Math 634: Algebraic Topology I, Fall 2015
Solutions to Homework #7

Exercises from Hatcher: Chapter 4.2, Problems 1, 2, 30, 31, 32, 39.

1. We know that $\pi_k(\mathbb{R}P^k) \cong \pi_k(S^k) \cong \mathbb{Z}$ and $\pi_k(\mathbb{R}P^n) \cong \pi_k(S^n) \cong \{1\}$. If $\mathbb{R}P^k$ were a retract of $\mathbb{R}P^n$, we would have maps $\pi_k(\mathbb{R}P^k) \to \pi_k(\mathbb{R}P^n) \to \pi_k(\mathbb{R}P^k)$ that compose to the identity, which is impossible.

2. By the first problem on the previous assignment (Problem 4 from Chapter 4.1), the action of the nontrivial element $\gamma \in \pi_1(\mathbb{R}P^n)$ on $\pi_n(\mathbb{R}P^n) \cong \pi_n(S^n) \cong \mathbb{Z}$ is the same as the action induced by the antipodal map on S^n. (Note that we are allowed to be sloppy about basepoints here because $\pi_1(S^n)$ is trivial, so $\pi_n(S^n, x_0)$ is canonically isomorphic to $\pi_n(S^n, -x_0)$.) Corollary 4.25 says that the isomorphism $\pi_n(S^n) \cong \mathbb{Z}$ is given by the degree map, and the antipodal map multiplies degree by $(-1)^{n+1}$.

30. The fiber over 0 would have to be homeomorphic to $\mathbb{R}$, so the only candidates are $\emptyset$ (which is trivial), $(-\infty, a]$, $[b, \infty)$, and $(-\infty, a] \cup [b, \infty)$ for $a < b$. In fact, all of these possibilities are okay. I'll prove that it works for $(-\infty, 0]$ and $(-\infty, -\pi/2] \cup [\pi/2, \infty)$; the other cases can be easily derived from these.

First let's do $(-\infty, 0]$. I will show that $\mathbb{R}^2 \smallsetminus \{0\} \times (-\infty, 0] \cong \mathbb{R} \times \mathbb{R}_{>0}$ via a homomorphism compatible with projection onto the first coordinate. I did this in class with lots of explanation of why I was doing what I was doing, but here I will just pull the homeomorphism out of a hat. Define

$$\varphi : \mathbb{R} \times \mathbb{R}_{>0} \to \mathbb{R}^2 \smallsetminus \{0\} \times (-\infty, 0]$$

by the formula

$$\varphi(x, y) = \left(x, y^2 - \left(\frac{x}{2y}\right)^2\right)$$

and

$$\psi : \mathbb{R}^2 \smallsetminus \{0\} \times (-\infty, 0] \to \mathbb{R} \times \mathbb{R}_{>0}$$

by the formula

$$\psi(x, z) = \left(x, \frac{z + \sqrt{x^2 + z^2}}{2}\right).$$

One can check explicitly that these maps are inverse to each other.

Okay, now let's do $(-\infty, -\pi/2] \cup [\pi/2, \infty)$. I will show that

$$\mathbb{R}^2 \smallsetminus \{0\} \times ((-\infty, -\pi/2] \cup [\pi/2, \infty)) \cong \mathbb{R}^2$$

via a homomorphism compatible with projection onto the first coordinate. Define

$$\varphi : \mathbb{R}^2 \to \mathbb{R}^2 \smallsetminus \{0\} \times ((-\infty, -\pi/2] \cup [\pi/2, \infty))$$

by the formula

$$\varphi(s, t) = (s, s^2 t + \tan^{-1} t).$$

It is easy to verify that this is a continuous bijection. To show that its inverse is also continuous, we appeal to the inverse function theorem, which says that the inverse is smooth (a much stronger conclusion!) provided that the derivative at any point is invertible. The derivative of φ at a point (s,t) is given by the matrix

$$\begin{pmatrix} 1 & 0 \\ 2st & (1+t^2)^{-1} \end{pmatrix},$$

which clearly has nonzero determinant.

31. Choose a basepoint $x_0 \in F$ with image $b_0 \in B$. The hypothesis implies that the map $\pi_n(F,x_0) \to \pi_n(E,x_0)$ in the long exact sequence of Theorem 4.41 is trivial for all $n > 0$, so we get a short exact sequence

$$0 \to \pi_n(E,x_0) \to \pi_n(B,b_0) \to \pi_{n-1}(F,x_0) \to 0$$

for all $n > 1$. Since $n > 1$, $\pi_n(B,b_0)$ is abelian, so it is sufficient to construct a right splitting of this sequence. Choose a nullhomotopy $H : F \times I \to E$ of the inclusion of F into E. That is, let H be a map such that $H(x,1) = x$ and $H(x,0) = x_0$. Fix a point $s \in S^{n-1}$, and consider a map $f : (S^{n-1}, s) \to (F, x_0)$ representing a class $[f] \in \pi_{n-1}(F,x_0)$. We can extend f to a map $\tilde{f} : D^n \to E$ by putting $\tilde{f}(v) = H(f(v/|v|), |v|)$ if $v \neq 0$ and $\tilde{f}(0) = x_0$, which lifts $[f]$ to a class $[\tilde{f}] \in \pi_n(E,F,x_0) \cong \pi_n(B,b_0)$. It is straightforward to check that this lifting is well-defined (it doesn't change when we replace f with another map f' that is homotopic to f rel $\{s\}$) and that it is a homomorphism.

32. We know that $m = k + n$, so it is sufficient to prove that $k = n - 1$. We know that $k > 0$, for otherwise S^n would be a 2-sheeted cover of itself, which is impossible since S^n is simply connected. We also know that $n > 0$, for otherwise the base would be disconnected and the total space connected.

Since $k < n$, any inclusion of S^k into S^n is nullhomotopic, so the previous problem implies that $\pi_n(S^n) \cong \pi_n(S^m) \oplus \pi_{n-1}(S^k)$. Since $m > n$, $\pi_n(S^m)$ is trivial and $\mathbb{Z} \cong \pi_n(S^n) \cong \pi_{n-1}(S^k)$. This tells us that $n - 1 \geq k$.

We also have an exact sequence $\pi_{k+1}(S^n) \to \pi_k(S^k) \to \pi_k(S^m)$. Since $k < m$, $\pi_k(S^m)$ is trivial, so $\pi_{k+1}(S^n)$ surjects onto $\pi_k(S^k) \cong \mathbb{Z}$. This implies that $k + 1 \geq n$. Put together, these two inequalities tell us that $k = n - 1$.