



Convergence of integrals on moduli spaces of curves and cographical matroids*

Alexander Polishchuk and Nicholas Proudfoot

Abstract. We determine the convergence regions of certain local integrals on the moduli spaces of curves in neighborhoods of fixed stable curves in terms of the combinatorics of the corresponding graphs.

1 Introduction

Let $\overline{\mathcal{M}}_g$ denote the moduli space of stable curves of genus g , which is a compactification of the moduli space of smooth curves constructed by Deligne and Mumford in [4]. We denote by $\Delta^{ns} \subset \overline{\mathcal{M}}_g$ denote the non-separating node boundary divisor, i.e., the closure of the locus corresponding to irreducible curves with one node.

One records the combinatorial type of a connected nodal curve C by the corresponding dual marked graph Γ : vertices of Γ correspond to irreducible components of C , marked by the genus of the corresponding component, and edges correspond to nodes. Note that multiple edges and loops are both allowed. Stability of C corresponds to the following notion of stability for Γ (see [1, Ch. X, Lem. (3.1)]): for every vertex v one should have $2g_v - 2 + l_v > 0$, where g_v is the marking of the vertex and l_v is the number of half-edges emanating from v . The arithmetic genus of C is equal to the genus of Γ computed as $b(\Gamma) + \sum_v g_v$, where $b(\Gamma)$ is the first Betti number of Γ . We denote by $\mathcal{M}_\Gamma \subset \overline{\mathcal{M}}_g$ the stratum consisting of curves with a given dual stable graph Γ .

Let φ be a top holomorphic form on $\overline{\mathcal{M}}_g$ defined in a neighborhood in $\overline{\mathcal{M}}_g$ of a point $C_0 \in \mathcal{M}_\Gamma$, with a pole of order 1 along Δ^{ns} . We are interested in the region of convergence in $s \in \mathbb{C}$ of the integral

$$I_\Gamma(\varphi, s) = \int_{B_\Gamma} \frac{\varphi \wedge \overline{\varphi}}{|\det(\tau - \overline{\tau})|^s},$$

where B_Γ is a sufficiently small ball in an étale neighborhood of C_0 in $\overline{\mathcal{M}}_g$, and τ is the period matrix.

Recall that the period matrix depends on a local choice of a symplectic basis of $H_1(C)$ (see Equation (2.1)). This choice changes the function $I_\Gamma(\varphi, s)$, but Lemma 2.2 implies that it does not change its region of convergence. Globally, $\det(\tau - \overline{\tau})$ is related to the

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natural hermitian form on the determinant line bundle $\det H^0(C, \omega_C)$. The latter line bundle shows up in Mumford's calculation of the canonical line bundle on \mathcal{M}_g and its superanalog (see [5]). For this reason, the integrals of the form $I_\Gamma(\varphi, s)$ (for $s = 5$) model local behavior of integrals calculating vacuum amplitudes in superstring theory after integrating out the odd variables and using the GSO projection (see [7]).

The point is that $\det(\tau - \bar{\tau})$ has a logarithmic growth near Δ^{ns} that offsets the poles of $\varphi \wedge \bar{\varphi}$ for sufficiently large s . The precise region of convergence of $I_\Gamma(\varphi, s)$ depends on a graph Γ .

For example, if Γ has a single vertex of genus $g - 1$ with a loop, i.e., we are at the generic point of Δ^{ns} , then $\det(\tau - \bar{\tau}) = u \cdot \log |t|$, where u is invertible and $t = 0$ is a local equation of Δ^{ns} . So the convergence of our integral for such Γ is the same as for $\frac{dt \wedge d\bar{t}}{t \cdot \bar{t} \cdot (\log |t|)^s}$. Thus, in this case the integral absolutely converges for $\text{Re}(s) > 1$ and diverges for $s = 1$.

Our main result, Theorem A below, determines the convergence threshold for each Γ . In particular, it shows that for each genus ≥ 6 , there exists a stable graph Γ such that $I_\Gamma(\varphi, 5)$ diverges. This means that the definition of superstring vacuum amplitudes for $g \geq 6$ requires some additional regularization procedure at the non-separating node boundary divisor in addition to the GSO projection.

Using the asymptotics of τ near C_0 (controlled by the monodromy around branches of Δ^{ns}), in the case when C_0 has rational components, we relate the integral I_Γ to another integral defined in terms of the graph Γ .

Let $E(\Gamma)$ denote the set of edges of a connected graph Γ . We introduce independent variables x_e associated $e \in E(\Gamma)$. Let $b = b(\Gamma)$ denote the 1st Betti number of Γ (which is equal to g , the arithmetic genus of C_0), and let c_1, \dots, c_b be a collection of simple cycles in Γ giving a basis in $H_1(\Gamma)$. For each edge $e \in E(\Gamma)$, we define the index

$$(c_i, c_j)_e = \begin{cases} 1, & c_i \text{ and } c_j \text{ pass through } e \text{ in the same direction,} \\ -1, & c_i \text{ and } c_j \text{ pass through } e \text{ in the opposite directions,} \\ 0, & \text{otherwise,} \end{cases}$$

and consider the $b \times b$ -matrix $A = (a_{ij})$ with

$$a_{ij} = \sum_{e \in E(\Gamma)} (c_i, c_j)_e \cdot x_e. \quad (1.1)$$

Let $\psi_\Gamma \in \mathbb{Z}[x_e]_{e \in E(\Gamma)}$ denote the Kirchhoff polynomial (aka the first Symanzik polynomial) of Γ (see [3, Sec. 2]), defined as the determinant

$$\psi_\Gamma = \det(a_{ij}).$$

This polynomial also has an expansion

$$\psi_\Gamma = \sum_T \prod_{e \notin T} x_e, \quad (1.2)$$

where T runs over all spanning trees of Γ (see [3, Prop. 3.4] and Prop. 2.1 below). Note that ψ_Γ is homogeneous of degree $b(\Gamma)$.

Let $E'(\Gamma) \subset E(\Gamma)$ denote the set of edges which are not bridges. We consider the integral

$$J_\Gamma(s) := \int_B \frac{\prod_{e \in E(\Gamma)} dz_e d\bar{z}_e}{|\psi_\Gamma(\ln |z_\bullet|)|^s \cdot \prod_{e \in E'(\Gamma)} z_e \bar{z}_e},$$

where B is a small ball around 0 in $\mathbb{C}^{E(\Gamma)}$, (z_e) are complex coordinates on $\mathbb{C}^{E(\Gamma)}$.

Given a connected graph Γ without bridges (not necessarily stable), we define a rational constant $c(\Gamma) > 0$ as follows. Consider the vector space $\mathbb{R}^{E(\Gamma)}$ with the basis corresponding to edges of Γ , and let $v \in \mathbb{R}^{E(\Gamma)}$ denote the sum of all basis vectors. For each spanning tree T , consider the vector

$$v_T := \sum_{e \in T} e \in \mathbb{R}^{E(\Gamma)}.$$

Now we set

$$c(\Gamma) = \inf \left\{ \sum_T c_T \mid \sum_T c_T v_T \geq v, c_T \in \mathbb{R}_{\geq 0} \right\},$$

where $w \geq v$ means that $w - v \in \mathbb{R}_{\geq 0}^{E(\Gamma)}$.

It is easy to see that

$$c(\Gamma) \geq \frac{e(\Gamma)}{b(\Gamma)}, \quad (1.3)$$

where $e(\Gamma) = |E(\Gamma)|$ is the number of edges and $b(\Gamma)$ is the 1st Betti number of Γ . Indeed, let $\varphi : \mathbb{R}^{E(\Gamma)} \rightarrow \mathbb{R}$ denote the map given by the sum of all coordinates. Then $\varphi(v) = e(\Gamma)$, $\varphi(v_T) = b(\Gamma)$, so the inequality $\sum_T c_T v_T \geq v$ implies $b(\Gamma) \cdot (\sum_T c_T) \geq e(\Gamma)$.

For an arbitrary connected Γ (possibly with bridges), we set $c(\Gamma) := c(\Gamma')$, where Γ' is obtained from Γ by contracting all the bridges.

Theorem A. *The integral $J_\Gamma(s)$ converges for $\operatorname{Re}(s) > c(\Gamma)$ (for a sufficiently small ball around 0) and diverges for $s = c(\Gamma)$ (and any ball around 0). If Γ is stable and all components of C_0 are rational, the same assertions hold for the integral $I_\Gamma(\varphi, s)$.*

A simple example is the n -gon graph $\Gamma = C_n$. It is easy to see that one has $c(C_n) = n$. Indeed, the vectors v_T are just the basis vectors e_i and the condition $\sum c_i e_i \geq v$ means that $c_i \geq 1$, so the minimal $\sum c_i$ is n . There are similar stable graphs with all vertices of genus 0. For example, if Γ is a $2n$ -gon with every other side doubled then $c(\Gamma) = n$. Note that the genus of this graph is $n + 1$. This shows that the boundary $\operatorname{Re}(s) = c$ of the convergence halfplane for $I_\Gamma(\varphi, s)$ can have arbitrary large c as genus grows.

The proof of Theorem A uses some combinatorics of the *cographical matroid* associated to Γ to reduce to the case of graphs Γ for which inequality (1.3) becomes an equality (recall that the bases of the cographical matroid are complements to spanning trees, see [6, Sec. 2.3]). The key combinatorial result needed for Theorem A is that starting from any graph Γ , one can contract some edges in Γ so that the resulting graph $\bar{\Gamma}$ satisfies $c(\bar{\Gamma}) = e(\bar{\Gamma})/b(\bar{\Gamma})$ (see Cor. 3.8).

Note that the integral $J_\Gamma(s)$ is similar to the Euler-Mellin integrals considered in [2] (but with a different integration domain) and our convergence result is similar to [2, Thm. 2.2] (see Remark 3.4).

Convention. By a graph we mean a connected undirected multigraph, possibly with multiple edges and loops.

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2 Kirchhoff polynomial and the asymptotics of the period matrix

2.1 Kirchhoff polynomial

We are going to give a proof of the determinant identity relating two definitions of ψ_Γ , using Cauchy-Binet theorem, and also get a bit more information about the corresponding matrix. There is also a simple recursive proof of this determinant formula in [3].

Let us consider a more general setup where we are given a surjective morphism of free \mathbb{Z} -modules of finite rank

$$\alpha : \mathbb{Z}^E \rightarrow H,$$

with the property that, for every subset $S \subset E$, $\text{coker}(\alpha_S)$ is a free \mathbb{Z} -module, where $\alpha_S : \mathbb{Z}^S \rightarrow H$ is the restriction of α .¹

We will apply it in the case when $E = E(\Gamma)$, $H = H^1(\Gamma)$, and α is the natural projection. Note that in this case $\text{coker}(\alpha_S)$ is canonically identified with $H^1(\Gamma^S)$, where Γ^S is obtained from Γ by deleting edges in S .

With α as above we associate a symmetric bilinear form on H^* with entries which are linear forms in independent variables $(x_e)_{e \in E}$:

$$B(\xi_1, \xi_2) = \sum_{e \in E} \xi_1(\alpha(e)) \cdot \xi_2(\alpha(e)) \cdot x_e.$$

Thus, we have a well defined discriminant $\det(B) \in R := \mathbb{Z}[x_e \mid e \in E]$.

Let us say that $S \subset E$ is a basis if $\alpha_S : \mathbb{Z}^S \rightarrow H$ is an isomorphism. We denote by \mathcal{B} the set of all bases.

Proposition 2.1 (i) One has $\det(B) = \sum_{S \in \mathcal{B}} \prod_{e \in S} x_e$.

(ii) Let us view B as a nondegenerate form over the field $\mathbb{Q}R$, the fraction field over R , and let B^{-1} denote the corresponding bilinear form on the dual space H^\vee with values in $\mathbb{Q}R$. Then setting $x_e = \ln(y_e)$, where $y_e > 0$, we have

$$\lim_{y \rightarrow 0} B^{-1}(\ln(y_e)) = 0.$$

Proof (i) Let f_1, \dots, f_h be a \mathbb{Z} -basis of H , so we can view α as a matrix $E \times h$ -matrix. Then the matrix of B is given by

$$B = \alpha \cdot D(x) \cdot \alpha^t,$$

where $D(x)$ is the diagonal $E \times E$ -matrix with the entries (x_e) . We can calculate $\det(B)$ by applying the Cauchy-Binet's theorem to the decomposition of B in α and $D(x)\alpha^t$.

¹Such a morphism is known as a unimodular collection of vectors in H .

In this theorem we need to sum over subsets $S \subset E$, where $|S| = h$, such that the corresponding minor of α is nonzero. This condition is equivalent to the condition that $\alpha_S : \mathbb{Z}^S \rightarrow H$ has nonzero determinant. Since $\text{coker}(\alpha_S)$ is free by assumption, this is equivalent to S being a basis, in which case the corresponding minor is ± 1 . Since the corresponding minor of α^t is the same sign times $\prod_{e \in S} x_e$, the assertion follows.

(ii) In the dual basis (f_i^*) the matrix of B^{-1} is the inverse matrix of B . Hence, it is enough to prove that every $(h-1) \times (h-1)$ -minor M of B satisfies

$$\lim_{y \rightarrow 0} \frac{M(\ln(y))}{\det B(\ln(y))} = 0.$$

Due to the formula for the determinant, it is enough to prove that every monomial appearing in M is a constant multiple of $\prod_{e \in S} x_e$, for some $S \subset E$ with $|S| = h-1$ and S a basis. Indeed, without loss of generality we can assume that M corresponds to the rows $1, \dots, h-1$. Then by the Cauchy-Binet's formula, the monomials appearing in M would correspond to some subsets $S \subset E$ with $|S| = h-1$ such that the map $\mathbb{Z}^S \rightarrow H/\mathbb{Z} \cdot f_h$ induced by α_S is nondegenerate. But this implies that α_S is injective, and so the image of α_S is of rank $h-1$. Since α is surjective, there exists an element $s \in E \setminus S$ such that $\alpha(s)$ is not contained in the image of α_S , hence for $S' = S \cup \{s\}$, the map $\alpha_{S'}$ is nondegenerate, i.e., S' is a basis. ■

We are interested in the case of the natural projection $\alpha : \mathbb{Z}^{E(\Gamma)} \rightarrow H^1(\Gamma)$ for a graph Γ . In this case, \mathcal{B} is the set of bases of the cographical matroid associated with Γ . Choosing a basis of simple cycles (c_i) , we can view the rows of α as coefficients of the edges in c_i (with respect to a fixed orientation of all edges). Then the matrix of the symmetric form B will be exactly the matrix (1.1). Hence, $\det(B)$ gets identified with the polynomial ψ_Γ and we derive the expansion (1.2) Note that in this case bases are exactly complements to spanning trees. In particular, if an edge e is a bridge then it is not contained in any bases, so ψ_Γ depends only on variables corresponding to *non-separating edges* (i.e., edges that are not bridges).

2.2 Asymptotics of the period matrix

Assume that Γ is a stable graph of genus g , with all vertices of genus 0, and let C_0 be a stable curve with rational components with the dual graph Γ . It is well known that the arithmetic genus of C_0 is g , so we can view C_0 as a point of $\overline{\mathcal{M}}_g$. Then the set of edges $E(\Gamma)$ is in bijection with the branches of the boundary divisor through C_0 , and the subset $E^{ns}(\Gamma) \subset E(\Gamma)$ of non-separating edges is in bijection with the branches of Δ^{ns} through C_0 .

Note that if the graph Γ is trivalent then the corresponding stratum \mathcal{M}_Γ is a point and $e(\Gamma) = 3g - 3$. Otherwise, the stratum \mathcal{M}_Γ has positive dimension.

For every non-separating edge e , there is a vanishing cycle $\alpha_e \in H_1(C)$, where C is a smooth curve close to C_0 , and the corresponding monodromy transformation M_e on $H_1(C)$ has form

$$M_e(x) = (x \cdot \alpha_e)\alpha_e + x.$$

More precisely, C is obtained from the collection of spheres numbered by the set $V(\Gamma)$ of vertices of Γ by connecting them with tubes numbered by the set of edges $E(\Gamma)$.

We fix an orientation of Γ and let β_e denote a path along the tube corresponding to e going in the direction of e . Then we define α_e as the class of a circle around the tube corresponding to e , so that $(\beta_e \cdot \alpha_e) = 1$.

The subgroup $A \subset H_1(C)$ generated by (α_e) is maximal isotropic, and we have natural identifications

$$H_1(C)/A \simeq H_1(C_0) \simeq H_1(\Gamma)$$

(see [3, Sec. 6]).

Let $(\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g)$ be a symplectic basis of $H_1(C)$ (so $(\beta_j \cdot \alpha_i) = \delta_{ij}$, $(\alpha_i \cdot \alpha_j) = (\beta_i \cdot \beta_j) = 0$) such that $A = \mathbb{Z}\alpha_1 + \dots + \mathbb{Z}\alpha_g$. In particular, M_e does not change α_i 's. Let $\omega_1, \dots, \omega_g$ be the basis of $H^0(C, \omega_C)$, normalized by $\int_{\alpha_i} \omega_j = \delta_{ij}$. The period matrix $\tau = (\tau_{ij})$ is given by

$$\tau_{ij} = \int_{\beta_i} \omega_j \quad (2.1)$$

The monodromy M_e preserves (ω_j) and acts on the periods by the transformation

$$M_e : \tau_{ij} \mapsto \int_{M_e(\beta_i)} \omega_j = \tau_{ij} + (\beta_i \cdot \alpha_e) \cdot (\alpha_e)_j = \tau_{ij} + (\beta_i \cdot \alpha_e) \cdot (\beta_j \cdot \alpha_e),$$

where the integers $(\alpha_e)_j$ are determined from $\alpha_e = \sum_j (\alpha_e)_j \cdot \alpha_j$.

Let z_e denote a local equation of the branch of the boundary divisor near C_0 corresponding to the edge e . Then the monodromy M_e acts on $\ln(z_e)/(2\pi i)$ as $\ln(z_e)/(2\pi i) \mapsto \ln(z_e)/(2\pi i) + 1$. Hence,

$$\tau'_{ij} := \tau_{ij} - \sum_{e \in E} \frac{\ln(z_e)}{2\pi i} \cdot (\beta_i \cdot \alpha_e) \cdot (\beta_j \cdot \alpha_e)$$

are invariant under all monodromy transformations M_e . In other words, we have

$$\tau = \tau_0 \left(\frac{\ln(z)}{2\pi i} \right) + \tau', \quad (2.2)$$

where $\tau = (\tau_{ij})$, $\tau' = (\tau'_{ij})$, and $\tau_0(x)$ is the matrix with coefficients in $\mathbb{Z}[x_e \mid e \in E]$ with the entries

$$\tau_0(x)_{ij} = \sum_{e \in E} x_e \cdot (\beta_i \cdot \alpha_e) \cdot (\beta_j \cdot \alpha_e).$$

In fact, it follows from the nilpotent orbit theorem (see [3, Sec. 9]) that in Eq. (2.2) the term τ' is regular near C_0 . We will use this to compute the asymptotics for $\det(\tau - \bar{\tau})$ near C_0 .

Lemma 2.2 *Near C_0 one has*

$$\det(\tau - \bar{\tau}) = \psi_\Gamma \left(\frac{\ln(|z|)}{\pi i} \right) \cdot (1 + f),$$

where $z = (z_e)$ and $f \rightarrow 0$ as $C \rightarrow C_0$.

Proof Let us choose a symplectic basis of $H_1(C)$ as follows. First, let $(c_i)_{i=1, \dots, g}$ be a basis of simple cycles in $H_1(\Gamma)$, and let $(\alpha_i)_{i=1, \dots, g}$ be the dual basis of A with respect to

the pairing between A and $H_1(C)/A \simeq H_1(\Gamma)$ induced by the intersection pairing. Let $(\beta_i)_{i=1, \dots, g}$ be a set of mutually orthogonal classes in $H_1(C)$ projecting to (c_i) under the projection $H_1(C) \rightarrow H_1(\Gamma)$ (such a set exists since the intersection pairing is perfect). Then (α_i, β_i) is a standard symplectic basis. Furthermore, due to our definition of α_e , the intersection index $(\beta_i \cdot \alpha_e)$ is 1 (resp., -1) exactly when c_i passes through e in the direction of the orientation (resp., in the opposite direction).

It follows that $\tau_0(x)$ coincides with the matrix $A = (a_{ij})$ given by (1.1). Since $\tau_0(x)$ depends on (x_e) linearly with integer coefficients, (2.2) gives

$$\tau - \bar{\tau} = A \left(\frac{\ln |z|}{\pi i} \right) + \tau' - \bar{\tau}' = A \left(\frac{\ln |z|}{\pi i} \right) \cdot \left(1 + A \left(\frac{\ln |z|}{\pi i} \right)^{-1} \cdot (\tau' - \bar{\tau}') \right),$$

where $\tau' - \bar{\tau}'$ is regular near C_0 . Now by Proposition 2.1(ii), we have

$$\lim_{C \rightarrow C_0} A \left(\frac{\ln |z|}{2\pi i} \right)^{-1} \cdot (\tau' - \bar{\tau}') = 0,$$

and the assertion follows. ■

3 Convergence region

3.1 Elementary observations

Lemma 3.1 *Let $P(x_1, \dots, x_n)$ be a (nonzero) polynomial with non-negative coefficients. Then for $s \in \mathbb{C}$, the integral*

$$\int_B \frac{\prod_{i=1}^n dz_i d\bar{z}_i}{|P(-\ln |z_1|, \dots, -\ln |z_n|)|^s \cdot \prod_{i=1}^n z_i \bar{z}_i}$$

over a sufficiently small ball B around the origin in \mathbb{C}^n converges if and only if the integral

$$\int_{[C, +\infty)^n} \frac{dx_1 \dots dx_n}{P(x_1, \dots, x_n)^s}$$

converges for sufficiently large C .

Proof This follows from the change of variables $z_i = e^{-x_i + i\phi_i}$, with $x_i > C$ and $\phi_i \in (0, 2\pi)$. ■

Lemma 3.2 *Let $C > 0$. The integral*

$$\int_{[C, +\infty)^n} \frac{dx_1 \dots dx_n}{(x_1 + \dots + x_n)^s}$$

converges for $\operatorname{Re}(s) > n$ and diverges for $s = n$.

Proof Convergence for $\operatorname{Re}(s) > n$ follows from the inequality

$$(x_1 + \dots + x_n)^n \geq x_1 \dots x_n.$$

To prove the divergence for $s = n$, we will use induction on n . The base case $n = 1$ is well known. Now let $n > 1$. It is enough to prove the divergence of the integral

$$\int_{x_1 > C_1} \cdots \int_{x_n > C_n} \frac{dx_1 \cdots dx_n}{(x_1 + \cdots + x_n)^n},$$

for any large C_1, \dots, C_n . Performing the integration in x_n we get

$$\frac{1}{n-1} \int_{x_1 > C_1} \cdots \int_{x_{n-1} > C_{n-1}} \frac{dx_1 \cdots dx_{n-1}}{(x_1 + \cdots + x_{n-1} + C_n)^{n-1}}.$$

The proof now follows from the change of variables $x_1 \mapsto x_1 + C_n$ and the inductive hypothesis. ■

For a vector $w = (w_1, \dots, w_n) \in \mathbb{Z}_{\geq 0}^n$, we denote $x^w := x_1^{w_1} \cdots x_n^{w_n}$.

Lemma 3.3 (i) Let $C > 0$. Suppose for a collection of vectors $v_1, \dots, v_N \in \mathbb{Z}_{\geq 0}^n$ and scalars $c_1, \dots, c_N \in \mathbb{R}_{\geq 0}$ one has

$$\sum c_i v_i \geq v := (1, \dots, 1) \in \mathbb{R}^n.$$

Then the integral

$$\int_{[C, +\infty)^n} \frac{dx_1 \cdots dx_n}{(x^{v_1} + \cdots + x^{v_N})^s} \quad (3.1)$$

converges for $\operatorname{Re}(s) > c_1 + \cdots + c_N$. In particular, for Γ without bridges, the integral $J_\Gamma(s)$ converges for $s > c(\Gamma)$.

(ii) The integral $J_\Gamma(s)$ diverges for $s = e(\Gamma)/b(\Gamma)$.

Proof (i) The first assertion follows from the inequality

$$(x^{v_1} + \cdots + x^{v_N})^{c_1 + \cdots + c_N} \geq (x^{v_1})^{c_1} \cdots (x^{v_N})^{c_N} \geq x^v = x_1 \cdots x_n$$

for $x_i \geq 1$. The assertion about J_Γ follows from this using Lemma 3.1.

(ii) Set $n = e(\Gamma)$, $b = b(\Gamma)$. Since ψ_Γ has degree b , we have

$$\psi_\Gamma(x_1, \dots, x_n) \leq (x_1 + \cdots + x_n)^b.$$

Now the divergence for $s = n/b$ follows immediately from Lemmas 3.1 and 3.2. ■

Remark 3.4 The convergence statement of Lemma 3.3(i) is similar to the convergence statement [2, Thm. 2.2] about more general Euler-Mellin integrals. Note however that our domain of integration is $[C, +\infty)^n$, where $C > 0$, so our condition on s is different from the one in [2, Thm. 2.2] where the integration is over $(0, +\infty)^n$. The converse of Lemma 3.3(i) is false, i.e., the integral (3.1) can converge even when there are no $c_i \in \mathbb{R}_{\geq 0}$ with $\sum c_i v_i \geq v$ and $\operatorname{Re}(s) > \sum c_i$. For example, consider the integral

$$I(s) = \int_{[C, +\infty)} \frac{dx_1 dx_2}{(x_1 x_2^2 + x_1^4 x_2^3)^s}.$$

Then conditions of Lemma 3.3(i) hold only for $\operatorname{Re}(s) > 2/5$, however, we claim that $I(s)$ converges for $\operatorname{Re}(s) > 1/3$. Indeed, changing the variables by $x_i = y_i + C$, we get

$$I(s) = \int_{[0,+\infty)} \frac{dy_1 dy_2}{P(y_1, y_2)^s},$$

where $P(y_1, y_2) = (C + y_1)(C + y_2)^2 + (C + y_1)^4(C + y_2)^3$. Since the Newton polytope of P is the rectangle with the vertices $(0, 0)$, $(4, 0)$, $(0, 3)$ and $(4, 3)$, convergence for $\operatorname{Re}(s) > 1/3$ follows from [2, Thm. 2.2].

Recall that we have an inequality $c(\Gamma) \geq e(\Gamma)/b(\Gamma)$ (see (1.3)).

Definition 3.1 We say that Γ *optimal* if $c(\Gamma) = e(\Gamma)/b(\Gamma)$.

Lemma 3.3 proves our assertion about convergence/divergence of $J_\Gamma(s)$ in the case of optimal Γ . Below we will reduce the case of a general Γ to that of optimal Γ .

3.2 Combinatorial statement

Let M be a loopless matroid on the ground set E . For all $S \subset E$, consider the linear functional

$$\varphi_S : \mathbb{R}^E \rightarrow \mathbb{R}$$

taking an element to the sum of its coordinates in S . The **base polytope** $P(M)$ consists of the vectors $w \in \mathbb{R}^E$ such that $0 \leq \varphi_S(w) \leq \operatorname{rk} S$ for all $S \subset E$ and $\varphi_E(w) = \operatorname{rk} E$.

Let

$$m = m(M) := \max\{|S|/\operatorname{rk} S \mid S \neq \emptyset\},$$

and let \mathcal{T}_0 be the (nonempty) collection of subsets of E that attain this maximum.

Lemma 3.5 If $S, T \in \mathcal{T}_0$, then $S \cup T \in \mathcal{T}_0$. In other words, \mathcal{T}_0 has a maximal element.

Proof For any set $U \subset E$ of nonzero rank, we have $|U| \leq m \operatorname{rk} U$. Applying this inequality to $U = S \cap T$, we find that

$$\begin{aligned} |S \cup T| &= |S| + |T| - |S \cap T| \\ &\geq m \operatorname{rk} S + m \operatorname{rk} T - m \operatorname{rk} S \cap T \\ &= m(\operatorname{rk} S + \operatorname{rk} T - \operatorname{rk} S \cap T) \\ &\geq m \operatorname{rk} S \cup T. \end{aligned}$$

Applying it next to $U = S \cup T$, we find that $|S \cup T| = m \operatorname{rk} S \cup T$, thus $S \cup T \in \mathcal{T}_0$. ■

Let

$$c = c(M) := \min\{t \mid \text{there exists } w \in tP(M) \text{ with } w_e \geq 1 \text{ for all } e \in E\},$$

and let w be an element of $cP(M)$ with $w_e \geq 1$ for all $e \in E$ (a witness for c).

Proposition 3.6 *We have $c = m$.*

Proof Let T_0 be the maximal element of \mathcal{T}_0 (which exists by Lemma 3.5). Since $w_e \geq 1$ for all $e \in E$ and $w \in cP(M)$, we have

$$|T_0| \leq \varphi_{T_0}(w) \leq c \operatorname{rk} T_0,$$

and therefore $c \geq |T_0|/\operatorname{rk} T_0 = m$.

Now we must prove the opposite inequality. We will do it by constructing an element $w \in mP(M)$ with $w_e \geq 1$ for all $e \in E$. This construction will proceed in stages.

First let $w(0) = (1, \dots, 1)$. We then have $\varphi_S(w(0)) = |S| \leq m \operatorname{rk} S$ for all subsets $S \subset E$, with equality if and only if $S \in \mathcal{T}_0$. In particular, it is not necessarily the case that $\varphi_E(w(0)) = m \operatorname{rk} E$. If $T_0 = E$, then $\varphi_E(w(0)) = m \operatorname{rk} E$, so $w(0) \in mP(M)$ and we are done. If not, choose $e_0 \in E \setminus T_0$. Then we have $e_0 \notin S$ for all $S \in \mathcal{T}_0$. That means that there exists $\epsilon > 0$ such that $\varphi_S(w(0) + \epsilon x_{e_0}) \leq m \operatorname{rk} S$ for all $S \subset E$. Choose the largest such ϵ , and let $w(1) = w(0) + \epsilon x_{e_0}$.

Now let \mathcal{T}_1 be the collection of subsets $S \subset E$ with $\varphi_S(w(1)) = m \operatorname{rk} S$. By an argument identical to that of Lemma 3.5, there is a maximal element T_1 of \mathcal{T}_1 . If $T_1 = E$, then $w(1) \in mP(M)$ and we are done. If not, choose $e_1 \in E \setminus T_1$, and repeat the procedure to produce a new vector $w(2)$.

Eventually we will have $T_k = E$, and this process will terminate with $w(k) \in mP(M)$ and $w(k)_e \geq 1$ for all $e \in E$. ■

Proposition 3.7 *Given a loopless matroid M and any $T \in \mathcal{T}_0$, one has*

$$c(M) = c(M|_T) = |T|/\operatorname{rk} T.$$

In particular, this is true for the maximal element $T_0 \in \mathcal{T}_0$.

Proof Since the rank function on $M|_T$ is the same as that on M , it is clear that

$$m(M) = m(M|_T) = |T|/\operatorname{rk} T.$$

It remains to apply Proposition 3.6. ■

Corollary 3.8 *For any connected graph Γ without bridges, there exists a graph $\bar{\Gamma}$ obtained by contracting some edges in Γ , such that*

$$c(\Gamma) = c(\bar{\Gamma}) = \frac{e(\bar{\Gamma})}{b(\bar{\Gamma})}$$

(the second equality means that $\bar{\Gamma}$ is optimal).

Proof Let M denote the cographical matroid associated with Γ . Then $c(M) = c(\Gamma)$. The property that Γ is optimal is the property that $c(M) = |E|/\operatorname{rk} E$. Corollary 3.7 says that we can find an optimal deletion of M with the same constant c . That means that we can find an optimal contraction of Γ with the same constant c . ■

Example 3.9 Let Γ be an $2n$ -gon with every other side doubled, so $e(\Gamma) = 3n$. It is easy to see that $c(\Gamma) = n$. We can collapse all the doubled sides to get the n -gon $\bar{\Gamma}$, which is optimal and has $c(\bar{\Gamma}) = n$.

3.3 Proof of Theorem A

Lemma 2.2 shows that the case of the integral $I_\Gamma(\varphi, s)$ (for stable Γ and all components of C_0 rational) reduces to the case of $J_\Gamma(s)$. Also, since $\psi_\Gamma = \psi_{\Gamma'}$, where Γ' is obtained by contracting all the bridges, it is enough to consider the case when Γ has no bridges.

Due to Lemma 3.3(i), it remains to prove that $J_\Gamma(s)$ diverges for $s = c(\Gamma)$. We define $S \subset E(\Gamma)$ to be the set of edges that get contracted to get $\bar{\Gamma}$. By Fubini theorem, it is enough to prove that for any fixed values $z_i = c_i$ with $i \in S$, the integral

$$\int_{B'} \frac{\prod_{e \notin S} dz_e d\bar{z}_e}{\psi_\Gamma(\ln |z_\bullet|) \Big|_{z_i=c_i, i \in S} \cdot \prod_{e \notin S} z_e \bar{z}_e},$$

where B' is a small ball around the origin in $\mathbb{C}^{E(\Gamma) \setminus S}$, diverges. Thus, by Lemma 3.3(ii) applied to $\bar{\Gamma}$, it is enough to prove an inequality

$$\psi_\Gamma(x_\bullet) \Big|_{x_e=a_e, e \in S} \leq C \cdot \psi_{\bar{\Gamma}}(x_\bullet),$$

for $a_e > 0$, with some constant $C > 0$ depending on (a_e) . Furthermore, it is enough that this inequality holds for $x_e > C$. Now we observe that for every spanning forest $T \subset \Gamma$, the intersection $\bar{T} := T \cap \bar{\Gamma}$ is also a spanning forest. This implies that for every monomial $x_T = \prod_{e \in T} x_e$ of ψ_Γ , one has

$$x_T \Big|_{x_e=a_e, e \in S} = \prod_{e \in S} a_e \cdot x_{\bar{T}}.$$

This easily leads to the claimed inequality.

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