_{FG}}(t) \) for all \( F \leq G \), we immediately obtain the following proposition.

**Proposition 4.6.** The functions \( P \) and \( \hat{Q} \) are mutual inverses in \( I^W(L) \).

**References**


