A cohomological interpretation of the Heaviside filtration on the Varchenko–Gelfand ring of a pair

Galen Dorpalen-Barry
Faculty of Mathematics, Ruhr-Universität Bochum, D-44801 Bochum, Germany

Nicholas Proudfoot
Department of Mathematics, University of Oregon, Eugene, OR 97403

Jidong Wang
Department of Mathematics, University of Texas, Austin, TX 78712

Abstract. We give a cohomological interpretation of the Heaviside filtration on the Varchenko–Gelfand ring of a pair \((A, K)\), where \(A\) is a real hyperplane arrangement and \(K\) is a convex open subset. This builds on work of the first author, who studied the filtration from a purely algebraic perspective, and also work of Moseley, who gave a cohomological interpretation in the special case where \(K\) is the ambient vector space.

1 Introduction

Let \(V\) be a finite dimensional vector space over \(\mathbb{R}\) and \(A\) a finite set of hyperplanes in \(V\), and consider the complement

\[M_1(A) := V \setminus \bigcup_{H \in A} H,\]

which is simply the disjoint union of the chambers. The Varchenko–Gelfand ring \(\text{VG}(A)\) is defined as the ring of locally constant \(\mathbb{Z}\)-valued functions on \(M_1(A)\). This is a boring ring with an interesting filtration: it is generated as a ring by Heaviside functions, which take the value 1 on one side of a given hyperplane and 0 on the other side, and we define \(F_k(A) \subset \text{VG}(A)\) be the subgroup generated by polynomial expressions in the Heaviside functions of degree at most \(k\). Varchenko and Gelfand \([VGS7]\) computed the relations between the Heaviside functions.

For any ring \(R\) equipped with an increasing filtration \(F_0 \subset F_1 \subset \cdots \subset R\), one can define the associated graded

\[\text{gr } R := \bigoplus_{k \geq 0} F_k/F_{k-1}\]

and the Rees algebra

\[\text{Rees } R := \bigoplus_{k \geq 0} u^k F_k \subset R \otimes \mathbb{Z}[u].\]

The Rees algebra is a torsion-free graded module over the polynomial ring \(\mathbb{Z}[u]\), and we have

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canonical isomorphisms

\[ \text{Rees } R/\langle u - 1 \rangle \cong R \quad \text{and} \quad \text{Rees } R/\langle u \rangle \cong \text{gr } R^2 \]

The geometric meaning of the Heaviside filtration of \( \text{VG}(\mathcal{A}) \), along with its associated graded and its Rees algebra, was explained in a paper of Moseley \cite{Mos17}. Let

\[ M_3(\mathcal{A}) := V \otimes \mathbb{R}^3 \setminus \bigcup_{H \in \mathcal{A}} H \otimes \mathbb{R}^3. \]

This space admits an action of \( T := U(1) \) by identifying \( \mathbb{R}^3 \) with \( \mathbb{R} \times \mathbb{C} \) and letting \( T \) act on \( \mathbb{C} \) by scalar multiplication; the fixed point set of this action can be identified with the space \( M_1(\mathcal{A}) \). Moseley showed that we have isomorphisms

\[ \text{VG}(\mathcal{A}) \otimes \mathbb{Q} \cong H^*(M_3(\mathcal{A})^T; \mathbb{Q}) \]
\[ \text{gr } \text{VG}(\mathcal{A}) \otimes \mathbb{Q} \cong H^*(M_3(\mathcal{A}); \mathbb{Q}) \]
\[ \text{Rees } \text{VG}(\mathcal{A}) \otimes \mathbb{Q} \cong H^*_T(M_3(\mathcal{A}); \mathbb{Q}), \]

the latter being an isomorphism of graded algebras over \( H^*(BT; \mathbb{Q}) \cong \mathbb{Q}[u] \).

The first of the three isomorphisms above is immediate from the definition of \( \text{VG}(\mathcal{A}) \), and the second can be obtained by comparing the results of Varchenko and Gelfand with the presentation of \( H^*(M_3(\mathcal{A}); \mathbb{Q}) \) due to de Longueville and Schultz \cite[Corollary 5.6]{dS01}. The most interesting is the last isomorphism, which interpolates between the first two (see Section 2).

Our goal is to generalize these results to a larger class of rings and spaces, and also to work with coefficients in \( \mathbb{Z} \) rather than in \( \mathbb{Q} \). Fix an open, convex subset \( \mathcal{K} \subset V \), and consider the spaces

\[ M_1(\mathcal{A}, \mathcal{K}) := M_1(\mathcal{A}) \cap \mathcal{K} \quad \text{and} \quad M_3(\mathcal{A}, \mathcal{K}) := M_3(\mathcal{A}) \cap (\mathcal{K} \times \mathbb{C}). \]

Note that we still have an action of \( T \) on \( M_3(\mathcal{A}, \mathcal{K}) \) with fixed point set isomorphic to \( M_1(\mathcal{A}, \mathcal{K}) \). We define the \textbf{Varchenko–Gelfand ring of the pair} \( (\mathcal{A}, \mathcal{K}) \) to be the ring \( \text{VG}(\mathcal{A}, \mathcal{K}) \) of locally constant \( \mathbb{Z} \)-valued functions on \( M_1(\mathcal{A}, \mathcal{K}) \). We define the Heaviside functions and associated filtration as before. When \( \mathcal{K} \) is an intersection of open half spaces associated with hyperplanes in \( \mathcal{A} \), this filtered ring was introduced and studied by the first author \cite{DB}. Our first main result is the following theorem.

\textbf{Theorem 1.1.} \textit{We have canonical isomorphisms}

\[ \text{VG}(\mathcal{A}, \mathcal{K}) \cong H^*(M_3(\mathcal{A}, \mathcal{K})^T; \mathbb{Z}) \]
\[ \text{gr } \text{VG}(\mathcal{A}, \mathcal{K}) \cong H^*(M_3(\mathcal{A}, \mathcal{K}); \mathbb{Z}) \]
\[ \text{Rees } \text{VG}(\mathcal{A}, \mathcal{K}) \cong H^*_T(M_3(\mathcal{A}, \mathcal{K}); \mathbb{Z}), \]

\footnote{We will always take the degree of \( u \) to be 2, which means that the isomorphism \( \text{Rees } R/\langle u \rangle \cong \text{gr } R \) halves degrees.}
the latter being an isomorphism of graded algebras over $H^*(BT; \mathbb{Z}) \cong \mathbb{Z}[u]$.

**Remark 1.2.** If we take $\mathcal{K} = V$, then $M_1(\mathcal{A}, \mathcal{K}) = M_1(\mathcal{A})$ and $M_3(\mathcal{A}, \mathcal{K}) = M_3(\mathcal{A})$. We then recover Moseley’s result by tensoring with $\mathbb{Q}$.

**Remark 1.3.** Theorem 1.1 extends to affine arrangements. Let $\mathcal{A}$ be a collection of affine hyperplanes (not necessarily containing the origin) in $V$. For each $H \in \mathcal{A}$, let $H_0$ be the linear hyperplane obtained by translating $H$ to the origin, and let

$$H \otimes \mathbb{R}^3 := \{(x, y, z) \in V \otimes \mathbb{R}^3 \mid x \in H \text{ and } y, z \in H_0\}.$$ 

Then define $M_1(\mathcal{A}, K)$ and $M_3(\mathcal{A}, K)$ as above, and note that $T$ still acts on $M_3(\mathcal{A}, K)$ with fixed locus $M_1(\mathcal{A}, K)$. For each $H \in \mathcal{A}$, let $\text{cone}(H)$ be the linear hyperplane in $V \oplus \mathbb{R}$ whose intersection with $V \oplus \{1\}$ is equal to $H$, and let $\text{cone}(\mathcal{A})$ be the linear arrangement in $V \oplus \mathbb{R}$ consisting of $\text{cone}(H)$ for all $H \in \mathcal{A}$ along with the single additional hyperplane $V \oplus \{0\}$. For any convex open set $\mathcal{K} \subset V$, let

$$\text{cone}(\mathcal{K}) := \{(v, r) \in V \oplus \mathbb{R}_{>0} \mid v/r \in \mathcal{K}\}.$$ 

Then the inclusion of $M_1(\mathcal{A}, K)$ into $M_1(\text{cone}(\mathcal{A}), \text{cone}(\mathcal{K}))$ taking $v$ to $(v, 1)$ is a homotopy equivalence, and the corresponding inclusion of $M_3(\mathcal{A}, K)$ into $M_3(\text{cone}(\mathcal{A}), \text{cone}(\mathcal{K}))$ is a $T$-equivariant homotopy equivalence. In particular, the statement of Theorem 1.1 holds verbatim, and it follows from the corresponding statement for the pair $(\text{cone}(\mathcal{A}), \text{cone}(\mathcal{K}))$.

Our proof of Theorem 1.1 is purely topological, and does not require us to give presentations of any of the rings involved. That said, it is indeed possible to compute presentations, building on the work of the first author and generalizing the presentations in [VG87, dS01, Mos17] in the case where $\mathcal{K} = V$.

For notational convenience, we fix an orientation of $V/H$ for each $H \in \mathcal{A}$. Let $H^+ \subset V$ be the positive open half space associated with $H$ (that is, the preimage of the positive part of the line $V/H$), and let $H^- := V \setminus (H^+ \sqcup H)$ be the negative open half space. We define a signed set to be a pair $S = (S^+, S^-)$, where $S^+$ and $S^-$ are disjoint subsets of $\mathcal{A}$. For any signed set $S$, let

$$H_S := \bigcap_{H \in S^+} H^+ \cap \bigcap_{H \in S^-} H^-.$$ 

Consider the free graded $\mathbb{Z}[u]$-algebra

$$R_\mathcal{A} := \mathbb{Z}[u, e^+_{H}, e^-_{H}]_{H \in \mathcal{A}}/\left\langle e^+_{H} e^-_{H}, e^+_{H} + e^-_{H} - u \mid H \in \mathcal{A}\right\rangle,$$

with all generators having degree 2. For each signed set $S = (S^+, S^-)$, let

$$e_S := \prod_{H \in S^+} e^+_{H} \prod_{H \in S^-} (-e^-_{H}) \in R_\mathcal{A}.$$
Let $\tilde{S} := (S^-, S^+)$, be the opposite signed subset, and let

$$f_S := \frac{e_S - e_{\tilde{S}}}{u} \in R_A.$$ 

We define the ideals

$$I_{A,K} := \langle e_S \mid S \in S \rangle \subset R_A \quad \text{and} \quad J_{A,K} := \langle f_S \mid S, \tilde{S} \in K \rangle \subset R_A$$

where $S := \{S \subset A \mid H_S \cap K = \emptyset\}$.

For any $m \in \mathbb{Z}$, consider the quotient ring $R_{A,m} := R_A / \langle u - m \rangle$ (we will only be interested in the cases $m = 0$ and $m = 1$). Let $I_{A,K,m}$ and $J_{A,K,m}$ be the images of $I_{A,K}$ and $J_{A,K}$ in $R_{A,m}$.

**Theorem 1.4.** We have canonical isomorphisms

$$\text{VG}(A, K) \cong R_{A,1} / I_{A,K,1} + J_{A,K,1}$$

$$\text{gr VG}(A, K) \cong R_{A,0} / I_{A,K,0} + J_{A,K,0}$$

$$\text{Rees VG}(A, K) \cong R_A / I_{A,K} + J_{A,K}$$

given by identifying each $e^\pm_H$ with the corresponding Heaviside function.

**Remark 1.5.** The ideal $I_{A,K,1} + J_{A,K,1}$ is inhomogeneous, and it is clear that its initial ideal is contained in $I_{A,K,0} + J_{A,K,0}$. The fact that its initial ideal is equal to $I_{A,K,0} + J_{A,K,0}$ is not obvious; the proof of this fact is a substantial part of the proof of Theorem 1.4. This is equivalent to the statement that $R_A / I_{A,K} + J_{A,K}$ is a free module over $\mathbb{Z}[u]$.

**Remark 1.6.** When $K = H_S$ for some signed set $S$, the first two isomorphisms in Theorem 1.4 appear in [DB], and the third isomorphism can be derived from this result. When $K = V$, the second isomorphism in Theorem 1.4 appears in [dS01, Corollary 5.6], however even in this special case the proof here is new (and arguably more geometric).

**Remark 1.7.** If $H \in S^+$, then $e^+_H f_S = e_S$, and similarly if $H \in S^-$, then $-e^-_H f_S = e_S$. For this reason, we may replace the ideal $I_{A,K}$ with the ideal

$$I'_{A,K} := \langle e_S \mid S \in S, \tilde{S} \notin S \rangle$$

in the statement of Theorem 1.4. In the special case where $K = V$, we have $I'_{A,V} = 0$, thus we can eliminate the ideals $I_{A,K}$ and $I_{A,K,m}$ entirely from the statement of the theorem. This gives us the presentation for $\text{VG}(A)$ that appears in [VG87].

**Remark 1.8.** When $K = V$, the presentations in Theorem 1.4 depend only on the oriented matroid associated with $A$. Gelfand and Rybnikov extend the definition of the Varchenko–Gelfand algebra to arbitrary oriented matroids [GR89], and the associated graded was studied by Cordovil [Cor02]. For general $K$, the presentations in Theorem 1.4 depend on the oriented matroid along.
with the collection of maximal covectors corresponding to the chambers of \( \mathcal{A} \) that meet \( \mathcal{K} \). The right combinatorial structure to encode this information is that of a conditional oriented matroid \([BCK18]\).

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2 Equivariant cohomology

We begin by reviewing some basic facts about equivariant cohomology. Most of this material is well known, and standard references include \([\text{Bor60, AB84, GKM98}]\). However, it is sometimes difficult to find precise statements in the literature about equivariant cohomology with \( \mathbb{Z} \) coefficients, so we collect the results that we need here. For simplicity, we work only with the group \( T = U(1) \).

2.1 Basic definitions

Let \( ET := \mathbb{C}^\infty \setminus \{0\} \). This is a contractible space equipped with a free action of \( T \), with quotient

\[ BT := ET/T \cong \mathbb{C}P^\infty. \]

Let \( M \) be a \( T \)-space. We define the **Borel space**

\[ M_T := (M \times ET)/T, \]

where \( T \) acts diagonally on \( M \times ET \). The **equivariant cohomology** of \( M \) is defined to be the graded ring

\[ H^*_T(M; \mathbb{Z}) := H^*(M_T; \mathbb{Z}). \]

The \( T \)-equivariant projection from \( M \times ET \) to \( ET \) descends to a fiber bundle \( \pi : M_T \to BT \) with fiber \( M \). Pullback along \( \pi \) makes \( H^*_T(M; \mathbb{Z}) \) into an algebra over \( H^*(BT; \mathbb{Z}) \cong \mathbb{Z}[u] \). Any \( T \)-equivariant map from \( M \) to \( N \) induces a map from \( M_T \) to \( N_T \) that is compatible with the bundle projections. In particular, this means that \( H^*_T(\cdot; \mathbb{Z}) \) is a contravariant functor from the category of \( T \)-spaces with equivariant maps to the category of graded \( \mathbb{Z}[u] \)-algebras.

If \( N \subset M \) is a \( T \)-subspace, we define the relative \( \mathbb{Z}[u] \)-modules

\[ H^*_T(M, N; \mathbb{Z}) := H^*(M_T, N_T; \mathbb{Z}). \]

We then obtain a long exact sequence in equivariant cohomology for the pair \( (M, N) \) as the long exact sequence in ordinary cohomology for the associated Borel spaces.
Example 2.1. If $T$ acts trivially on $M$, then $M_T \cong M \times BT$, and

$$H^*_T(M; \mathbb{Z}) \cong H^*(M; \mathbb{Z}) \otimes H^*(BT; \mathbb{Z}) \cong H^*(M; \mathbb{Z}) \otimes \mathbb{Z}[u].$$

Example 2.2. If $T$ acts freely on $M$, then $M_T \cong M/T \times ET$, which is homotopy equivalent to $M/T$, so

$$H^*_T(M; \mathbb{Z}) \cong H^*(M/T; \mathbb{Z}).$$

More generally, if $N \subset M$ is a $T$-subspace, then $H^*_T(M,N; \mathbb{Z}) \cong H^*(M/T,N/T; \mathbb{Z}).$

2.2 Specializations

The inclusion of a fiber $\iota : M \to M_T$ defines a graded algebra homomorphism

$$\varphi := \iota^* : H^*_T(M; \mathbb{Z}) \to H^*(M; \mathbb{Z}),$$

called the forgetful homomorphism. Since $\pi \circ \iota$ is a constant map, we have

$$\varphi(\pi^*u) = \iota^*(\pi^*u) = (\pi \circ \iota)^*u = 0,$$

therefore $\varphi$ descends to a homomorphism

$$\bar{\varphi} : H^*_T(M; \mathbb{Z})/\langle \pi^*u \rangle \to H^*(M; \mathbb{Z}).$$

We will often abuse notation by using the symbol $u$ to denote $\pi^*u \in H^*_T(M; \mathbb{Z})$, so that we may instead write

$$\bar{\varphi} : H^*_T(M; \mathbb{Z})/\langle u \rangle \to H^*(M; \mathbb{Z}).$$

The inclusion of the fixed point set $\kappa : M^T \to M$ induces a graded algebra homomorphism

$$\psi := \kappa^* : H^*_T(M; \mathbb{Z}) \to H^*_T(M^T; \mathbb{Z}) \cong H^*(M^T; \mathbb{Z}) \otimes \mathbb{Z}[u].$$

Setting $u$ equal to 1, this descends to a homomorphism

$$\bar{\psi} : H^*_T(M; \mathbb{Z})/\langle u - 1 \rangle \to H^*(M^T; \mathbb{Z}).$$

2.3 Localization

We say that a $\mathbb{Z}[u]$-module $Q$ is torsion if, for all $x \in H$, there exists some $n \in \mathbb{N}$ such that $u^n x = 0$.

Lemma 2.3. Suppose that $M$ is a finite dimensional $T$-manifold with the property that $T$ acts freely on $M \setminus M^T$. Then the relative cohomology group $H^*_T(M,M^T; \mathbb{Z})$ is a torsion $\mathbb{Z}[u]$-module.

Proof. Let $N$ be a $T$-equivariant closed tubular neighborhood of $M^T$ in $M$, and let $U$ be the interior
of \(N\). Then

\[
H^*_T(M, M^T; \mathbb{Z}) \cong H^*_T(M, N; \mathbb{Z}) \cong H^*_T(M \setminus U, N \setminus U; \mathbb{Z}) \cong H^* \left( \left( M \setminus U \right) / T, \left( N \setminus U \right) / T; \mathbb{Z} \right),
\]

where the first isomorphism is induced by the retraction from \(N\) to \(M^T\), the second by excision, and the third by Example 2.2. Since \(T\) acts freely away from \(M\), \((\overline{M \setminus U}) / T, \overline{N \setminus U} / T\) is a manifold with boundary \((N \setminus U) / T\). Its cohomology vanishes in degrees greater than its dimension, and the lemma follows.

The long exact sequence of the pair \((M, M^T)\) gives us the following corollary.

**Corollary 2.4.** Suppose that \(M\) is a finite dimensional \(T\)-manifold with the property that \(T\) acts freely on \(M \setminus M^T\). Then the kernel and cokernel of the map

\[
\psi : H^*_T(M; \mathbb{Z}) \to H^*_T(M^T; \mathbb{Z})
\]

are torsion \(\mathbb{Z}[u]\)-modules.

**Remark 2.5.** If we drop the assumption that \(T\) acts freely on \(M \setminus M^T\), then Corollary 2.4 holds over \(\mathbb{Q}\), but not over \(\mathbb{Z}\). For example, consider the action of \(T\) on itself as multiplication by the square, so that the fixed locus is empty and the stabilizer of every point is \(\{\pm 1\} \subset T\). Then

\[
\text{Ker}(\psi) = H^*_T(T; \mathbb{Z}) \cong H^*(B\{\pm 1\}; \mathbb{Z}) \cong H^*(\mathbb{R}P^{\infty}; \mathbb{Z}) \cong \mathbb{Z}[u] / (2u).
\]

Only after tensoring with \(\mathbb{Q}\) does this become a torsion \(\mathbb{Z}[u]\)-module.

### 2.4 Equivariant formality

The \(T\)-space \(M\) is said to be **equivariantly formal over** \(\mathbb{Z}\) if the map \(\varphi : H^*_T(M; \mathbb{Z}) \to H^*(M; \mathbb{Z})\) is surjective.

**Proposition 2.6.** If \(M\) is equivariantly formal over \(\mathbb{Z}\), then the equivariant cohomology ring \(H^*_T(M; \mathbb{Z})\) is free as a module over \(\mathbb{Z}[u]\), and the map \(\bar{\varphi} : H^*_T(M; \mathbb{Z}) / \langle u \rangle \to H^*(M; \mathbb{Z})\) is an isomorphism.

**Proof.** This follows from the Leray–Hirsch theorem, applied to the fiber bundle \(\pi : M_T \to BT\).

**Proposition 2.7.** If \(M\) is equivariantly formal over \(\mathbb{Z}\) and \(x_1, \ldots, x_n \in H^*_T(M; \mathbb{Z})\) are homogeneous classes with the property that \(\varphi(x_1), \ldots, \varphi(x_n)\) generate \(H^*(M; \mathbb{Z})\) as a ring, then \(x_1, \ldots, x_n\) generate \(H^*_T(M; \mathbb{Z})\) as a \(\mathbb{Z}[u]\)-algebra.

**Proof.** Let \(R \subset H^*_T(M; \mathbb{Z})\) be the subalgebra generated by the classes \(x_1, \ldots, x_n\). Assume for the sake of contradiction that \(R \subsetneq H^*_T(M; \mathbb{Z})\), and let \(\alpha \in H^*_T(M; \mathbb{Z}) \setminus R\) be a homogeneous class of minimal degree. Since \(\varphi(R) = H^*(M; \mathbb{Z})\), there exists a class \(x \in R\) such that \(\varphi(x) = \varphi(\alpha)\), or equivalently \(\varphi(x - \alpha) = 0\). By Proposition 2.6, this means that there is a class \(\beta\) such that
$x - \alpha = u\beta$. By minimality of the degree of $\alpha$, we have $\beta \in R$. But thus contradicts the fact that $\alpha \notin R$.

**Proposition 2.8.** Suppose that $M$ is a finite dimensional $T$-manifold with the property that $T$ acts freely on $M \setminus M^T$, and that $M$ is equivariantly formal over $\mathbb{Z}$. Then

$$\psi : H^*_T(M; \mathbb{Z}) \to H^*_T(M^T; \mathbb{Z})$$

is injective and

$$\bar{\psi} : H^*_T(M; \mathbb{Z})/\langle u - 1 \rangle \to H^*(M^T; \mathbb{Z})$$

is an isomorphism.

**Proof.** Corollary 2.4 tells us that the kernel and cokernel of $\psi$ are torsion. Proposition 2.6 tells us that the domain of $\psi$ is a free $\mathbb{Z}[u]$-module, and a torsion submodule of a free module is zero, thus $\psi$ is injective. Consider the short exact sequence

$$0 \to H^*_T(M; \mathbb{Z}) \xrightarrow{\psi} H^*(M^T; \mathbb{Z}) \otimes \mathbb{Z}[u] \to \text{Coker}(\psi) \to 0.$$

Taking the tensor product over $\mathbb{Z}[u]$ with the module $Q := \mathbb{Z}[u]/\langle u - 1 \rangle$, we obtain the sequence

$$\text{Tor}_1(\text{Coker}(\psi), Q) \to H^*_T(M; \mathbb{Z})/\langle u - 1 \rangle \xrightarrow{\bar{\psi}} H^*(M^T; \mathbb{Z}) \to \text{Coker}(\psi) \otimes_{\mathbb{Z}[u]} Q \to 0.$$

Since Coker$(\psi)$ is torsion, Coker$(\psi) \otimes_{\mathbb{Z}[u]} Q = 0$. Since Coker$(\psi)$ is graded,

$$\text{Tor}_1(\text{Coker}(\psi), Q) \cong \{ x \in \text{Coker}(\psi) \mid (u - 1)x = 0 \} = 0,$$

as well. Thus $\bar{\psi}$ is an isomorphism.

2.5 The equivariant filtration

Suppose that $M$ satisfies the hypotheses of Proposition 2.8 and in particular that $\bar{\psi}$ is an isomorphism. Suppose also that the cohomology of $M$ vanishes in odd degree. By Proposition 2.6, the same is true of the equivariant cohomology. Define

$$F_k(M) \subset H^*(M^T; \mathbb{Z}) \cong H^*_T(M; \mathbb{Z})/\langle u - 1 \rangle$$

to be the set of classes that can be lifted to $H^{2k}_T(M; \mathbb{Z})$. Note that any class that can be lifted to $\alpha \in H^{2k}_T(M; \mathbb{Z})$ can also be lifted to $u^i\alpha \in H^{2(k+i)}_T(M; \mathbb{Z})$ for any $i \geq 0$, so we obtain a filtration

$$F_0(M) \subset F_1(M) \subset \cdots \subset H^*(M^T; \mathbb{Z}).$$

We will call this the equivariant filtration. The following proposition is immediate from the definition of the equivariant filtration.
Proposition 2.9. If $M$ satisfies the hypotheses of Proposition 2.8 and has vanishing odd cohomology, then the image of the inclusion

$$\psi : H^*_T(M; \mathbb{Z}) \to H^*_T(M^T; \mathbb{Z}) \cong H^*(M^T; \mathbb{Z}) \otimes \mathbb{Z}[u]$$

is equal to the Rees algebra of to the equivariant filtration, thus $\psi$ induces an isomorphism

$$H^*_T(M; \mathbb{Z}) \cong \text{Rees } H^*(M^T; \mathbb{Z})$$

of graded $\mathbb{Z}[u]$-algebras.

Remark 2.10. If one wants to drop the assumption that the odd cohomology vanishes, one can alternatively define a filtration of $H^*(M^T; \mathbb{Z})$ by taking the $k$th filtered piece to be the images of classes of degree $\leq k$ (rather than $2k$) in $H^*_T(M; \mathbb{Z})$. The Rees algebra of this filtration will be isomorphic to the algebra $H^*_T(M; \mathbb{Z}) \otimes_{\mathbb{Z}[u]} \mathbb{Z}[u^{1/2}]$, where now the Rees parameter corresponds to $u^{1/2}$ rather than $u$.

2.6 Classes represented by submanifolds

Suppose that $M$ is a manifold and $N \subset M$ is a closed submanifold of codimension $k$. A coorientation of $N$ (that is, a choice of orientation of the normal bundle) determines a cohomology class $[N] \in H^k(M; \mathbb{Z})$. One construction of this class is as follows. Let $U$ be a tubular neighborhood of $N$ in $M$, and let $\bar{U} := M/(M \setminus U)$. Then $\bar{U}$ is isomorphic to the Thom space of the normal bundle to $N$, and we therefore have the Thom isomorphism $H^*(N; \mathbb{Z}) \cong H^{*+k}(\bar{U}; \mathbb{Z})$. The class $[N]$ is obtained by pulling back the class $1 \in H^0(N; \mathbb{Z}) \cong H^k(\bar{U}; \mathbb{Z})$ to $M$.

If $T$ acts on both $M$ and $N$, then we define

$$[N]_T = [N_T] \in H^k(M_T; \mathbb{Z}) = H^k_T(M; \mathbb{Z}).$$

This construction has the following properties. The first four follow from the corresponding non-equivariant statements, while (v) follows from (iv) applied to the inclusion of $M$ into $M_T$.

(i) Reversing the coorientation of a submanifold negates the equivariant cohomology class.

(ii) If $N_1$ and $N_2$ are disjoint closed cooriented $T$-submanifolds, then $[N_1 \cup N_2]_T = [N_1]_T + [N_2]_T$.

(iii) If $N_1$ and $N_2$ are transverse closed cooriented $T$-submanifolds, then $[N_1 \cap N_2]_T = [N_1]_T \cdot [N_2]_T$.

(iv) If $N$ is a cooriented closed $T$-submanifold of $M$ and $f : M' \to M$ is a $T$-equivariant map that is transverse to $N$, then $f^*([N]_T) = [f^{-1}N]_T \in H^*(M'; \mathbb{Z})$.

(v) This construction is compatible with the forgetful homomorphism. That is, $\varphi([N]_T) = [N]$.

\footnote{We mean closed in the sense that $N$ is a closed subset of $M$. We do not mean to imply that $N$ is compact, which is what topologists sometimes mean by the phrase “closed manifold”.}
Example 2.11. Let $M = \mathbb{C}$ with the standard action of $T$. Then $M_T$ is homotopy equivalent to $BT$, so $H^*_T(M; \mathbb{Z}) \cong \pi^*(BT; \mathbb{Z}) \cong \mathbb{Z}[u]$. The class $[\{0\}]_T \in H^3_T(M; \mathbb{Z})$ is a generator, and can therefore be identified with $u$.

Example 2.12. Let $M = \mathbb{R}^3 \setminus \{0\} = \mathbb{R} \times \mathbb{C} \setminus \{(0,0)\}$. Let

$$e^+ := [\mathbb{R}_{>0} \times \{0\}]_T \in H^2_T(M; \mathbb{Z}) \quad \text{and} \quad e^- := [\mathbb{R}_{<0} \times \{0\}]_T \in H^2_T(M; \mathbb{Z}).$$

Note that $\varphi(e^+) = -\varphi(e^-)$ generates $H^2(M; \mathbb{Z})$, so $M$ is equivariantly formal over $\mathbb{Z}$. Since $e^+$ and $e^-$ are represented by disjoint submanifolds, we have $e^+ e^- = 0$.

Consider the projection $f : M \to \mathbb{C}$. Since $f$ is transverse to the submanifold $\{0\}$, we have the following identities in $H^2(M; \mathbb{Z})$:

$$u = f^* u = f^*(\{0\})_T = [f^{-1}(0)]_T = [\mathbb{R}_{>0} \times \{0\} \cup \mathbb{R}_{<0} \times \{0\}]_T = [\mathbb{R}_{>0} \times \{0\}]_T + [\mathbb{R}_{<0} \times \{0\}]_T = e^+ + e^-.$$

So the class $e^+ + e^-$ is in the kernel of $\varphi$, but it is not trivial. In fact, we have

$$H^*_T(M; \mathbb{Z}) \cong \mathbb{Z}[e^+, e^-]/\langle e^+ e^- \rangle.$$

Lemma 2.13. If $N_1, \ldots, N_r$ are (not necessarily transverse) closed cooriented $T$-submanifolds of $M$ with $N_1 \cap \cdots \cap N_r = \emptyset$, then $[N_1]_T \cdots [N_r]_T = 0$.

Proof. The class $[N_1]_T \cdots [N_r]_T$ is equal to the image of the class $[N_1]_T \otimes \cdots \otimes [N_r]_T$ under the composition

$$H^*_T(M; \mathbb{Z}) \otimes \cdots \otimes H^*_T(M; \mathbb{Z}) \to H^*_T(M^r; \mathbb{Z}) \xrightarrow{\Delta^*} H^*_T(M; \mathbb{Z}),$$

where $\Delta$ is the diagonal map. Let $\tilde{N}_i := \{(p_1, \ldots, p_r) \in M^r \mid p_i \in N_i\}$. Then the image of $[N_1]_T \otimes \cdots \otimes [N_r]_T$ in $H^*_T(M^r; \mathbb{Z})$ is equal to the product of the classes $[\tilde{N}_1]_T$, which is equal to zero because the submanifolds $\tilde{N}_i$ are pairwise transverse and their total intersection is empty. \hfill $\square$

3 Proof of Theorem 1.1

Let $V$ be a finite dimensional real vector space, $A$ a finite set of hyperplanes in $V$, and $K \subset A$ a convex open set. For any $H \in A$, we define the deletion $(A', K')$, which is a pair consisting of an arrangement and a convex open subset in $V$, by putting $A' = A \setminus \{H\}$ and $K' = K$. We also define the contraction $(A'', K'')$, which is a pair consisting of an arrangement and a convex open subset in $H$, by putting $A'' = \{H' \cap H \mid H' \in A'\}$ and $K'' = K \cap H$. Note that we have an open inclusion from $M_3(A, K)$ to $M_3(A', K')$, and the complement is equal to $M_3(A'', K'')$. The following lemma is standard in the case where $K = V$, and the proof in this more general setting is identical.

Lemma 3.1. We have a canonical isomorphism of graded abelian groups

$$H^i(M_3(A', K'), M_3(A, K); \mathbb{Z}) \cong H^{i-2}(M_3(A'', K''); \mathbb{Z}).$$
Proof. The key observation is that the normal bundle to $M_3(A', K')$ inside of $M_3(A', K')$ is a trivial bundle of rank 3. By the Tubluar Neighborhood Theorem, the Excision Theorem, and the Künneth Theorem, this implies that

$$H^i(M_3(A', K'), M_3(A, K); \mathbb{Z}) \cong H^{i-2}(M_3(A'', K''); \mathbb{Z}) \otimes H^2(\mathbb{R}^3, \mathbb{R}^3 \setminus \{0\}; \mathbb{Z})$$

$$\cong H^{i-2}(M_3(A'', K''); \mathbb{Z}).$$

This completes the proof.

Corollary 3.2. We have a map of long exact sequences

$$\cdots \to H^i(M_3(A'; \mathbb{Z}) \to H^i(M_3(A'; \mathbb{Z}) \to H^{i-2}(M_3(A''); \mathbb{Z}) \to H^{i+1}(M_3(A'; \mathbb{Z}) \to \cdots$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$\cdots \to H^i(M_3(A', K'; \mathbb{Z}) \to H^i(M_3(A, K); \mathbb{Z}) \to H^{i-2}(M_3(A'', K''); \mathbb{Z}) \to H^{i+1}(M_3(A', K'); \mathbb{Z}) \to \cdots.$$

Proof. The bottom row is obtained by applying Lemma 3.1 to the long exact sequence of the pair $(M_3(A', K'), M_3(A, K))$. The top row comes from the pair $(M_3(A'), M_3(A))$, which is the special case where $K = V$. The vertical maps are restriction maps.

Proposition 3.3. The restriction map $H^*(M_3(A'; \mathbb{Z}) \to H^*(M_3(A, K); \mathbb{Z})$ is surjective.

Proof. We proceed by induction on the cardinality of $A$. The base case is where $A$ is empty, in which case the result follows from the fact that $K$ is convex. If there is at least one hyperplane, then we may apply Corollary 3.2 and our inductive hypothesis tells us that two thirds of the downward arrows are surjective. The fact that the remaining downward arrows are surjective follows from a diagram chase.

For each hyperplane $H \in A$, let $f_H : V \to \mathbb{R}$ be a linear form with kernel $H$, and consider the induced map

$$g_H : M_3(A, K) \to \mathbb{R}^3 \setminus \{0\}.$$

Let

$$e^+_H := g^*_H(e^+) \quad \text{and} \quad e^-_H := g^*_H(e^-),$$

where $e^+, e^- \in H^2_T(\mathbb{R}^3 \setminus \{0\}; \mathbb{Z})$ are defined in Example 2.12.

Remark 3.4. We note that $f_H$ induces an orientation of $V/H$, and the classes $e^+_H \in H^2_T(M_3(A, K); \mathbb{Z})$ will eventually be identified with the images of the classes $e^+_H \in R_A$ via the isomorphisms of Theorems 1.1 and 1.4.

Proposition 3.5. The $T$-space $M_3(A, K)$ is equivariantly formal over $\mathbb{Z}$, and $H^*_T(M_3(A, K); \mathbb{Z})$ is generated as a $\mathbb{Z}[u]$-algebra by the classes $\{e^+_H | H \in A\}$.
Proof. By Proposition\textsuperscript{2.7}, it is sufficient to prove that $H^*(M(A, \mathcal{K}); \mathbb{Z})$ is generated as a ring by the classes $\{\varphi(e^\pm_H) \mid H \in \mathcal{A}\}$. These classes clearly extend to $H^*(M(A); \mathbb{Z})$ (because the maps $g_H$ extend), thus Proposition\textsuperscript{3.3} tells us that it is sufficient to consider the case where $\mathcal{K} = V$. This case is proved in \cite[Corollary 5.6]{dS01}.

Proof of Theorem\textsuperscript{1.1}. The first isomorphism follows from the definition of $\text{VG}(A, \mathcal{K})$ and the fact that $M_3(A, \mathcal{K})^T \cong M_1(A, \mathcal{K})$. Propositions\textsuperscript{2.9} and \textsuperscript{3.5} imply that we have an isomorphism of graded $\mathbb{Z}[u]$-algebras

$$H^*_T(M_3(A, \mathcal{K}); \mathbb{Z}) \cong \text{Rees} H^*(M_1(A, \mathcal{K}); \mathbb{Z}),$$

and Proposition\textsuperscript{2.6} tells us that setting $u = 0$ gives the isomorphism

$$H^*(M_3(A, \mathcal{K}); \mathbb{Z}) \cong \text{gr} H^*(M_1(A, \mathcal{K}); \mathbb{Z}).$$

The one subtlety is that the isomorphism coming from Proposition\textsuperscript{2.9} involves the Rees algebra of the equivariant filtration, and Theorem\textsuperscript{1.1} is about the Heaviside filtration. Thus we need to check that these two filtrations coincide.

By definition, the $k$th piece of the Heaviside filtration consists of classes that can be expressed as polynomials of degree at most $k$ in the Heaviside functions. On the other hand, the second half of Proposition\textsuperscript{3.5} says that the $k$th piece of the equivariant filtration consists of classes that can be expressed as polynomials of degree at most $k$ in the restrictions of $\{e^\pm_H \mid H \in \mathcal{A}\}$ to $M_1(A, \mathcal{K})$, with $u$ specialized to 1. It is therefore sufficient to observe that the restriction of $e^\pm_H$ is precisely the Heaviside function that takes the value 1 on $H^\pm$ and 0 on $H^\mp$.

4 Proof of Theorem\textsuperscript{1.4}

Our main focus in this section is on the $\mathbb{Z}[u]$-algebra homomorphism $R_A \to \text{Rees} \text{VG}(A, \mathcal{K})$ taking $e^\pm_H$ to $u$ times the corresponding Heaviside function. It is clear that this homomorphism is surjective, and also that $e_S$ is in the kernel for all $S \in \mathcal{S}$.

Remark 4.1. It is also possible to see this by using Theorem\textsuperscript{1.1} to interpret $\text{Rees} \text{VG}(A, \mathcal{K})$ as the equivariant cohomology ring $H^*_T(M_3(A, \mathcal{K}); \mathbb{Z})$. From this perspective, our homomorphism takes $e^\pm_H$ to the class $[\pm g_H^{-1} \mathbb{R}_{>0}]_T \in H^2_T(M_3(A, \mathcal{K}); \mathbb{Z})$. Surjectivity follows from Proposition\textsuperscript{3.5} and the fact that $e_S$ is in the kernel follows from Lemma\textsuperscript{2.13}.

If $S, \bar{S} \in \mathcal{S}$, then $e_S$ and $e_{\bar{S}}$ are both in the kernel, and therefore so is their difference. Since $\text{Rees} \text{VG}(A, \mathcal{K})$ is torsion-free, $f_S$ must also be contained in the kernel. We thus obtain an induced surjective homomorphism

$$R_A / I_{A, \mathcal{K}} + J_{A, \mathcal{K}} \to \text{Rees} \text{VG}(A, \mathcal{K}).$$

Our goal in this section is to prove that this homomorphism is also injective; the rest of Theorem\textsuperscript{1.4} will follow from specializing $u$ to 0 or 1.
Choose a linear ordering \(<\) on \(A\) and a degree monomial order \(\prec\) on \(\mathbb{Z}[e_H^+]|_{H \in A} \cong R_{A,1}\) such that \(e_H^+ < e_{H'}^+\) if and only if \(H < H'\). For any polynomial \(f \in R_{A,1}\), we will write \(\text{in}(f)\) to denote its leading monomial. For any signed set \(S = (S^+, S^-)\), let \(\min(S)\) denote the minimal element of \(S^+ \sqcup S^-\), and let

\[
\hat{S} := S^+ \sqcup S^- \setminus \{\min(S)\} \subset A.
\]

Note that \(\hat{S}\) is not a signed set, but rather an ordinary subset of \(A\). Recall that we defined elements \(e_S, f_S \in R_A\); we now use the same notation to denote the images of these elements in \(R_{A,1}\). We have

\[
\text{in}(e_S) = \prod_{H \in S^+ \cup S^-} e_H^+ \quad \text{and} \quad \text{in}(f_S) = \pm \prod_{H \in \hat{S}} e_H^+,
\]

where we have a minus sign in the second set of relations if and only if \(\min(S) \in S^-\). We will say that a monomial in the variables \(\{e_H^+\}\) is \(\mathcal{K}\)-nbc if it is square-free and is not a multiple of \(\text{in}(e_S)\) for any \(S \in S\) or of \(\text{in}(f_S)\) for any \(S\) such that \(S, \hat{S} \in S\). When \(\mathcal{K} = V\), this gives the usual notion of an nbc monomial, where nbc stands for “no broken circuit”. This definition is motivated by the observation that \(R_A/I_{A,K} + J_{A,K}\) is spanned as a \(\mathbb{Z}[u]\)-module by the classes of \(\mathcal{K}\)-nbc monomials.

An intersection of some subset of the hyperplanes in \(A\) is called a flat of \(A\). We define \(\mathcal{L}(A, \mathcal{K})\) to be the set of flats whose intersection with \(\mathcal{K}\) is nonempty. We define the support of a monomial to be the intersection of those \(H \in A\) for which \(e_H^+\) divides the monomial. This is by definition a flat, and one can easily check that the support of a \(\mathcal{K}\)-nbc monomial is an element of \(\mathcal{L}(A, \mathcal{K})\).

**Lemma 4.2.** The number of \(\mathcal{K}\)-nbc monomials is equal to the number of connected components of \(M_1(A, \mathcal{K})\).

**Proof.** Let \(\mu\) denote the Möbius function of the poset \(\mathcal{L}(A, \mathcal{K})\), where the order is given by reverse inclusion. The number of connected components of \(M_1(A, \mathcal{K})\) is \(\sum_{F \in \mathcal{L}(\mathcal{K})} \mu(V, F)|\text{[Zas77, Theorem 3.2(A) and Example A]}|\). Since the lower interval \([V, F]\) is a geometric lattice for any \(F \in \mathcal{L}(A, \mathcal{K})\), a theorem of Rota \([\text{Sag95, Theorem 1.1]}\) says that \(\mu(V, F)|\) is equal to the number of \(\mathcal{K}\)-nbc monomials with support \(F\). Taking the sum over all \(F\), we obtain the number of \(\mathcal{K}\)-nbc monomials. \(\square\)

**Proof of Theorem 1.4.** As observed above, it is sufficient to show that the surjection from the ring \(R_A/I_{A,K} + J_{A,K}\) to \(\text{Rees VG}(A, \mathcal{K})\) is in fact an isomorphism. Let \(r\) be the number of \(\mathcal{K}\)-nbc monomials, which is also equal to the number of connected components of \(M_1(A, \mathcal{K})\) by Lemma 4.2. Then we have \(\mathbb{Z}[u]\)-module surjections

\[
\mathbb{Z}[u]^r \rightarrow R_A/I_{A,K} + J_{A,K} \rightarrow \text{Rees VG}(A, \mathcal{K}),
\]

where the first map takes the \(r\) basis vectors to the \(r\) \(\mathcal{K}\)-nbc monomials. By Theorem 1.1 and Propositions 2.6 and 3.5, the Rees algebra \(\text{Rees VG}(A, \mathcal{K})\) is a free \(\mathbb{Z}[u]\)-module of rank \(r\). Any surjective homomorphism from one free \(\mathbb{Z}[u]\)-module to another of the same rank is an isomorphism, hence both homomorphisms must be isomorphisms. \(\square\)

\(^4\)In general, Rees algebras are torsion-free but not necessarily free. The fact that it is free here is special.
References


