Equivariant topology of real hyperplane arrangements

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Abstract. Given a real hyperplane arrangement $\mathcal{A}$, the complement $\mathcal{M}(\mathcal{A})$ of the complexification of $\mathcal{A}$ admits an action of $\mathbb{Z}_2$ by complex conjugation. In this note we survey the results of [P1] and [P2], in which the equivariant cohomology and KO-theory of $\mathcal{M}(\mathcal{A})$ are described combinatorially.

1 Introduction

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an arrangement of n hyperplanes in $\mathbb{C}^d$, with $H_i = \omega_i^{-1}(0)$ for some affine linear map $\omega_i : \mathbb{C}^d \to \mathbb{C}$. Let $\mathcal{M}(\mathcal{A})$ denote the complement of $\mathcal{A}$ in $\mathbb{C}^d$. It is a fundamental problem in the study of hyperplane arrangements to determine the extent to which the topology of $\mathcal{M}(\mathcal{A})$ is controlled by the combinatorics of $\mathcal{A}$, by which we mean its *pointed matroid.* Geometrically, the pointed matroid encodes two types of data:

1. which subsets $S \subseteq \{1, \ldots, n\}$ have the property that $\bigcap_{i\in S} H_i = \emptyset$, and
2. which subsets $S \subseteq \{1, \ldots, n\}$ have the property that $\operatorname{codim} \bigcap_{i\in S} H_i < |S|$.

The first major success of this program, due to Orlik and Solomon, is a combinatorial presentation of the cohomology ring of $\mathcal{M}(\mathcal{A})$.

Definition 1.1 The *Orlik-Solomon algebra* $A(\mathcal{A}; R)$ is the cohomology ring $H^*(\mathcal{M}(\mathcal{A}); R)$ of the complement of the complexified arrangement with coefficients in the ring R.

For each $i \leq n$, let $e_i = \omega_i^*[\mathbb{R}^+] \in A(\mathcal{A}; R)$ be the pullback of the generator $[\mathbb{R}^+] \in H^1(\mathbb{C}^*; R)$ under the map $\omega_i : \mathcal{M}(\mathcal{A}) \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. The following theorem, due to Orlik and Solomon, states that the elements $e_1, \ldots, e_n$ generate $A(\mathcal{A}; R)$, and gives explicit relations in terms of the pointed matroid of $\mathcal{A}$. We give here a simplified version by working only with the coefficient ring $R = \mathbb{Z}_2$, because this is the version that will extend well to the equivariant setting.

Theorem 1.2 [OS] *Consider the linear map $\partial = \sum_{i=1}^n \frac{\partial}{\partial e_i}$ from $\mathbb{Z}_2[e_1, \ldots, e_n]$ to itself, lowering degree by 1. The Orlik-Solomon algebra $A(\mathcal{A}; \mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2[e_1, \ldots, e_n]/\mathcal{I}$, where $\mathcal{I}$ is generated by the following three families of relations:*

$$1) \quad e_i^2 \quad \textit{for} \quad i \in \{1, \ldots, n\}$$

$$2) \quad \prod_{i\in S} e_i \quad \textit{if} \quad \bigcap_{i\in S} H_i = \emptyset$$

$$3) \quad \partial \prod_{i\in S} e_i \quad \textit{if} \quad \bigcap_{i\in S} H_i \textit{ is nonempty with codimension less than } |S|.$$

Now suppose that our arrangement $\mathcal{A}$ is the complexification of a real hyperplane arrangement, i.e. that ω_i restricts to a map $\omega_i : \mathbb{R}^d \to \mathbb{R}$ for all i. This allows us to define a richer combinatorial object called the *pointed oriented matroid* of $\mathcal{A}$. Let

$$H_i^+ = \{p \in \mathbb{R}^d \mid \omega_i(p) \geq 0\} \quad \text{and} \quad H_i^- = \{p \in \mathbb{R}^d \mid \omega_i(p) \leq 0\},$$

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both half-spaces in $\mathbb{R}^d$ with boundary H_i. Like the pointed matroid, the pointed oriented matroid also encodes two types of geometrical data:

1. which pairs of subsets $S^+, S^- \subseteq \{1, \ldots, n\}$ have the property that $\bigcap_{i \in S^+} H_i^+ \cap \bigcap_{j \in S^-} H_j^- = \emptyset$, and
2. which pairs of subsets $S^+, S^- \subseteq \{1, \ldots, n\}$ have the property that $\bigcap_{i \in S^+} H_i^+ \cap \bigcap_{j \in S^-} H_j^-$ is nonempty and contained in some hyperplane.

In addition to enhancing our notion of the combinatorics of a complexified hyperplane arrangement, we may also enhance our notion of the topology of its complement. The space $\mathcal{M}(\mathcal{A})$ is now equipped with an action of the group $\mathbb{Z}_2 = Gal(\mathbb{C}/\mathbb{R})$, given by complex conjugation. It is therefore natural to make the following definition.

Definition 1.3 The *equivariant Orlik-Solomon algebra* $\mathcal{A}_2(\mathcal{A}, \mathbb{Z}_2)$ of a complexified hyperplane arrangement is the equivariant cohomology ring $H^*_{\mathbb{Z}_2}(\mathcal{M}(\mathcal{A}); \mathbb{Z}_2)$.

The purpose of this note is to announce the results of [P1] and [P2], in which we describe the equivariant cohomology and *KO*-rings of $\mathcal{M}(\mathcal{A})$ in terms of the pointed oriented matroid. Along the way we will interpret $\mathcal{A}_2(\mathcal{A}, \mathbb{Z}_2)$ as a deformation from the ordinary Orlik-Solomon algebra $A(\mathcal{A}; \mathbb{Z}_2)$ to the Varchenko-Gelfand ring $VG(\mathcal{A}; \mathbb{Z}_2)$ of locally constant functions from the real locus of $\mathcal{M}(\mathcal{A})$ to $\mathbb{Z}_2$, thus proving topologically the well-known fact that the dimension of the Orlik-Solomon algebra is equal to the number of components of the complement of the real arrangement.

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2 Equivariant cohomology and K-theory

In this section we review some basic definitions and results in equivariant algebraic topology. Let X be a topological space equipped with an action of a group G.

Definition 2.1 Let EG be a contractible space with a free G-action. Then we put

$$X_G := X \times_G EG = (X \times EG)/G$$

(well-defined up to homotopy equivalence), and define the G-equivariant cohomology of X

$$H^*_G(X) := H^*(X_G).$$

The G-equivariant map from X to a point induces a map on cohomology in the other direction, hence $H^*_G(X)$ is a module over $H^*_G(pt) \cong H^*(BG)$, where $BG = EG/G$ is the classifying space for G. Indeed, H^*_G is a contravariant functor from the category of G-spaces to the category of $H^*_G(pt)$-modules.

Example 2.2 If $G = \mathbb{Z}_2$, then we may take $EG = S^\infty$ and $BG = S^\infty/\mathbb{Z}_2 = \mathbb{R}P^\infty$. Then $H^*_{\mathbb{Z}_2}(pt; \mathbb{Z}_2) = H^*(\mathbb{R}P^\infty; \mathbb{Z}_2) \cong \mathbb{Z}_2[x]$.

The following theorem is a consequence of [Bo, IV.3.7(b) and XII.3.5]; it says that we may interpret $H^*_{\mathbb{Z}_2}(X; \mathbb{Z}_2)$ as a deformation of $H^*(X; \mathbb{Z}_2)$ into $H^*(F; \mathbb{Z}_2)$ over the $\mathbb{Z}_2$ affine line.

Theorem 2.3 *Suppose that $F = X^{\mathbb{Z}_2}$ is nonempty, the induced action of $\mathbb{Z}_2$ on $H^*(X; \mathbb{Z}_2)$ is trivial, and $H^*(X; \mathbb{Z}_2)$ is generated in degree 1. Then $H^*_{\mathbb{Z}_2}(X; \mathbb{Z}_2)$ is a free module over $\mathbb{Z}_2[x]$, and we have*

$$H^*(X; \mathbb{Z}_2) \cong H^*_{\mathbb{Z}_2}(X; \mathbb{Z}_2)/\langle x \rangle$$

and

$$H^*(F;\mathbb{Z}_2) \cong H^*_{\mathbb{Z}_2}(X;\mathbb{Z}_2)/\langle x-1\rangle.$$

Remark 2.4 If $X = \mathcal{M}(\mathcal{A})$ for some complexified hyperplane arrangement $\mathcal{A}$, then $F = \mathcal{M}_{\mathbb{R}}(\mathcal{A})$ is equal to the complement in $\mathbb{R}^d$ of the real parts of the hyperplanes. In this case, Theorem 2.3 says that the equivariant Orlik-Solomon algebra of $\mathcal{A}$ is a deformation of the ordinary Orlik-Solomon algebra with coefficients in $\mathbb{Z}_2$ into the Varchenko-Gelfand ring

$$VG(\mathcal{A};\mathbb{Z}_2) := \operatorname{Maps}\big(\mathcal{M}_{\mathbb{R}}(\mathcal{A});\mathbb{Z}_2\big) = H^*(\mathcal{M}_{\mathbb{R}}(\mathcal{A});\mathbb{Z}_2).$$

In the following example we take X to be $\mathbb{C}^*$, the simplest instance of the complement of a hyperplane arrangement.

Example 2.5 Let $X = \mathbb{C}^*$, with $\mathbb{Z}_2$ acting by complex conjugation. Since X deformation-retracts equivariantly onto the compact space S^1, Theorem 2.3 applies. The image of x under the standard map $\mathbb{Z}_2[x] = H^*_{\mathbb{Z}_2}(pt,\mathbb{Z}_2) \to H^*_{\mathbb{Z}_2}(X;\mathbb{Z}_2)$ is the $\mathbb{Z}_2$-equivariant Euler class of the topologically trivial real line bundle with a nontrivial $\mathbb{Z}_2$ action. This bundle has a $\mathbb{Z}_2$-equivariant section, transverse to the zero section, vanishing exactly on the real points of X, and is therefore represented by the submanifold $\mathbb{R}^* \subseteq \mathbb{C}^*$. Let $e = [\mathbb{R}^+] \in H^*_{\mathbb{Z}_2}(X;\mathbb{Z}_2)$. Then $x - e$ is represented by $\mathbb{R}^-$, therefore $e(x-e) = 0$. Theorem 2.3 tells us that we have found all of the generators and relations.

The equivariant *KO*-ring of a G-space X is easier to define than the equivariant cohomology, because it does not require passage to the Borel space.

Definition 2.6 The equivariant *KO*-ring $KO_G(X)$ is the Grothendieck ring of G-equivariant real vector bundles on X.

Let $\mathcal{A}$ be a complexified arrangement. This ring $KO_{\mathbb{Z}_2}\big(\mathcal{M}(\mathcal{A})\big)$ has the advantage over the equivariant Orlik-Solomon algebra that it is well behaved even with coefficients in the integers, rather than $\mathbb{Z}_2$. It is, however, much more difficult to calculate. For this reason, we consider the subring $Line(\mathcal{A}) \subseteq KO_{\mathbb{Z}_2}\big(\mathcal{M}(\mathcal{A})\big)$ generated by the classes of line bundles, which we will compute in Theorem 3.5.

3 The results

A celebrated theorem of Salvetti [Sa] states that if $\mathcal{A}$ is a complex hyperplane arrangement defined over the real numbers, then the homotopy type of $\mathcal{M}(\mathcal{A})$ is determined by the pointed oriented matroid of $\mathcal{A}$. More precisely, one may use the pointed oriented matroid to construct a poset $\operatorname{Sal}(\mathcal{A})$, and the order complex of this poset is homotopy equivalent to $\mathcal{M}(\mathcal{A})$. Our first result is an extension of this theorem to the equivariant setting.

Theorem 3.1 [P1, 4.1] *The poset* $\operatorname{Sal}(\mathcal{A})$ *admits a combinatorially defined* $\mathbb{Z}_2$ *action, such that its order complex is* $\mathbb{Z}_2$*-equivariantly homotopy equivalent to* $\mathcal{M}(\mathcal{A})$.

Remark 3.2 Theorem 3.1 provides an explanation for the recent discovery of Huisman that the equivariant fundamental group of a line arrangement is determined by its pointed oriented matroid [Hu].

Theorem 3.1 tells us in particular that the rings $A_2(\mathcal{A};\mathbb{Z}_2)$ and $Line(\mathcal{A})$ are combinatorially determined. They may be explicitly described as follows.[2]

[2] A special case of this presentation first appeared in [HP, 5.5], using the geometry of hypertoric varieties.

Theorem 3.3 [P1, 3.1] *The equivariant Orlik-Solomon algebra* $A_2(\mathcal{A};\mathbb{Z}_2) = H^*_{\mathbb{Z}_2}\big(\mathcal{M}(\mathcal{A});\mathbb{Z}_2\big)$ *is isomorphic to* $\mathbb{Z}_2[e_1,\ldots,e_n,x]/\mathcal{J}$*, where* $\mathcal{J}$ *is generated by the following three families of relations:*[3]

$$1)\quad e_i(x-e_i) \quad \textit{for} \quad i \in \{1,\ldots,n\}$$

$$2)\quad \prod_{i\in S^+} e_i \times \prod_{j\in S^-}(x-e_j) \quad \textit{if} \quad \bigcap_{i\in S^+} H_i^+ \cap \bigcap_{j\in S^-} H_j^- = \emptyset$$

$$3)\quad x^{-1}\left(\prod_{i\in S^+} e_i \times \prod_{j\in S^-}(x-e_j) - \prod_{i\in S^+}(x-e_i) \times \prod_{j\in S^-} e_j\right)$$
$$\textit{if} \quad \bigcap_{i\in S^+} H_i^+ \cap \bigcap_{j\in S^-} H_j^- \textit{ is nonempty and contained in some hyperplane } H_k.$$

Remark 3.4 By setting $x = 0$, we obtain the presentation of $A(\mathcal{A};\mathbb{Z}_2)$ given in Theorem 1.2. By setting $x = 1$, we recover the interesting presentation of the boring ring $VG(\mathcal{A};\mathbb{Z}_2)$ studied in [VG]. In particular, we explain topologically the fact, observed in [VG], that $VG(\mathcal{A};\mathbb{Z}_2)$ admits a filtration with associated graded $A(\mathcal{A};\mathbb{Z}_2)$.

Theorem 3.5 [P2, 3.1] *The subring* $Line(\mathcal{A}) \subseteq KO_{\mathbb{Z}_2}\big(\mathcal{M}(\mathcal{A})\big)$ *generated by line bundles is isomorphic to* $\mathbb{Z}[e_1,\ldots,e_n,x]/\mathcal{J}$*, where* $\mathcal{J}$ *is generated by the following five families of relations:*

$$1)\quad x^2 - 2x$$

$$2)\quad e_i^2 - 2e_i \quad \textit{for} \quad i \in \{1,\ldots,n\}$$

$$3)\quad e_i(x-e_i) \quad \textit{for} \quad i \in \{1,\ldots,n\}$$

$$4)\quad \prod_{i\in S^+} e_i \times \prod_{j\in S^-}(x-e_j) \quad \textit{if} \quad \bigcap_{i\in S^+} H_i^+ \cap \bigcap_{j\in S^-} H_j^- = \emptyset$$

$$5)\quad x^{-1}\left(\prod_{i\in S^+} e_i \times \prod_{j\in S^-}(x-e_j) - \prod_{i\in S^+}(x-e_i) \times \prod_{j\in S^-} e_j\right)$$
$$\textit{if} \quad \bigcap_{i\in S^+} H_i^+ \cap \bigcap_{j\in S^-} H_j^- \textit{ is nonempty and contained in some hyperplane.}$$

Remark 3.6 Note that $A_2(\mathcal{A};\mathbb{Z}_2)$ is almost identical to the associated graded of $Line(\mathcal{A})$; only the degree zero parts are different.

We conclude with an example of two arrangements $\mathcal{A}$ and $\mathcal{A}'$ such that $M(\mathcal{A})$ is homotopy equivalent to $M(\mathcal{A}')$, but not equivariantly. We demonstrate this fact by showing that their equivariant Orlik-Solomon algebras are not isomorphic.

Example 3.7 Consider the two line arrangements shown in Figure 1.[4] These two arrangements are related by a flip (parallel translation of a hyperplane), hence they have homotopy equivalent complements [Fa]. We

[3] *Note that all of these relations are polynomial; the* x^{-1} *in the third relation cancels.*
[4] This example appeared first in [HP].

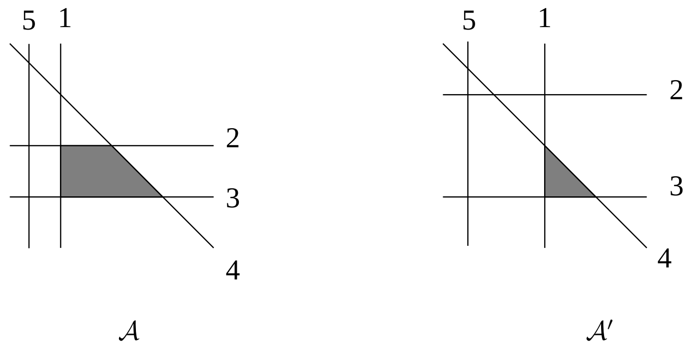

Figure 1: Two arrangements whose complements are homotopy equivalent only nonequivariantly.

have

$$A_2(\mathcal{A};\mathbb{Z}_2) \cong \mathbb{Z}_2[\vec{e},x] \Big/ \left\langle \begin{array}{c} e_1(x-e_1), e_2(x-e_2), e_3(x-e_3), e_4(x-e_4), \\ e_5(x-e_5), e_2e_3, (x-e_1)e_5, e_1(x-e_2)e_4, \\ e_1e_3e_4, (x-e_2)e_4e_5, e_3e_4e_5 \end{array} \right\rangle$$

and

$$A_2(\mathcal{A}';\mathbb{Z}_2) \cong \mathbb{Z}_2[\vec{e},x] \Big/ \left\langle \begin{array}{c} e_1(x-e_1), e_2(x-e_2), e_3(x-e_3), e_4(x-e_4), \\ e_5(x-e_5), e_2e_3, (x-e_1)e_5, (x-e_1)e_2(x-e_4), \\ e_1e_3e_4, (x-e_2)e_4e_5, e_3e_4e_5 \end{array} \right\rangle.$$

Using Macaulay 2 [M2], we find that the annihilator of the element $e_2 \in A_2(\mathcal{A};\mathbb{Z}_2)$ is generated by two linear elements (namely e_3 and $x-e_2$) and nothing else, while none of the (finitely many) elements of $A_2(\mathcal{A}';\mathbb{Z}_2)$ has this property. Hence the two rings are not isomorphic, and $\mathcal{M}(\mathcal{A})$ is not equivariantly homotopy equivalent to $\mathcal{M}(\mathcal{A}')$. From this example we conclude that the equivariant Orlik-Solomon algebra of an arrangement is *not* determined by the pointed *un*oriented matroid.

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