The categorical graph minor theorem

Dane Miyata, Nicholas Proudfoot, and Eric Ramos
Department of Mathematics, University of Oregon, Eugene, OR 97403

Abstract. We define the graph minor category and prove that the category of contravariant representations of the graph minor category over a Noetherian ring is locally Noetherian. This can be regarded as a categorification of the Robertson–Seymour graph minor theorem. In addition, we generalize Sam and Snowden's Gröbner theory of categories to the setting of pairs consisting of a category along with a functor to sets, and we apply this theory to the edge functor on the graph minor category. As an application, we study homology groups of unordered configuration spaces of graphs, improving upon various finite generation results in this subject.

1 Introduction

We study a category G whose objects are finite connected graphs and whose morphisms are built out of automorphisms, deletions, and contractions. A precise definition of this category will appear in Section 3.1, but the main property that we want to stress for the purposes of this introduction is that there exists a morphism from a connected graph G to another connected graph G' if and only if G' is isomorphic to a minor of G.

1.1 Noetherianity

Let C be a category and k a ring. We define $\mathrm{Rep}_k(\mathrm{C})$ to be the category of functors from C to the category of k-modules. A module $\mathcal{M} \in \mathrm{Rep}_k(\mathrm{C})$ is called **finitely generated** if there exist finitely many objects $c_1, \ldots, c_r$ of C along with elements $v_i \in \mathcal{M}(c_i)$ such that, for any object c of C, $\mathcal{M}(c)$ is spanned over k by the images of the elements v_i along the maps induced by all possible morphisms $\varphi_i : c_i \to c$. If every submodule of $\mathcal{M}$ is finitely generated, then $\mathcal{M}$ is said to be **Noetherian**. If every finitely generated module is Noetherian, the category $\mathrm{Rep}_k(\mathrm{C})$ is said to be **locally Noetherian**.

Sam and Snowden have developed powerful machinery for proving that module categories are locally Noetherian. They define what it means for C to be **quasi-Gröbner**, and they show that, if C is quasi-Gröbner, then $\mathrm{Rep}_k(\mathrm{C})$ is locally Noetherian for any left-Noetherian ring k [SS17]. Examples of quasi-Gröbner categories include the following:

- The category FI of finite sets with injections [SS17, Theorem 7.1.4].
- The category $\mathrm{FS}^{\mathrm{op}}$, where FS is finite sets with surjections [SS17, Theorem 8.1.2].
- For any finite field $\mathbb{F}$, the category $\mathrm{VI}_{\mathbb{F}}$ of finite dimensional vector spaces over $\mathbb{F}$ with linear inclusions [SS17, Theorem 8.3.1].
- The full subcategory $\mathrm{G}_g^{\mathrm{op}} \subset \mathrm{G}^{\mathrm{op}}$ consisting of graphs with fixed genus g [PRa, Theorem 1.1].

Remark 1.1. The fact that $\mathrm{Rep}_k(\mathrm{FI})$ is locally Noetherian when k is a field of characteristic zero was originally proved by Church–Ellenberg–Farb [CEF15, Theorem 2.60] and independently by Snowden [Sno13, Theorem 2.3]. For an arbitrary Noetherian ring k, it was originally proved by Church–Ellenberg–Farb–Nagpal [CEFN14, Theorem A]. The fact that $\mathrm{Rep}_k(\mathrm{VI}_{\mathbb{F}})$ is locally Noetherian for an arbitrary Noetherian ring k was originally proved by Putman and Sam [PS17, Theorem A].

Our first theorem is the analogous result for the category G^{op}.

Theorem 1.2. *The category* G^{op} *is quasi-Gröbner.*

Remark 1.3. Theorem 1.2 is a strengthening of the graph minor theorem of Robertson and Seymour [RS04], which they proved over a series of 20 papers. That theorem says that, given a sequence $(G_i \mid i \in \mathbb{N})$ of graphs, it is possible to find $i < j$ such that G_i is isomorphic to a minor of G_j. To see that this is implied by Theorem 1.2, let $\mathcal{M} \in \mathrm{Rep}_{\mathrm{C}}(\mathrm{G}^{\mathrm{op}})$ be the module that assigns the 1-dimensional vector space C to every graph and the identity map to every morphism. Given a list list $\{G_i \mid i \in \mathbb{N}\}$, let $\mathcal{N} \subset \mathcal{M}$ be the submodule that assigns C to any graph that has some G_i as a minor and 0 to any graph that does not. If the graph minor theorem failed, then $\mathcal{N}$ would not be finitely generated. For this reason, we refer to Theorem 1.2 as the **categorical graph minor theorem**. Note that we do not claim that Theorem 1.2 provides a new proof of the graph minor theorem; indeed, the graph minor theorem is one of the main ingredients of our proof of Theorem 1.2. However, this observation implies that Theorem 1.2 is at least as nontrivial as the graph minor theorem.

Remark 1.4. As we will explain below (Propositions 3.9, 3.10, and 3.11), the statements that FI, $\mathrm{FS}^{\mathrm{op}}$, $\mathrm{VI}_{\mathbb{F}}$, and $\mathrm{G}_g^{\mathrm{op}}$ are quasi-Gröbner are all implied by Theorem 1.2. Philosophically, this can be attributed to the universality of the ordinary graph minor theorem. The proofs that FI, $\mathrm{FS}^{\mathrm{op}}$, and $\mathrm{VI}_{\mathbb{F}}$ are quasi-Gröbner all rely on Higman's lemma, and the proof that $\mathrm{G}_g^{\mathrm{op}}$ is quasi-Gröbner relies on Kruskal's tree theorem; both Higman's lemma and Kruskal's tree theorem can be regarded as special cases of the graph minor theorem.

1.2 Modules over algebras over categories

Let C be a category equipped with a functor $S : \mathrm{C} \to \mathrm{FI}$ and let k be a commutative ring. There is a natural functor from C to k-algebras taking an object c to the polynomial ring

$$\mathcal{R}(c) := k[x_e \mid e \in S(c)].$$

Equivalently, we can think of $\mathcal{R} \in \mathrm{Rep}_k(\mathrm{C})$ as a module equipped with a product $\mathcal{R} \otimes \mathcal{R} \to \mathcal{R}$ that is both associative and commutative. Let $\mathrm{Rep}_k(\mathrm{C}, S)$ be the category of modules over $\mathcal{R}$. Formally, an object of $\mathrm{Rep}_k(\mathrm{C}, S)$ is an object $\mathcal{M} \in \mathrm{Rep}_k(\mathrm{C})$ along with a multiplication $\mathcal{R} \otimes \mathcal{M} \to \mathcal{M}$ such that the two natural maps $\mathcal{R} \otimes \mathcal{R} \otimes \mathcal{M} \to \mathcal{M}$ coincide. More intuitively, an object $\mathcal{M}$ of $\mathrm{Rep}_k(\mathrm{C}, S)$ consists of an $\mathcal{R}(c)$-module $\mathcal{M}(c)$ for each object c of C and an $\mathcal{R}(c)$-module map $\mathcal{M}(c) \to \mathcal{M}(c')$ for

each morphism $\varphi : c \to c'$, where $\mathcal{M}(c')$ is an $\mathcal{R}(c)$-module via the ring homomorphism $\mathcal{R}(c) \to \mathcal{R}(c')$ induced by φ.

A module $\mathcal{M} \in \mathrm{Rep}_k(\mathrm{C}, S)$ is called **finitely generated** if there exist finitely many objects $c_1, \ldots, c_r$ of C along with elements $v_i \in \mathcal{M}(c_i)$ such that, for any object c of C, $\mathcal{M}(c)$ is spanned over $\mathcal{R}(c)$ by the images of the elements v_i along the maps induced by all possible morphisms $\varphi_i : c_i \to c$. If every submodule of $\mathcal{M}$ is finitely generated, then $\mathcal{M}$ is said to be **Noetherian**. If every finitely generated module is Noetherian, the category $\mathrm{Rep}_k(\mathrm{C}, S)$ is said to be **locally Noetherian**. We define what it means for the pair (C, S) to be **quasi-Gröbner**, and we prove the following generalization of the result of Sam and Snowden.

Theorem 1.5. *Let* C *be a category and* $S : \mathrm{C} \to \mathrm{FI}$ *a functor. If the pair* (C, S) *is quasi-Gröbner, then* $\mathrm{Rep}_k(\mathrm{C}, S)$ *is locally Noetherian for any Noetherian commutative ring* k*.*

Remark 1.6. Theorem 1.5 is motivated by the work of Nagel and Römer [NR19]. Though they do not make these definitions in the same generality, they essentially prove that the pair $(\mathrm{FI}, \mathrm{id})$ is quasi-Gröbner, and they use this result to show that $\mathrm{Rep}_k(\mathrm{FI}, \mathrm{id})$ is locally Noetherian for any Noetherian commutative ring k. Moreover, they show that if $S_d : \mathrm{FI} \to \mathrm{FI}$ is the functor taking a set T to the set of unordered d-tuples of distinct elements of T, then the pair (FI, S_d) is quasi-Gröbner and the category $\mathrm{Rep}_k(\mathrm{FI}, \mathrm{id})$ is locally Noetherian if and only if $d \leq 1$ [NR19, Proposition 4.8].

Our main application is to the edge functor on G^{op}. If G' is a minor of G, there is a natural inclusion from the edges of G' to the edges of G, which gives us a functor $E : \mathrm{G}^{\mathrm{op}} \to \mathrm{FI}$.

Theorem 1.7. *The pair* $(\mathrm{G}^{\mathrm{op}}, E)$ *is quasi-Gröbner.*

Remark 1.8. Just as the statement that FI is quasi-Gröbner can be seen as a consequence of the statement that G^{op} is quasi-Gröbner, the statement that the pair $(\mathrm{FI}, \mathrm{id})$ is quasi-Gröbner can be seen as a consequence of Theorem 1.7; see Proposition 3.9.

Remark 1.9. Note that we have $\mathrm{Rep}_k(\mathrm{C}) = \mathrm{Rep}_k(\mathrm{C}, \emptyset)$, where $\emptyset : \mathrm{C} \to \mathrm{FI}$ is the constant functor that takes every object of C to the empty set. Furthermore, the category C is quasi-Gröbner if and only if the pair $(\mathrm{C}, \emptyset)$ is quasi-Gröbner. Thus our results say that the pair $(\mathrm{G}^{\mathrm{op}}, S)$ is quasi-Gröbner for both $S = \emptyset$ (Theorem 1.2) and $S = E$ (Theorem 1.7).

1.3 Homology of configuration spaces of graphs

Given a graph G and a natural number n, the ***n*-stranded unordered configuration space of *G*** is the topological space

$$U_n(G) := \big\{(x_1, \ldots, x_n) \in G^n \;\big|\; x_i \neq x_j\big\}/\Sigma_n.$$

The homology groups of these spaces have been extensively studied in settings both theoretical [Abr00, ADCK19, KP12] and applied [Far08].

Let $\mathcal{R}$ be the algebra associated with the pair $(\mathrm{G}^{\mathrm{op}}, E)$ with coefficients in $\mathbb{Z}$. That is, for each graph G, $\mathcal{R}(G)$ is the polynomial ring $\mathbb{Z}[x_e \mid e \in E(G)]$. Given an edge $e \in E(G)$, there is a **stabilization map** $\sigma_{e,n} : U_n(G) \to U_{n+1}(G)$ that adds a a point to the edge e [ADCK]. If we fix a homological degree i, these maps induce a graded $\mathcal{R}(G)$-module structure on

$$\mathcal{H}_i(G) := \bigoplus_{n \in \mathbb{N}} H_i\big(U_n(G); \mathbb{Z}\big),$$

where the variable x_e acts by the map in homology induced by the stabilization maps $\sigma_{e,n}$. Furthermore, any morphism $\varphi : G \to G'$ in G induces a graded $\mathcal{R}(G')$-module homomorphism [ADCK19, Lemma C.7]

$$\varphi^* : \mathcal{H}_i(G') \to \mathcal{H}_i(G).$$

Equivalently, there is a canonical graded module $\mathcal{H}_i \in \mathrm{Rep}_{\mathbb{Z}}(\mathrm{G}^{\mathrm{op}}, E)$ that takes a graph G to the graded $\mathcal{R}(G)$-module $\mathcal{H}_i(G)$.

Remark 1.10. If φ is an automorphism or a deletion, the geometric meaning of the map φ^* is clear. If φ is a contraction, it is less obvious why it should induce a map on homology of configuration spaces from the smaller graph to the bigger graph. There is no canonical map at the level of spaces themselves, but there is a canonical homotopy class of maps. One way to see this is to observe that there is a map from the fundamental group of $U_n(G')$ to the fundamental group of $U_n(G)$ obtained by lifting paths in a canonical way, and then use the fact that these spaces are Eilenberg-MacLane spaces [Abr00] to conclude that this induces a homotopy class of maps of spaces.

Our next theorem says that the aforementioned module is finitely generated.

Theorem 1.11. *For any natural number i, the module $\mathcal{H}_i \in \mathrm{Rep}_{\mathbb{Z}}(\mathrm{G}^{\mathrm{op}}, E)$ is finitely generated.*

Remark 1.12. Theorem 1.11 generalizes many previous results in the literature. The fact that $H_i(G)$ is a finitely generated module over $\mathcal{R}(G)$ for each graph G is due to An, Drummond-Cole, and Knudsen [ADCK, Theorem 1.1]. The idea to fix the homological degree i and the number n of points and to vary the graph first appeared in [Lüt] and was further explored in [RW, PRb, PRa]. The idea of using the category $\mathrm{Rep}_{\mathbb{Z}}(\mathrm{FI}, \mathrm{id})$ of Nagel and Römer to simultaneously vary the graph and the number of points was exploited in [Ram]. Each of these papers contains a finite generation result that can be regarded as a special case or a consequence of Theorem 1.11.

We know from the work of Ko and Park that the only torsion that can appear in the group $H_1\big(U_n(G); \mathbb{Z}\big)$ is 2-torsion [KP12, Corollary 3.6]. Furthermore, this torsion carries extremely interesting combinatorial information about the graph G: it is trivial if and only if $n \geq 2$ and G is planar. One concrete consequence of Theorems 1.5, 1.7, and 1.11 is that similar bounds on torsion exponents must exist in higher homological degrees.

Corollary 1.13. *For any natural number i, there exists a natural number ϵ_i such that for every graph G and every natural number n, the torsion part of $H_i\big(U_n(G); \mathbb{Z}\big)$ is annihilated by ϵ_i.*

Proof. Let $\mathcal{T}_i \subset \mathcal{H}_i$ be the $\mathcal{R}$-submodule consisting of all torsion classes. Theorem 1.11 says that $\mathcal{H}_i$ is finitely generated, thus Theorems 1.5 and 1.7 imply that $\mathcal{T}_i$ is finitely generated, as well. We may then take ϵ_i to be the least common multiple of the exponents of the classes that generate $\mathcal{T}_i$. $\square$

Remark 1.14. As stated above, Ko and Park prove that we may take $\epsilon_1 = 2$. It is known that 2-torsion can exist in arbitrary homological degree, so ϵ_i must be even for all i. It is not known whether or not we can take $\epsilon_i = 2$ for all i. That is, it is not known whether or not there exists a graph with nontrivial odd torsion or 4-torsion in the homology of one of its unordered configuration spaces.

As a byproduct of the proof of Theorem 1.11, we obtain the following result. For a nonempty connected graph G, let $e(G)$ denote the number of edges of G, and let $g(G)$ denote the genus of G, which is defined to be the number of edges minus the number of vertices plus one.

Proposition 1.15. *For any natural number i, there exists a natural number α_i such that for every connected graph G and every natural number n,*

$$\dim H_i\big(U_n(G);\mathbb{Z}\big) \leq \alpha_i\, e(G)^{i+n+g(G)}.$$

Remark 1.16. An analogue of Corollary 1.13 appears in [PRa, Corollary 1.3] and an analogue of Proposition 1.15 appears in [PRa, Theorem 1.2 and Proposition 4.3]. In those two results, the constants ϵ_i and α_i depend on i, n, and the genus g, whereas we now show that they can be taken to depend only on i.

Acknowledgments: DM and NP are supported by NSF grant DMS-1954050. ER is supported by NSF grant DMS-1704811. ER would like to thank Melody Chan for numerous conversations that aided in the development of this paper.

2 Modules over algebras over categories

In this section we define Gröbner and quasi-Gröbner pairs and prove Theorem 1.5.

2.1 Finite generation and Noetherianity

We begin with some basic facts about finitely generated modules and Noetherian modules, the proofs of which are completely standard. Let C be a category, $S : \mathrm{C} \to \mathrm{FI}$ a functor, and k a commutative ring. For any object c of C, define the **principal projective** $\mathcal{P}_c \in \mathrm{Rep}_k(\mathrm{C}, S)$ to be the module that takes an object c' to the free $\mathcal{R}(c')$-module spanned by the set $\mathrm{Hom}_{\mathrm{C}}(c, c')$, with maps defined via composition.

Lemma 2.1. *A module $\mathcal{M} \in \mathrm{Rep}_k(\mathrm{C}, S)$ is finitely generated if and only if there exists a surjection*

$$\bigoplus_{i=1}^{r} \mathcal{P}_{c_i} \twoheadrightarrow \mathcal{M}$$

for some list of (not necessarily distinct) objects $c_1, \ldots, c_r$ of C.

Proof. Suppose that $c_1, \ldots, c_r$ are objects of C and $v_i \in \mathcal{M}(c_i)$ for all i. These classes generate $\mathcal{M}$ if and only if the map

$$\bigoplus_{i=1}^{r} \mathcal{P}_{c_i} \to \mathcal{M}$$

taking $\mathrm{id}_{c_i} \in \mathcal{P}_{c_i}(c_i)$ to v_i is surjective. □

Recall from the introduction that a module $\mathcal{M} \in \mathrm{Rep}_k(\mathrm{C}, S)$ is **Noetherian** if every submodule of $\mathcal{M}$ is finitely generated.

Lemma 2.2. *A module $\mathcal{M}$ is Noetherian if and only if every ascending chain of submodules of $\mathcal{M}$ eventually stabilizes.*

Proof. Suppose that $\mathcal{M}$ is Noetherian and $(\mathcal{N}_i \mid i \in \mathbb{N})$ is an ascending chain of submodules of $\mathcal{M}$. Let $\mathcal{N} := \bigcup_{i \in \mathbb{N}} \mathcal{N}_i \subset \mathcal{M}$. Since $\mathcal{M}$ is Noetherian, $\mathcal{N}$ is finitely generated. If we choose i large enough so that $\mathcal{N}_i$ contains all of the finitely many generating classes, then we have $\mathcal{N}_i = \mathcal{N}$.

Conversely, suppose that $\mathcal{M}$ has a submodule $\mathcal{N} \subset \mathcal{M}$ that is not finitely generated. We will define an ascending chain of finitely generated submodules $(\mathcal{N}_i \mid i \in \mathbb{N})$ as follows. Let $\mathcal{N}_0 = 0$. Once we have defined $\mathcal{N}_i$, the fact that $\mathcal{N}_i$ is finitely generated means that $\mathcal{N}_i \subsetneq \mathcal{N}$, so we may choose an object c_i of C and an element $v_i \in \mathcal{N}(c) \setminus \mathcal{N}_i(c)$. Let $\mathcal{N}_i$ be the smallest submodule of $\mathcal{N}$ containing both $\mathcal{N}_i$ and v_i. This chain of submodules clearly does not stabilize. □

Lemma 2.3. *Suppose that $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ is short exact sequence in $\mathrm{Rep}_k(\mathrm{C}, S)$. Then $\mathcal{M}$ is Noetherian if and only if both $\mathcal{M}'$ and $\mathcal{M}''$ are Noetherian.*

Proof. If $\mathcal{M}$ is Noetherian, then $\mathcal{M}'$ is Noetherian by definition. If $\mathcal{N}'' \subset \mathcal{M}''$ is a submodule, let $\mathcal{N} \subset \mathcal{M}$ be the preimage of $\mathcal{N}''$ in $\mathcal{M}$. Since $\mathcal{M}$ is Noetherian, $\mathcal{N}$ is finitely generated, thus so is $\mathcal{N}''$ by Lemma 2.1.

Conversely, suppose that both $\mathcal{M}'$ and $\mathcal{M}''$ are Noetherian, and let $(\mathcal{N}_i \mid i \in \mathbb{N})$ be an ascending chain of submodules of $\mathcal{M}$. For each i, let $\mathcal{N}'_i := \mathcal{N}_i \cap \mathcal{M}'$ and let $\mathcal{N}''_i$ be the image of $\mathcal{N}_i$ in $\mathcal{M}''$. Since $\mathcal{M}'$ and $\mathcal{M}''$ are both Noetherian, Lemma 2.2 tells us that there is an index n such that, for all $i > n$, $\mathcal{N}'_i = \mathcal{N}'_{i+1}$ and $\mathcal{N}''_i = \mathcal{N}''_{i+1}$. We can then conclude that $\mathcal{N}_i = \mathcal{N}_{i+1}$ by applying the Five Lemma to the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{N}'_i \arrow[r] \arrow[d, "="'] & \mathcal{N}_i \arrow[r] \arrow[d] & \mathcal{N}''_i \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \mathcal{N}'_{i+1} \arrow[r] & \mathcal{N}_{i+1} \arrow[r] & \mathcal{N}''_{i+1} \arrow[r] & 0
\end{tikzcd}$$

Thus $\mathcal{M}$ satisfies the ascending chain condition, and is therefore Noetherian by Lemma 2.2. □

2.2 Gröbner pairs

Let OI be the category whose objects are totally ordered finite sets and whose morphisms are ordered inclusions, and let $\Psi : \mathrm{OI} \to \mathrm{FI}$ be the functor that forgets the order on a finite set. Let D be an essentially small category and $T : \mathrm{D} \to \mathrm{OI}$ any functor. The purpose of this section is to define what it means for the pair (D, T) to be Gröbner.

A **quartet** for the pair (D, T) is a quadruple $\mu = (d, d', \varphi, m)$, where d and d' are objects of D, $\varphi : d \to d'$ is a morphism, and $m : T(d') \to \mathbb{N}$ is a map of sets. For any morphism $\psi : d' \to d''$ in D, we will write $T(\psi) : T(d') \to T(d'')$ for the induced morphism in OI, and we will write

$$\psi(\mu) := (d, d'', \psi \circ \varphi, m_\psi),$$

where m_ψ is determined by the conditions that $m_\psi \circ T(\psi) = m$ and m_ψ is identically zero outside of the image of $T(\psi)$. For any map $n : T(d') \to \mathbb{N}$, we will write

$$\mu + n := (d, d', \varphi, m + n).$$

If $\mu_1 = (d, d_1', \varphi_1, m_1)$ and $\mu_2 = (d, d_2', \varphi_2, m_2)$, we say that $\mu_1 \leq \mu_2$ if there exists a morphism $\psi : d_1' \to d_2'$ and a map $n : T(d'') \to N$ such that $\mu_2 = \psi(\mu_1) + n$.

Remark 2.4. The motivation for these definitions is that, once we choose a commutative ring k, the quartet μ determines a monomial

$$x^m := \prod_{a \in T(d')} x_a^{m(a)} \in \mathcal{R}(d')$$

along with an element

$$b_\mu := x^m \cdot \varphi \in \mathcal{P}_d(d') \in \mathrm{Rep}_k(\mathrm{D}, \Psi \circ T).$$

Then $\mu_1 \leq \mu_2$ if and only if φ_2 factors through φ_1 via a map ψ and we have

$$b_{\mu_2} = x^n \psi(b_{\mu_1})$$

for some monomial $x^n \in \mathcal{R}(d_2')$.

We say that μ_1 and μ_2 are **equivalent** if $\mu_1 \leq \mu_2 \leq \mu_1$. For each object d of D, let $|\mathrm{D}_d^T|$ denote the poset of equivalence classes of quartets with first coordinate d. Given a quartet $\mu = (d, d', \varphi, m)$, we will write $[\mu]$ to denote its equivalence class in $|\mathrm{D}_d^T|$. A well-order $\preceq$ of $|\mathrm{D}_d^T|$ is called **admissible** if, given two quartets $\mu_1 = (d, d', \varphi_1, m_1)$ and $\mu_2 = (d, d', \varphi_2, m_2)$ with the same source and target along with a morphism $\psi : d' \to d''$ and a map $n : T(d'') \to \mathbb{N}$, we have

$$[\mu_1] \preceq [\mu_2] \implies [\psi(\mu_1) + n] \preceq [\psi(\mu_2) + n].$$

We say that the pair (D, T) satisfies **property (G1)** if, for every object d of D, the poset $|\mathrm{D}_d^T|$ admits an admissible well-order. A poset P is said to be **Noetherian** if, for any sequence

$(p_i \mid i \in \mathbb{N})$ in P, there exist natural numbers $i < j$ such that $p_i \leq p_j$. We say that the pair (D, T) satisfies **property (G2)** if, for every object d of D, the poset $|\mathrm{D}_d^T|$ is Noetherian. The category D is said to be **directed** if, for any object d of D, the only morphism from d to d is the identity. We call the pair (D, A) **Gröbner** if D is directed and (D, A) satisfies properties (G1) and (G2).

Remark 2.5. Property (G1) for the pair $(\mathrm{D}, \emptyset)$ is equivalent to property (G1) for D as defined in [SS17, Section 1.1], and similarly property (G2) for the pair $(\mathrm{D}, \emptyset)$ is equivalent to property (G2) for D. Thus a directed category D is Gröbner in the sense of [SS17] if and only if the pair $(\mathrm{D}, \emptyset)$ is Gröbner.

The following Proposition says that the functor T does not add anything interesting to property (G1). In other words, the distinction between a Gröbner category and a Gröbner pair lies entirely in the property (G2).

Proposition 2.6. *The pair* (D, T) *satisfies property (G1) if and only if the pair* $(\mathrm{D}, \emptyset)$ *satisfies property (G1).*

Proof. If $\preceq$ is an admissible order of $|\mathrm{D}_d^T|$, then restriction to quartets with $m = 0$ gives an admissible order of $|\mathrm{D}_d^\emptyset|$. Conversely, if we have an admissible order of $|\mathrm{D}_d^\emptyset|$, we can compare the classes of two quartets $\mu_1 = (d, d_1', \varphi_1, m_1)$ and $\mu_2 = (d, d_2', \varphi_2, m_2)$ for (D, T) by first comparing the classes of the quartets $(d, d_1', \varphi_1, 0)$ and $(d, d_2', \varphi_2, 0)$ for $(\mathrm{D}, \emptyset)$ and then, if they are equal, breaking the tie by comparing m_1 and m_2 lexicographically. □

2.3 Gröbner bases

Let D be an essentially small category and $T : \mathrm{D} \to \mathrm{OI}$ a functor such that the pair (D, T) is Gröbner, and choose an admissible well-order $\preceq$ of $|\mathrm{D}_d^T|$ for each object d of D as in the definition of property (G1). For any pair of objects d and d' in D, let $Q_{d,d'}$ be the set of quartets of the form $\mu = (d, d', \varphi, m)$. The fact that D is directed implies that the natural map from $Q_{d,d'}$ to $|\mathrm{D}_d^T|$ is injective, thus $Q_{d,d'}$ is well-ordered by $\preceq$.

Fix a commutative ring k, so that we may define the representation category $\mathrm{Rep}_k(\mathrm{D}, \Psi \circ T)$. For any nonzero element

$$p \;=\; \sum_{\mu \in Q_{d,d'}} \lambda_\mu b_\mu \;\in\; \mathcal{P}_d(d'),$$

we define the **leading quartet** $\mathrm{LQ}(p)$ to be the maximal μ with respect to the well-order $\preceq$ such that the coefficient $\lambda_\mu \in k$ is nonzero. If $\mu = \mathrm{LQ}(p)$, we define the **leading term** $\mathrm{LT}(p) := \lambda_\mu b_\mu \in \mathcal{P}_d(d')$ and the **leading coefficient** $\mathrm{LC}(p) := \lambda_\mu \in k$.

Lemma 2.7. *Suppose we have a morphism* $\psi : d' \to d''$*, a map* $n : T(d') \to \mathbb{N}$*, and an element* $0 \neq p \in \mathcal{P}_d(d')$*. Then*

$$\mathrm{LT}(x^n \psi(p)) = x^n \psi(\mathrm{LT}(p)).$$

Proof. This is precisely the definition of admissibility of the well-order $\preceq$. □

Given a submodule $\mathcal{N} \subset \mathcal{P}_d$, we define a subset

$$\mathrm{LQ}(\mathcal{N}) := \left\{ [\mathrm{LQ}(p)] \;\middle|\; 0 \neq p \in \mathcal{N}(d') \right\} \subset |\mathrm{D}_d^T|.$$

For each object d' of D, we define

$$\mathrm{LT}(\mathcal{N})(d') := \{0\} \cup \left\{ \mathrm{LT}(p) \mid 0 \neq p \in \mathcal{N}(d') \subset \mathcal{P}_d(d') \right\}.$$

For each quartet $\mu = (d, d', \varphi, m)$, we define the ideal

$$\mathrm{LC}(\mathcal{N}, \mu) := \{0\} \cup \left\{ \mathrm{LC}(p) \mid 0 \neq p \in \mathcal{N}(d') \text{ and } \mathrm{LQ}(p) = \mu \right\} \subset k.$$

Lemma 2.7 implies that $\mathrm{LT}(\mathcal{N}) \subset \mathcal{P}_d$ is a submodule and that we have an inclusion of ideals $\mathrm{LC}(\mathcal{N}, \mu_2) \subset \mathrm{LC}(\mathcal{N}, \mu_1)$ whenever $[\mu_1] \leq [\mu_2]$.

Suppose we are given a finite set $B = \{(d'_1, p_1), \ldots, (d'_r, p_r)\}$ of pairs with $0 \neq p_i \in \mathcal{N}(d'_i)$ for all i. We say that B is a **Gröbner basis** for $\mathcal{N}$ if the module $\mathrm{LT}(\mathcal{N})$ is generated by the classes $\mathrm{LT}(p_i)$ for $1 \leq i \leq r$.

Lemma 2.8. *If B is a Gröbner basis for $\mathcal{N}$, then B generates $\mathcal{N}$.*

Proof. If not, choose an element $p \in \mathcal{N}(d')$ that is not in the submodule generated by B, and choose it in such a way that the leading quartet $\mathrm{LQ}(p)$ is minimal with respect to the admissible well-order on $|\mathrm{D}_d^T|$. Since B is a Gröbner basis, we may choose an index i, a morphism $\psi : d'_i \to d'$, a function $n : T(d') \to \mathbb{N}$, and a scalar $\lambda \in k$ such that $\mathrm{LT}(p) = \lambda x^n \psi(\mathrm{LT}(p_i))$. By Lemma 2.7, this is equal to $\mathrm{LT}(\lambda x^n \psi(p_i))$. But then $p - \lambda x^n \psi(p_i)$ is not in the submodule generated by B and has a leading quartet strictly smaller than $\mathrm{LQ}(p)$, which gives a contradiction. □

Proposition 2.9. *Suppose that the ring k is Noetherian. For every object d of* D*, the principal projective $\mathcal{P}_d \in \mathrm{Rep}_k(\mathrm{D}, \Psi \circ T)$ is Noetherian.*

Proof. By Lemma 2.8, it is sufficient to show that every submodule $\mathcal{N} \subset \mathcal{P}_d$ has a Gröbner basis. By property (G2), the set $\mathrm{LQ}(\mathcal{N}) \subset |\mathrm{D}_d^T|$ has only finitely many minimal elements with respect to the partial order. Choose finitely many quartets $\mu_1, \ldots, \mu_r$ representing these minimal classes, and write $\mu_i = (d, d'_i, \varphi_i, m_i)$. For each i, the fact that k is Noetherian implies that the ideal $\mathrm{LC}(\mathcal{N}, \mu_i)$ is generated by finitely many elements

$$\lambda_i^1, \ldots, \lambda_i^{s_i} \in k.$$

For each $j \leq s_i$, choose an element $0 \neq p_i^j \in \mathcal{N}(d'_i)$ with $\mathrm{LT}(p_i^j) = \lambda_i^j b_{\mu_i}$, and let

$$B := \{(d'_i, p_i^j) \mid 1 \leq i \leq r,\ 1 \leq j \leq s_i\}.$$

We claim that B is a Gröbner bases for $\mathcal{N}$.

Let $0 \neq p \in \mathcal{N}(d')$ be given; we will show that $\mathrm{LT}(p)$ is in the submodule of $\mathcal{P}_d$ generated by the classes $\mathrm{LT}(p_i^j)$. Let $\nu := \mathrm{LQ}(p)$. By definition of the quartets $\mu_1, \ldots, \mu_r$, there exists an index i such that $[\mu_i] \leq [\nu]$. That means that we can choose a morphism $\psi : d_i' \to d'$ and a map $n : T(d') \to \mathbb{N}$ such that $\nu = \psi(\mu_i) + n$. Since $[\mu_i] \leq [\nu]$, we have $\mathrm{LC}(\mathcal{N}, \nu) \subset \mathrm{LC}(\mathcal{N}, \mu_i)$, and therefore there exist elements $\zeta_i^1, \ldots, \zeta_i^{s_i} \in k$ such that

$$\mathrm{LC}(p) = \zeta_i^1 \lambda_i^1 + \cdots + \zeta_i^{s_i} \lambda_i^{s_i}.$$

Then

$$\begin{aligned}
\mathrm{LT}(p) &= \mathrm{LC}(p) b_\nu \\
&= (\zeta_i^1 \lambda_i^1 + \cdots + \zeta_i^{s_i} \lambda_i^{s_i}) b_{\psi(\mu_i)+n} \\
&= x^n \psi(\zeta_i^1 \lambda_i^j b_{\mu_i} + \cdots + \zeta_i^{s_i} \lambda_i^j b_{\mu_i}) \\
&= x^n \psi\left(\zeta_i^1 \,\mathrm{LT}(p_i^1) + \cdots + \zeta_i^{s_i} \,\mathrm{LT}(p_i^{s_i})\right)
\end{aligned}$$

is in the submodule of $\mathcal{P}_d$ generated by the classes $\mathrm{LT}(p_i^j)$. □

Corollary 2.10. *Let* D *be an essentially small category,* $T : \mathrm{D} \to \mathrm{OI}$ *a functor, and* k *a Noetherian commutative ring. If the pair* (D, T) *is Gröbner, then* $\mathrm{Rep}_k(\mathrm{D}, \Psi \circ T)$ *is locally Noetherian.*

Proof. Suppose that $\mathcal{M} \in \mathrm{Rep}_k(\mathrm{D}, \Psi \circ T)$ if finitely generated. By Lemma 2.1, $\mathcal{M}$ is a quotient of a direct sum of principal projectives. Proposition 2.9 tells us that each of these principal projectives is Noetherian, and Lemma 2.3 then tells us that the same is true of $\mathcal{M}$. □

2.4 Quasi-Gröbner pairs

Let $\Phi : \mathrm{D} \to \mathrm{C}$ be a functor. Following Sam and Snowden [SS17, Definition 3.2.1], we say that Φ has **property (F)** if, for any object c of C, there exist finitely many objects $d_1, \ldots, d_r$ of D along with morphisms $\varphi_i : c \to \Phi(d_i)$ such that, for any object d of D and any morphism $\psi : c \to \Phi(d)$, there exists an i and a morphism $\rho : d_i \to d$ such that $\psi = \Phi(\rho) \circ \varphi_i$. Given a functor $S : \mathrm{C} \to \mathrm{FI}$, we say that the pair (C, S) is **quasi-Gröbner** if there exists a Gröbner pair (D, T) and an essentially surjective functor $\Phi : \mathrm{D} \to \mathrm{C}$ with property (F) such that $S \circ \Phi$ is naturally isomorphic to $\Psi \circ T$.

Remark 2.11. The pair $(\mathrm{C}, \emptyset)$ is quasi-Gröbner if and only if the category C is quasi-Gröbner in the sense of [SS17].

Let $\Phi : \mathrm{D} \to \mathrm{C}$ and $S : \mathrm{C} \to \mathrm{FI}$ be any functors. For any commutative ring k, we have an exact functor $\Phi^* : \mathrm{Rep}_k(\mathrm{C}, S) \to \mathrm{Rep}_k(\mathrm{D}, S \circ \Phi)$ that takes a module $\mathcal{M} \in \mathrm{Rep}_k(\mathrm{C}, S)$ to

$$\Phi^* \mathcal{M} := \mathcal{M} \circ \Phi \in \mathrm{Rep}_k(\mathrm{D}, S \circ \Phi).$$

Proposition 2.12. *Let* $\Phi : \mathrm{D} \to \mathrm{C}$ *be a functor with property* F*, let* $S : \mathrm{C} \to \mathrm{FI}$ *be any functor, and let* k *be a commutative ring. If* $\mathcal{M} \in \mathrm{Rep}_k(\mathrm{C}, S)$ *is finitely generated, then* $\Phi^* \mathcal{M} \in \mathrm{Rep}_k(\mathrm{D}, S \circ \Phi)$ *is finitely generated.*

Proof. Since $\mathcal{M}$ is finitely generated, Lemma 2.1 tells us that $\mathcal{M}$ is a quotient of a direct sum of principal projectives. Since Φ^* is exact, $\Phi^*\mathcal{M}$ is a quotient of a direct sum of pullbacks of principal projectives. Thus, it is sufficient to show that, for any object c of C, $\Phi^*\mathcal{P}_c$ is finitely generated. Choose finitely many objects $d_1, \ldots, d_r$ of D along with morphisms $\varphi_i : c \to \Phi(d_i)$ as in the definition of property (F). Consider the maps

$$\mathcal{P}_{d_i} \to \Phi^*\mathcal{P}_{\Phi(d_i)} \to \Phi^*\mathcal{P}_c,$$

where the first map is induced by Φ and the second is induced by φ_i. Property (F) says precisely that the direct sum map

$$\bigoplus_{i=1}^{r} \mathcal{P}_{d_i} \to \Phi^*\mathcal{P}_c$$

is surjective, which implies that $\Phi^*\mathcal{P}_c$ is finitely generated. □

Proof of Theorem 1.5. Let (C, S) be quasi-Gröbner pair. That means that there exists a Gröbner pair (D, T), an essentially surjective functor $\Phi : \mathrm{D} \to \mathrm{C}$ with property (F), and a natural isomorphism $\Psi \circ T \cong S \circ \Phi$. Fix a commutative ring k, a finitely generated module $\mathcal{M} \in \mathrm{Rep}_k(\mathrm{C}, S)$, and a submodule $\mathcal{N} \subset \mathcal{M}$. We need to prove that $\mathcal{N}$ is finitely generated, as well.

Proposition 2.12 tells us that $\Phi^*\mathcal{M} \in \mathrm{Rep}_k(\mathrm{D}, S \circ \Phi) \simeq \mathrm{Rep}_k(\mathrm{D}, \Psi \circ T)$ is finitely generated, and Corollary 2.10 then implies that $\Phi^*\mathcal{N} \subset \Phi^*\mathcal{M}$ is also finitely generated. Choose a generating set consisting of objects $d_1, \ldots, d_r$ of D and elements $v_i \in \Phi^*\mathcal{N}(d_i)$. This means that, for any object d of D, $\Phi^*\mathcal{N}(d)$ is spanned over $\mathcal{R}(d)$ by the images of the elements v_i along the maps induced by all possible morphisms $\varphi_i : d \to d_i$. This is equivalent to saying the $\mathcal{N}(\Phi(d))$ is spanned over $\mathcal{R}(\Phi(d))$ by the images of the elements $v_i \in \mathcal{N}(\Phi(d_i))$ along the maps induced by all morphisms $\Phi(\varphi_i) : \Phi(d) \to \Phi(d_i)$. Since Φ is essentially surjective, this means that $\mathcal{N}$ is finitely generated. □

3 Graphs

In this section we formally define the category G and prove Theorems 1.2 and 1.7.

3.1 Defining the graph categories

A **directed graph** is a quadruple (V, A, h, t), where V and A are finite sets (vertices and arrows), and h and t are each maps from A to V (head and tail). A **graph** is a quintuple (V, A, h, t, σ), where (V, A, h, t) is a directed graph and σ is a fixed-point-free involution of A with the property that $h = t \circ \sigma$. If (V, A, h, t, σ) is a graph, elements of the quotient A/σ are called **edges**. Given a directed graph $D = (V, A, h, t)$, we define the **underlying graph** $\bar{D} = (V, \bar{A}, h, t, \sigma)$, where $\bar{A} = A \times \{\pm 1\}$, $h(a, 1) = h(a) = t(a, -1)$, $t(a, 1) = t(a) = h(a, -1)$, and σ acts by toggling the second factor. We will usually suppress h and t from the notation and simply write (V, A) for a directed graph or (V, A, σ) for a graph.

Remark 3.1. This might seem to be an unnecessarily complicated definition of a graph. For example, one might try defining a graph to consist of a vertex set, and edge set, and a map from edges to unordered pairs of vertices. However, we want a graph with a loop to have a nontrivial automorphism that reverses the orientation of the loop. It is difficult to formalize this with the unordered pair definition.

If $G = (V, A, \sigma)$ is a graph and $v, v' \in V$, a **walk** in G from v to v' is a finite sequence $(a_1, \ldots, a_n)$ of arrows with $t(a_1) = v$, $h(a_n) = v'$, and $h(a_i) = t(a_{i+1})$ for all $1 \leq i < n$. A **path** in G from v to v' is a walk from v to v' of minimal length. We say that G is **connected** if there exists at least one path between any pair of vertices, and we say that G is a **forest** if there exists at most one path between any pair of vertices. A nonempty connected forest is called a **tree**. We say that a directed graph D is connected (or that it is a forest or that it is a tree) if and only if the underlying graph $\bar{D}$ is connected (or is a forest or is a tree).[1]

Let $D = (V, A)$ and $D' = (V', A')$ be directed graphs. A **minor morphism** from D to D' is a map

$$\varphi : V \sqcup A \sqcup \{\star\} \to V' \sqcup A' \sqcup \{\star\}$$

satisfying the following properties:

- $\varphi(\star) = \star$.
- For every vertex $v \in V$, $\varphi(v) \in V'$.
- For every arrow $a' \in A'$, there exists a unique arrow $a \in A$ with $\varphi(a) = a'$.
- If $\varphi(a) \in A$, then $\varphi \circ h(a) = h' \circ \varphi(a)$ and $\varphi \circ t(a) = t' \circ \varphi(a)$.
- If $\varphi(a) \in V$, then $\varphi \circ h(a) = \varphi(a) = \varphi \circ t(a)$.
- For every $v' \in V'$, $\varphi^{-1}(v')$ consists of the edges and vertices of a tree.

If $\varphi(a) \in V$ we say that a is a **contracted arrow**. If $\varphi(a) = \star$, we say that a is a **deleted arrow**. Note that the second condition says exactly that the arrows of D that are neither deleted nor contracted map bijectively to the arrows of D'. In particular, φ induces an injection $\varphi^* : A' \to A$.

Remark 3.2. If D and D' are connected directed graphs, then a minor morphism $\varphi : D \to D'$ is determined by its restriction to the set of arrows of D. (Connectedness is necessary because the directed graph with two vertices and no arrows has a nontrivial automorphism swapping the vertices.) However, it is convenient to define φ on the whole set $V \sqcup A \sqcup \{\star\}$ so that minor morphisms can be composed simply by composing functions. Note that φ is *not* determined by the arrow map φ^*. To see this, consider the example where D has two vertices and two parallel arrows between them, and D' consists of a single vertex with no arrows. There are two minor morphisms from D to D', corresponding to the choice of which arrow is deleted and which arrow is contracted.

[1] Note that this is not the same as requiring that there is at least (or at most or exactly) one directed path between any two vertices.

If $G = (V, A)$ and $G' = (V', A')$ are graphs, a minor morphism from G to G' is an S_2-equivariant map

$$\varphi : V \sqcup E \sqcup \{\star\} \to V' \sqcup E' \sqcup \{\star\}$$

satisfying the same list of properties appearing in the definition for directed graphs, where S_2 acts on E and E' via σ and acts trivially on V, V', and $\{\star\}$. An **ordered directed graph** is a directed graph equipped with a linear order on the set of arrows. A minor morphism between two ordered directed graphs is a minor morphism φ with the property that φ^* preserves orders. Let G denote the category whose objects are nonempty connected graphs and whose objects are minor morphisms, and let OD denote the category whose objects are nonempty connected ordered directed graphs and whose objects are minor morphisms.

Remark 3.3. Our category G is closely related to the category Gr defined by Borisov and Manin in [BM08, Definition 1.2.1]. What we call a graph is equivalent to what they call a graph with no tails. Furthermore, there is a functor from G to Gr whose essential image is the full subcategory of Gr consisting of graphs with no tails. The maps on morphisms induced by this functor are not injective, because morphisms in the category Gr do not distinguish between deleted and contracted arrows. For example, there are n distinct morphisms in G from an n-cycle to a point (we delete one edge and contract the rest), and these n morphisms all have the same image in Gr.

Let $\Phi : \mathrm{OD}^{\mathrm{op}} \to \mathrm{G}^{\mathrm{op}}$ be the functor that forgets the ordering on the arrows and takes a directed graph D to its underlying graph $\bar{D}$. We have an **arrow functor**

$$A : \mathrm{OD}^{\mathrm{op}} \to \mathrm{OI}$$

taking $D = (V, A)$ to $A(D) := A$ and an **edge functor**

$$E : \mathrm{G}^{\mathrm{op}} \to \mathrm{FI}$$

taking $G = (V, A, \sigma)$ to $E(G) := A/\sigma$.

Proposition 3.4. *The functor $\Phi : \mathrm{OD}^{\mathrm{op}} \to \mathrm{G}^{\mathrm{op}}$ is essentially surjective and has property (F). Furthermore, there is a natural isomorphism of functors*

$$\Psi \circ A \cong E \circ \Phi : \mathrm{OD}^{\mathrm{op}} \to \mathrm{FI}\,.$$

Proof. Essential surjectivity and the natural isomorphism are both clear. To prove property (F), we need to show that for any graph G, there exist finitely many ordered directed graphs $D_1, \ldots, D_r$ and minor morphisms $\varphi_i : \Phi(D_i) \to G$ such that, for any ordered directed graph D and any minor morphism $\varphi : \Phi(D) \to G$, there exists an index i and a minor morphism $\psi : D \to D_i$ such that $\varphi = \varphi_i \circ \Phi(\psi)$. Indeed, it is easy to see that this condition is satisfied if we take $D_1, \ldots, D_r$ to be representatives of all of the (finitely many) isomorphism classes of ordered directed graphs whose underlying graphs are isomorphic to G. $\square$

In the next section, we will need one more variation on the notion of minor morphisms. For any poset Q, a ***Q*-labeling** of an ordered directed graph (V, A) is a function $q : A \to Q$. Suppose that $D = (V, A, q)$ and $D' = (V', A', q')$ are Q-labeled ordered directed graphs. An ordered minor morphism $q : D \to D'$ is said to be compatible with the Q-labelings if $q \circ \varphi^*(a') \leq q'(a')$ for all $a' \in A'$.

3.2 Gröbner properties

In this section we prove that the the category $\mathrm{OD}^{\mathrm{op}}$ is Gröbner and that the pair $(\mathrm{OD}^{\mathrm{op}}, A)$ is Gröbner, and thereby prove Theorems 1.2 and 1.7.

Lemma 3.5. *Both the category* $\mathrm{OD}^{\mathrm{op}}$ *and the pair* $(\mathrm{OD}^{\mathrm{op}}, A)$ *have property (G1).*

Proof. It is clear that the set of isomorphism classes $|\mathrm{OD}^{\mathrm{op}}|$ is countable, and we therefore may choose a well-order $\preceq$ on $|\mathrm{OD}^{\mathrm{op}}|$. Fix an ordered directed graph $D = (V, A)$, and let D_1 and D_2 be ordered directed graphs equipped with minor morphisms $\varphi_1 : D_1 \to D$ and $\varphi_2 : D_2 \to D$. We endow $|\mathrm{OD}^{\mathrm{op}}_D| = |(\mathrm{OD}^{\mathrm{op}})^{\emptyset}_D|$ with a well-order by putting $[\varphi_1] \prec [\varphi_2]$ if $[D_1] \prec [D_2]$ or there exists an isomorphism $\psi : D_1 \to D_2$ and $\varphi_1^*(a) < \psi^* \circ \varphi_2^*(a)$ for the smallest arrow $A \in A$ such that $\varphi_1^*(a) \neq \psi^* \circ \varphi_2^*(a)$. (If no such arrow a exists, then $[\varphi_1] = [\varphi_2]$ by Remark 3.2.) It is clear that $\prec$ is a well-order and that it is compatible with precomposition in OD, or equivalently postcomposition in $\mathrm{OD}^{\mathrm{op}}$. This proves that the category $\mathrm{OD}^{\mathrm{op}}$ has property (G1). The fact that the pair $(\mathrm{OD}^{\mathrm{op}}, A)$ also has property (G1) follows from Lemma 2.6. □

Lemma 3.6. *The category* $\mathrm{OD}^{\mathrm{op}}$ *has property (G2).*

Proof. Fix an ordered directed graph $D = (V, A)$ and let $Q := V \sqcup E \sqcup \{\star\}$ with the trivial poset structure (two elements are comparable if and only if they are equal). Any minor morphism $\varphi : D' \to D$ of ordered directed graphs induces a Q-labeling of D'. Furthermore, if $\varphi_1 : D_1 \to D$, $\varphi_2 : D_2 \to D$, and $\psi : D_2 \to D_1$ are all minor morphisms, Remark 3.2 implies that $\varphi_2 = \varphi_1 \circ \psi$ if and only if ψ is compatible with the Q-labelings. Thus it will be sufficient to show that the poset of isomorphism classes of Q-labeled ordered directed graphs is Noetherian, where $[D_1] \leq [D_2]$ if and only if there exists a minor morphism from D_2 to D_1 that is compatible with the Q-labelings.

By [RS10, 1.7], given an infinite sequence $(D_i)_{i \in \mathbb{N}}$ of S-labeled ordered directed graphs with $D_i = (V_i, A_i)$, there exists a pair of indices $i < j$ along with the following data:[2]

- For each $v \in V_i$, a connected subgraph $D_v \subset D_j$, with D_v disjoint from D_w for all $v \neq w$.
- An injective map $\eta : A_i \to A_j$ that preserves both the orders and the Q-labels.

Furthermore, the map η has the properties that $h \circ \eta(a) \in D_{h(a)}$, $t \circ \eta(a) \in D_{t(a)}$, and if a is a loop at the vertex v, then $\eta(a) \notin D_v$.

[2] In the notation of [RS10], we take $E(\Omega) = Q \times \mathbb{N}$, where $(q, n) \leq (q', n')$ if and only if $q = q'$ and $n \leq n'$. We give each vertex the label $(\star, 0)$ and we give each edge a label indicating its Q-label along with its position in the linear order on E.

We claim that we can further assume that every vertex of D_j lies in the subgraph D_v for some $v \in V_j$. Indeed, if $w \in V_j$ does not lie in any D_v, construct a path (allowing backward as well as forward motion along arrows) that starts at w and ends in some D_v without passing through any other $D_{v'}$. We can then add this path to D_v without breaking any of the above properties. By iterating this procedure, we can arrange for every vertex to lie in some D_v.

Choose arbitrarily a spanning tree $T_v \subset D_v$ for all $v \in V_i$. To construct a minor morphism from D_j to D_i, we send $\eta(a)$ to a for all $a \in A_i$, send the edges and vertices of T_v to the vertex v for all $v \in V_i$, and delete all of the remaining edges. □

Lemma 3.7. *The pair* $(\mathrm{OD}^{\mathrm{op}}, A)$ *has property (G2).*

Proof. The proof is very similar to the proof of Lemma 3.6. Fix an ordered directed graph $D = (V, A)$, let $Q := V \sqcup E \sqcup \{\star\}$ with the trivial poset structure, and let $\tilde{Q} := Q \times \mathbb{N}$. If $D' = (V', A')$ is an ordered directed graph, If $\varphi : D' \to D$ is a minor morphism, and $m : A' \to \mathbb{N}$ is a map, we obtain a $\tilde{Q}$-labeling of D' by attaching the label $(\varphi(a'), m \circ \varphi(a'))$ to the arrow $a' \in A'$. Furthermore, if $\mu_1 = (D, D_1, \varphi_1, m_1)$ and $\mu_2 = (D, D_2, \varphi_2, m_2)$ are quartets, then $\mu_1 \leq \mu_2$ if and only if there exists a minor morphism $\psi : D_2 \to D_1$ that is compatible with the $\tilde{Q}$-labelings. Thus it will be sufficient to show that the poset of isomorphism classes of $\tilde{Q}$-labeled ordered directed graphs is Noetherian, where $[D_1] \leq [D_2]$ if and only if there exists a minor morphism from D_2 to D_1 that is compatible with the $\tilde{Q}$-labelings. The rest of the proof is identical to that of Lemma 3.6. □

Corollary 3.8. *The category* $\mathrm{OD}^{\mathrm{op}}$ *is Gröbner and the pair* $(\mathrm{OD}^{\mathrm{op}}, A)$ *is Gröbner.*

Proof. It is clear that $\mathrm{OD}^{\mathrm{op}}$ is an essentially small directed category, so the first statement follows from Lemmas 3.5 and 3.6 and the second statement follows Lemmas 3.5 and 3.7. □

Proof of Theorems 1.2 and 1.7. These follow from Proposition 3.4 and Corollary 3.8. □

3.3 Relating G^{op} and $\mathrm{OD}^{\mathrm{op}}$ to other categories

We now show how our theorems about G^{op} and $\mathrm{OD}^{\mathrm{op}}$ imply various previously known results about other categories, as discussed in Remarks 1.4 and 1.8.

Proposition 3.9. *The category* OI *is Gröbner and the category* FI *is quasi-Gröbner. In addition, the pair* (OI, Ψ) *is Gröbner and the pair* $(\mathrm{FI}, \mathrm{id})$ *is quasi-Gröbner.*

Proof. Consider the functor from OI to $\mathrm{OD}^{\mathrm{op}}$ that takes an ordered set A to the ordered directed graph (V, A), where $V = A \sqcup \{*\}$, $h(a) = a$ for all $a \in A$, and $t(a) = *$ for all $a \in A$. In other words, A is sent to the star graph with central vertex $*$, satellite vertices A, and all arrows directed outward. This functor is fully faithful and therefore realizes OI as a full subcategory of $\mathrm{OD}^{\mathrm{op}}$. Full subcategories of Gröbner categories are Gröbner, so this implies that OI is Gröbner. Since the arrow functor A on $\mathrm{OD}^{\mathrm{op}}$ restricts to the forgetful functor Ψ on OI, it also implies that the pair (OI, Ψ) is Gröbner. The fact that FI is quasi-Gröbner and that the pair $(\mathrm{FI}, \mathrm{id})$ is quasi-Gröbner follows because the forgetful functor $\Psi : \mathrm{OI} \to \mathrm{FI}$ is essentially surjective and has property (F). □

Let OD_g be the full subcategory of OD consisting of ordered directed graphs whose underlying graphs have genus g.

Proposition 3.10. *The category* $\mathrm{OD}_g^{\mathrm{op}}$ *is Gröbner and the category* $\mathrm{G}_g^{\mathrm{op}}$ *is quasi-Gröbner.*

Proof. Since $\mathrm{OD}_g^{\mathrm{op}}$ is a full subcategory of $\mathrm{OD}^{\mathrm{op}}$, it is Gröbner. Since $\Phi : \mathrm{OD}^{\mathrm{op}} \to \mathrm{G}^{\mathrm{op}}$ is essentially surjective and has property (F), the same is true for $\Phi_g : \mathrm{OD}_g^{\mathrm{op}} \to \mathrm{G}_g^{\mathrm{op}}$. □

Proposition 3.11. *The categories* $\mathrm{FS}^{\mathrm{op}}$ *and* $\mathrm{VI}_{\mathbb{F}}$ *for any finite field* $\mathbb{F}$ *are quasi-Gröbner.*

Proof. Let $\mathrm{K} \subset \mathrm{G}$ be the full subcategory consisting of simple complete graphs, and let $\mathrm{OT} \subset \mathrm{OD}$ be the full subcategory of OD consisting of objects that are sent to K by Φ.[3] Since $\mathrm{OD}^{\mathrm{op}}$ is Gröbner, so is $\mathrm{OT}^{\mathrm{op}}$. The restriction $\Psi : \mathrm{OT}^{\mathrm{op}} \to \mathrm{K}^{\mathrm{op}}$ still has property (F), hence K^{op} is quasi-Gröbner.

Consider the essentially surjective functor $\Theta : \mathrm{K}^{\mathrm{op}} \to \mathrm{FS}^{\mathrm{op}}$ that sends a graph to its vertex set. This functor is essentially surjective as well as surjective on morphisms, so it has property (F). This implies that $\mathrm{FS}^{\mathrm{op}}$ is quasi-Gröbner. The fact that $\mathrm{VI}_{\mathbb{F}}$ is quasi-Gröbner follows from the existence of an essentially surjective functor $\mathrm{FS}^{\mathrm{op}} \to \mathrm{VI}_{\mathbb{F}}$ [SS17, Theorem 8.3.1]. □

Remark 3.12. The functor $\Theta : \mathrm{K}^{\mathrm{op}} \to \mathrm{FS}^{\mathrm{op}}$ extends naturally to a functor from G^{op} to $\mathrm{FS}^{\mathrm{op}}$, but this functor does not have property (F).

4 Homology of configuration spaces

We now apply the results of the previous section to prove Theorem 1.11 and Proposition 1.15.

4.1 The reduced Świątkowski complex

Let $\mathcal{R}$ be the functor from G^{op} to rings associated with the functor $E : \mathrm{G}^{\mathrm{op}} \to \mathrm{FI}$. That is, for any graph $G = (V, A, \sigma)$, we have the polynomial ring

$$\mathcal{R}(G) := \mathbb{Z}[x_e \mid e \in E(G) = A/\sigma].$$

For any vertex $v \in V$, let $\mathcal{M}_v(G)$ denote the free $\mathcal{R}(G)$-module generated by the symbol $\emptyset$ along with the set

$$A_v := \{a \in A \mid h(a) = v\}.$$

We equip $\mathcal{M}_v(G)$ with a bigrading by putting $\deg x_e = (0, 1)$ for all $e \in E(G)$, $\deg \emptyset = (0, 0)$, and $\deg a = (1, 1)$ for all $a \in A_v$. Let $\widetilde{\mathcal{M}}_v(G) \subset \mathcal{M}_v(G)$ be the submodule generated by the elements $\emptyset$ and $a - a'$ for all $a, a' \in A_v$. We equip $\widetilde{\mathcal{M}}_v(G)$ with an $\mathcal{R}(G)$-linear differential ∂_v of degree $(-1, 0)$ by putting

$$\partial(a - a') := \big(x_{[a]} - x_{[a']}\big)\emptyset \qquad \text{and} \qquad \partial\emptyset = 0,$$

[3] The letters OT stand for "ordered tournament".

where $[a]$ denotes the image of a in $E(G)$. We then define the **reduced Świątkowski complex**

$$\widetilde{\mathcal{S}}(G) \ := \ \bigotimes_{v \in V} \widetilde{\mathcal{M}}_v,$$

where the tensor product is taken over the ring $\mathcal{R}(G)$. This is a bigraded free $\mathcal{R}(G)$-module with a differential ∂. Any minor morphism $\varphi : G \to G'$ induces a map $\varphi^* : \widetilde{\mathcal{S}}(G') \to \widetilde{\mathcal{S}}(G)$ of differential bigraded $\mathcal{R}(G')$-modules, thus making $\widetilde{\mathcal{S}}$ a differential bigraded object of $\mathrm{Rep}_{\mathbb{Z}}(\mathrm{G}^{\mathrm{op}}, E)$ [ADCK19, Lemma C.7]. More precisely, for each $i \in \mathbb{N}$, we have a graded object $\widetilde{\mathcal{S}}_{i,\bullet} \in \mathrm{Rep}_{\mathbb{Z}}(\mathrm{G}^{\mathrm{op}}, E)$, and we have a differential

$$\partial_i : \widetilde{\mathcal{S}}_{i,\bullet} \to \widetilde{\mathcal{S}}_{i-1,\bullet}$$

which is a morphism in $\mathrm{Rep}_{\mathbb{Z}}(\mathrm{G}^{\mathrm{op}}, E)$ and is compatible with the single grading. By taking homology, we obtain a graded object $H_i(\widetilde{\mathcal{S}}) := \ker(\partial_i)/\operatorname{im}(\partial_{i+1})$ of $\mathrm{Rep}_{\mathbb{Z}}(\mathrm{G}^{\mathrm{op}}, E)$. The following theorem appears in [ADCK19, Theorem 4.5 and Proposition 4.9].

Theorem 4.1. *For all $i > 0$, there exists a canonical graded module isomorphism $H_i(\widetilde{\mathcal{S}}) \cong \mathcal{H}_i$.*

Remark 4.2. The statement $H_0(\widetilde{\mathcal{S}}) \cong \mathcal{H}_0$ fails for a silly reason. If $*$ is a graph consisting of a single vertex with no edges, then $H_0(\widetilde{\mathcal{S}}(*)) = \mathbb{Z}$ concentrated in degree 0, whereas $\mathcal{H}_0(*) = \mathbb{Z} \oplus \mathbb{Z}$ concentrated in degrees zero and 1. In other words, the reduced Świątkowski complex fails to recognize that the degree zero homology of $U_1(*)$ is nontrivial. For all connected graphs G with at least one edge, we have $H_0(\widetilde{\mathcal{S}}(G)) \cong \mathcal{H}_0(G)$.

4.2 Finite generation

We are now ready to prove finite generation of the reduced Świątkowski complex.

Proposition 4.3. *The module $\widetilde{\mathcal{S}}_{i,\bullet} \in \mathrm{Rep}_{\mathbb{Z}}(\mathrm{G}^{\mathrm{op}}, E)$ is generated by $\widetilde{\mathcal{S}}_{i,i}(G)$ for all graphs G with at most $2i$ edges. More precisely, let $G_1, \ldots, G_r$ be representatives of each isomorphism class of graph with at most $2i$ edges. For each $1 \leq s \leq r$, choose a basis for the finite rank free abelian group $\widetilde{\mathcal{S}}_{i,i}(G_s)$. Taken all together, these classes generate $\widetilde{\mathcal{S}}_{i,\bullet}$.*

Proof. For each graph $G = (V, A, \sigma)$, $\mathcal{R}(G)$-module $\widetilde{\mathcal{S}}_{i,\bullet}(G)$ is generated by classes of the form

$$\xi \ := \ \bigotimes_{j=1}^{i} (a_j - a'_j) \quad \otimes \bigotimes_{v \notin \{v_1, \ldots, v_i\}} \emptyset,$$

where $v_1, \ldots, v_i$ are distinct vertices and, for each j, a_j and a'_j are arrows with

$$h(a_j) = v_j = h(a'_j).$$

We will call $v_1, \ldots, v_i \in V$ **distinguished vertices** and $[a_1], [a'_1], \ldots, [a_i], [a'_i] \in E(G)$ **distinguished edges**. If G has an edge e such that at least one vertex incident to e is not distinguished

and e is not a loop, then the class ξ may be pulled back from the graph obtained from G by contracting e [PRa, Section 5.3]. Furthermore, if e is not distinguished and not a bridge, then it is clear that the class ξ may be pulled back from the graph obtained from G by deleting e. Thus, if we are looking for generators for $\widetilde{\mathcal{S}}_{i,\bullet}$, we may restrict our attention to graphs G and classes $\xi \in \widetilde{\mathcal{S}}_{i,\bullet}(G)$ for which every edge of G is either a distinguished edge or a bridge between distinguished vertices. Finally, it is straightforward to check that ξ may be written as a linear combination of classes for which every bridge between distinguished vertices is itself a distinguished edge. Since there are at most $2i$ distinguished edges, it is sufficient to consider those graphs with at most $2i$ edges. □

Example 4.4. If $i = 1$, we see that there are seven connected graphs with at most two edges: the graph $*$ with no edges; the roses[4] R_1 and R_2; the paths[5] P_1 and P_2; the lollipop L; and the cycle C_2. We have $\widetilde{\mathcal{S}}_{1,1}(*) = 0 = \widetilde{\mathcal{S}}_{1,1}(P_1)$. We have $\widetilde{\mathcal{S}}_{1,1}(R_1) = \mathbb{Z} = \widetilde{\mathcal{S}}_{1,1}(P_2)$, and these classes cannot be pulled back under any minor morphisms. We have

$$\widetilde{\mathcal{S}}_{1,1}(R_2) = \mathbb{Z}^3, \qquad \widetilde{\mathcal{S}}_{1,1}(L) = \mathbb{Z}^2, \qquad \text{and} \qquad \widetilde{\mathcal{S}}_{1,1}(C_2) = \mathbb{Z}^2.$$

Each of these groups has a corank 1 direct summand consisting of classes that are spanned by the images of $\widetilde{\mathcal{S}}_{1,1}(R_1)$ under various minor morphisms. Thus a minimal set of generators for $\widetilde{\mathcal{S}}_{1,\bullet}$ consists of one class from each of the groups $\widetilde{\mathcal{S}}_{1,1}(R_1)$, $\widetilde{\mathcal{S}}_{1,1}(P_2)$, $\widetilde{\mathcal{S}}_{1,1}(R_2)$, $\widetilde{\mathcal{S}}_{1,1}(L)$, and $\widetilde{\mathcal{S}}_{1,1}(C_2)$.

Proof of Theorem 1.11. When $i = 0$, the theorem is trivial. When $i > 0$, Theorem 4.1 tells us that $\mathcal{H}_i$ is a subquotient of $\widetilde{\mathcal{S}}_{i,\bullet}$, which is finitely generated by Proposition 4.3. Quotients of finitely generated modules are always finitely generated, and submodules of finitely generated modules are also finitely generated by Theorems 1.5 and 1.7. □

Proof of Proposition 1.15. By Theorem 4.1, we have $\dim H_i\big(U_n(G);\mathbb{Z}\big) \leq \dim \widetilde{\mathcal{S}}_{i,n}(G)$. By Lemma 2.1 and Proposition 4.3, $\widetilde{\mathcal{S}}_{i,\bullet}(G)$ is isomorphic as a graded module to a quotient of a direct sum of shifted principal projective modules $\mathcal{P}_{G_j}[-i]$ for some finite list $G_1, \ldots, G_j$ of nonempty connected graphs, each with at most $2i$ edges. We will show that, for any nonempty connected graph G and any n, the dimension of the degree $n-i$ part of $\mathcal{P}_{G_j}(G)$ is bounded by $|\operatorname{Aut}(G_j)|\, e(G)^{i+n+g(G)}$. We can then take $\alpha_i = |\operatorname{Aut}(G_1)| + \cdots + |\operatorname{Aut}(G_r)|$, and the proof will be complete.

The dimension of the degree $n-i$ part of $\mathcal{P}_{G_j}(G)$ is equal to the number of degree $n-i$ monomials in the variables $\{x_e \mid e \in E(G)\}$ times the number of morphisms from G to G_j. The former is equal to

$$\binom{e(G)+n-i-1}{n-i} \leq e(G)^{n-i}.$$

A morphism from G to G_j is determined, up to post-composition by an automorphism of G_j, by a choice of $e(G_j)$ edges to neither delete nor contract along with a disjoint set of $g(G) - g(G_j)$ edges

[4]We write R_k to denote a graph with a single vertex and k loops.

[5]We write P_k to denote a path with k edges.

to delete. Hence the number of morphisms from G to G_j is bounded above by

$$|\operatorname{Aut}(G_j)| \binom{e(G)}{e(G_j)} \binom{e(G)-e(G_j)}{g(G)-g(G_j)} \leq |\operatorname{Aut}(G_j)|\, e(G)^{e(G_j)} e(G)^{g(G)} \leq |\operatorname{Aut}(G_j)|\, e(G)^{2i+g(G)}.$$

All together, this tells us that the dimension of the degree $n-i$ part of $\mathcal{P}_{G_j}(G)$ is bounded above by

$$e(G)^{n-i} \cdot |\operatorname{Aut}(G_j)|\, e(G)^{2i+g(G)} = |\operatorname{Aut}(G_j)|\, e(G)^{i+n+g(G)},$$

thus completing the proof. □

Remark 4.5. Proposition 4.3 does *not* imply that $\mathcal{H}_i$ is generated by $H_i\big(U_i(G);\mathbb{Z}\big)$ for all graphs G with at most $2i$ edges. For example, one of the two generators of $H_1(U_2(K_{1,3});\mathbb{Z}) \cong \mathbb{Z}$ must be included in any generating set for $\mathcal{H}_1$, despite the fact that $2 > 1$ and $K_{1,3}$ has $3 > 2$ edges. The point is that Theorem 1.5 tells us that a submodule of a finitely generated module is finitely generated, but it gives us no control over the generators of the submodule.

References

[Abr00] Aaron David Abrams, *Configuration spaces and braid groups of graphs*, ProQuest LLC, Ann Arbor, MI, 2000, Thesis (Ph.D.)–University of California, Berkeley.

[ADCK] Byung Hee An, Gabriel C. Drummond-Cole, and Ben Knudsen, *Edge stabilization in the homology of graph braid groups*, arXiv:1806.05585.

[ADCK19] ———, *Subdivisional spaces and graph braid groups*, Doc. Math. **24** (2019), 1513–1583.

[BM08] Dennis V. Borisov and Yuri I. Manin, *Generalized operads and their inner cohomomorphisms*, Geometry and dynamics of groups and spaces, Progr. Math., vol. 265, Birkhäuser, Basel, 2008, pp. 247–308.

[CEF15] Thomas Church, Jordan S. Ellenberg, and Benson Farb, *FI-modules and stability for representations of symmetric groups*, Duke Math. J. **164** (2015), no. 9, 1833–1910.

[CEFN14] Thomas Church, Jordan S. Ellenberg, Benson Farb, and Rohit Nagpal, *FI-modules over Noetherian rings*, Geom. Topol. **18** (2014), no. 5, 2951–2984.

[Far08] Michael Farber, *Invitation to topological robotics*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2008.

[KP12] Ki Hyoung Ko and Hyo Won Park, *Characteristics of graph braid groups*, Discrete Comput. Geom. **48** (2012), no. 4, 915–963.

[Lüt] Daniel Lütgehetmann, *Representation stability for configuration spaces of graphs*, arXiv:1701.03490.

[NR19] Uwe Nagel and Tim Römer, *FI- and OI-modules with varying coefficients*, J. Algebra **535** (2019), 286–322.

[PRa] Nicholas Proudfoot and Eric Ramos, *The contraction category of graphs*, arXiv:1907.11234.

[PRb] _____, *Functorial invariants of trees and their cones*, arXiv:1903.10592.

[PS17] Andrew Putman and Steven V. Sam, *Representation stability and finite linear groups*, Duke Math. J. **166** (2017), no. 13, 2521–2598.

[Ram] Eric Ramos, *An application of the theory of FI-algebras to graph configuration spaces*, arXiv:1805.05316.

[RS04] Neil Robertson and P. D. Seymour, *Graph minors. XX. Wagner's conjecture*, J. Combin. Theory Ser. B **92** (2004), no. 2, 325–357.

[RS10] Neil Robertson and Paul Seymour, *Graph minors XXIII. Nash-Williams' immersion conjecture*, J. Combin. Theory Ser. B **100** (2010), no. 2, 181–205.

[RW] Eric Ramos and Graham White, *Families of nested graphs with compatible symmetric-group actions*, arXiv:1711.07456.

[Sno13] Andrew Snowden, *Syzygies of Segre embeddings and Δ-modules*, Duke Math. J. **162** (2013), no. 2, 225–277.

[SS17] Steven V. Sam and Andrew Snowden, *Gröbner methods for representations of combinatorial categories*, J. Amer. Math. Soc. **30** (2017), no. 1, 159–203.