Let $M$ be a matroid. The Kazhdan-Lusztig polynomial $P_M(t) \in \mathbb{Z}[t]$ was introduced in [EPW16], and the closely related $Z$-polynomial $Z_M(t) \in \mathbb{Z}[t]$ was introduced in [PXY18]. Kazhdan-Lusztig polynomials of matroids are neither special cases nor generalizations of classical Kazhdan-Lusztig polynomials. Rather, both classes of polynomials are special cases of Kazhdan-Lusztig-Stanley polynomials; see [Pro18] for more details. The following conjecture appears in [GPY17b 3.2] and [PXY18 5.1].

**Conjecture 1.** The polynomials $P_M(t)$ and $Z_M(t)$ are real rooted.

**Remark 2.** We also have various conjectures that say that the roots of the Kazhdan-Lusztig polynomials or $Z$-polynomials of various matroids should interlace. For the conjectural statement about Kazhdan-Lusztig polynomials, see [GPY17b 3.4 and 3.5]. The $Z$-polynomial statement should roughly say that $Z_M(t)$ and $Z_{M/e}(t)$ have interlacing roots, but one has to rule out degenerate examples. For example, if $M$ is the thagomizer matroid of rank 4 and $e$ is the distinguished element of the ground set, then $M/e$ is Boolean, thus $Z_{M/e}(t) = (1 + t)^3$. But this does not interlace with $Z_M(t) = 1 + 11t + 21t^2 + 11t^3 + t^4$.

**Remark 3.** If there is a finite group $W$ acting on $M$, then these polynomials have equivariant analogues $P^W_M(t)$ [GPY17a] and $Z^W_M(t)$ [PXY18], whose coefficients isomorphism classes of representations of $W$. I’ve made various attempt to formulate equivariant versions of Conjecture 1 (involving minors of the Toeplitz matrix) and Remark 2 (involving minors of the Bézout matrix), but I keep finding counterexamples. For instance, $Z_{S_n U_{n-1,n}}^S(t)$ fails to be real rooted or to equivariantly interlace with $Z_{U_{n,n+1}}^{S_n}(t)$ when $n$ is large.

**References**


