Let OT_n be the subalgebra of rational functions on $\mathbb{C}^n$ generated by $\frac{1}{x_i - x_j}$ for all $1 \leq i < j \leq n$; this is called the **Orlik-Terao algebra** of the braid matroid of rank $n-1$. This algebra is naturally graded, and it admits an action of the symmetric group S_n by permuting coordinates. We would like to understand OT_n as a graded representation of S_n.

Let $R_n = \mathbb{C}[z_1, \ldots, z_n]/\langle z_1 + \cdots + z_n \rangle$. The polynomial ring R_n equivariantly injects into OT_n by sending z_i to $\sum_{j \neq i} \frac{1}{x_i - x_j}$. It turns out that OT_n is free of finite rank over R_n [PS06], which means that if we define

$$M_n := OT_n / \langle z_1, \ldots, z_n \rangle,$$

then we have a (non-canonical) S_n-equivariant graded R_n-module isomorphism $OT_n \cong R_n \otimes M_n$. Since it is easy to understand R_n as a graded representation, this reduces our problem to understanding the finite dimensional graded representation M_n.

Remark 1. The ring OT_n is isomorphic to the torus-equivariant intersection cohomology of the hypertoric variety associated with the braid arrangement, and M_n is isomorphic to the non-equivariant intersection cohomology [BP09]. This variety admits an action of the symmetric group S_n, and the isomorphisms are equivariant with respect to that action.

Let $G = \mathrm{SU}(2)$, and consider the space $\mathrm{Conf}(n, G)/G$ of ordered n-tuples of distinct elements of G up to right translation. By translating the last coordinate to the identity, we see that $\mathrm{Conf}(n, G)/G$ is S_{n-1}-equivariantly isomorphic to $\mathrm{Conf}(n-1, \mathbb{R}^3)$. However, the action of S_{n-1} on the latter space extends to an action of S_n on $\mathrm{Conf}(n, G)/G$. Let $D_n := H^*(\mathrm{Conf}(n, G)/G; \mathbb{C})$. The following conjecture appears in [MPY17, 2.10].

Conjecture 2. *There exists a graded S_n-equivariant isomorphism $M_n \cong D_n$.*

Remark 3. Conjecture 2 has been checked on a computer up to $n = 10$.

Remark 4. It is not hard to show that M_n and D_n have the same Poincaré polynomial, so Conjecture 2 is certainly true if we forget the action.

Remark 5. The rings D_n naturally form an FI-algebra, while the rings M_n naturally form an $\mathrm{FI}^{\mathrm{op}}$-algebra. More concretely, we have natural homomorphisms $D_n \to D_{n+1}$ (induced by forgetting a point) and $M_{n+1} \to M_n$ (induced by setting $x_n = \infty$) for each n. This means that it is most natural to look for an isomorphism $M_n \cong D_n^*$. (Since finite dimensional graded representations of S_n are self-dual, the existence of such an isomorphism is of course equivalent to Conjecture 2.)

Remark 6. We have ring isomorphisms

$$M_n \cong \mathbb{C}[e_{ij}] \Big/ \Big\langle \sum_j e_{ij} \;\forall\, i; \;\; e_{ij} + e_{ji} \;\forall\, i,j; \;\; e_{ij}e_{jk} + e_{jk}e_{kj} + e_{ki}e_{ij} \;\forall\, i,j,k \text{ distinct} \Big\rangle$$

and

$$D_n \cong \mathbb{C}[e_{ij}] \Big/ \Big\langle \sum_j e_{ij} \;\forall\, i; \;\; e_{ij} + e_{ji} \;\forall\, i,j; \;\; (e_{ij} + e_{jk} + e_{ki})^2 \;\forall\, i,j,k \text{ distinct} \Big\rangle.$$

It is therefore natural to define

$$E_n := \mathbb{C}[e_{ij}, t] \Big/ \Big\langle \sum_j e_{ij} \ \forall\, i; \ \ e_{ij} + e_{ji} \ \forall\, i, j; \ \ (e_{ij} + e_{jk} + e_{ki})^2 - t(e_{ij}^2 + e_{jk}^2 + e_{ki}^2) \ \forall\, i, j, k \text{ distinct} \Big\rangle$$

and note that $E_n/\langle t \rangle \cong D_n$ and $E_n/\langle t-1 \rangle \cong M_n$. If E_n were free over $\mathbb{C}[t]$, then we would have a flat deformation from D_n to M_n, which would prove Conecjture 2 by semisimplicity of the category of finite dimensional graded representations of S_n. Unfortunately, this family is not free; indeed, the $t = 0$ and $t = 1$ specializations are both larger (by the same amount) than the generic specialization. A closely related observation appears in [MPY17, 2.14].

References

[BP09] Tom Braden and Nicholas Proudfoot, *The hypertoric intersection cohomology ring*, Invent. Math. **177** (2009), no. 2, 337–379.

[MPY17] Daniel Moseley, Nicholas Proudfoot, and Ben Young, *The Orlik-Terao algebra and the cohomology of configuration space*, Exp. Math. **26** (2017), no. 3, 373–380.

[PS06] Nicholas Proudfoot and David Speyer, *A broken circuit ring*, Beiträge Algebra Geom. **47** (2006), no. 1, 161–166.