The Orlik-Terao algebra and the cohomology of configuration space

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Abstract. We give a recursive algorithm for computing the Orlik-Terao algebra of the Coxeter arrangement of type A_{n-1} as a graded representation of S_n, and we give a conjectural description of this representation in terms of the cohomology of the configuration space of n points in $SU(2)$ modulo translation. We also give a version of this conjecture for more general graphical arrangements.

1 Introduction

We consider the subalgebra OT_n of rational functions on $\mathbb{C}^n$ generated by $\frac{1}{x_i - x_j}$ for all $i \neq j$. This is a special case of a class of algebras called Orlik-Terao algebras, which have received much recent attention [Ter02, PS06, ST09, Sch11, VLR13, SSV13, DGT14, Le14, Liu, MP15, EPW]. Our interest is in understanding OT_n as a graded representation of the symmetric group S_n, which acts by permuting the indices.

Let C_n be the cohomology of the configuration space of n labeled points in $\mathbb{R}^3$, which is also acted on by S_n. The ring C_n is related to OT_n in two different ways. The first is that C_n is isomorphic to the quotient of OT_n by the ideal generated by the squares of the generators. This can be seen explicitly by computing presentations of the two rings, but there is also a much deeper geometric explanation. Braden and the second author proved that OT_n is isomorphic to the equivariant intersection cohomology of a certain hypertoric variety (Theorem 3.1), and C_n is isomorphic to the equivariant cohomology of a certain smooth open subset of that hypertoric variety; the map from OT_n to C_n is simply the restriction map in equivariant intersection cohomology. By exploring this geometric relationship further and considering not only the open subset in question but also other strata of higher codimension, we obtain a formula which allows us to recursively compute OT_n in terms of C_n (Theorem 3.2). Since the action of S_n on C_n is well understood, this allows us to compute the action of S_n on OT_n for arbitrary n.

Once we do these computations, a different and *a priori* unrelated relationship between OT_n and C_n becomes apparent. Let R_n be the symmetric algebra of the irreducible permutation representation of S_n, generated in degree two. The ring OT_n is naturally an algebra over R_n, and it

[1]Supported by NSF grant DMS-0950383.

is finitely generated and free as a graded module. Thus we may define $M_n := OT_n \otimes_{R_n} \mathbb{C}$, and we have an S_n-equivariant isomorphism $OT_n \cong R_n \otimes_{\mathbb{C}} M_n$. This reduces the problem of understanding OT_n to the problem of understanding M_n. Let D_n be the cohomology of the configuration space of n labeled points in $SU(2) \cong S^3$ modulo the action of $SU(2)$ by simultaneous left translation. It is easy to show that C_n and D_n are closely related; see Propositions 2.3 and 2.5 for precise statements. Our computations suggest the following result, which is the main conjecture in this paper (Conjecture 2.10):

Conjecture. There exists an isomorphism of graded S_n representations $M_n \cong D_n$.

Given that we have descriptions of both M_n and D_n in terms of C_n, one would think that this conjecture would be easy to prove. However, our recursive formula for M_n involves plethysms of symmetric functions, and while plethysms are fine for computing in SAGE, it is notoriously difficult to use them to prove anything.

Our paper is structured roughly in the reverse of the order in which it was presented above. We begin in Section 2 by giving a detailed account of our main conjecture, without any discussion of how to compute OT_n and M_n. We also generalize our conjecture to arbitrary graphs. In Section 3, we explain how to use the equivariant intersection cohomology of hypertoric varieties to compute OT_n. Our main result in this section is Theorem 3.2, but we also we also do some extra work to translate our recursive formula to the language of symmetric functions (Proposition 3.6), since this is the most convenient formulation for actually computing with SAGE. All of the code that was used for this project is available at `https://github.com/benyoung/ot`.

Acknowledgments: The authors are grateful to Nick Addington, Richard Stanley, Dan Petersen, and Ben Webster for their valuable suggestions and conversations. All computations were performed with the assistance of the computer algebra systems SAGE [S$^+$15] and Macaulay 2 [GS].

2 Conjectures

We begin by introducing the main players in our paper: the Orlik-Terao algebra OT_n and its finite dimensional quotient M_n (Section 2.1), the cohomology rings C_n and D_n of two closely related configuration spaces (Section 2.2), and our main conjecture relating them (Section 2.3). We also generalize our conjecture to arbitrary graphs (Section 2.4).

2.1 The Orlik-Terao algebra

Fix a positive integer n, and let OT_n be the subalgebra of rational functions on $\mathbb{C}^n$ generated by the elements $e_{ij} := \frac{1}{x_i - x_j}$ for all $i \neq j$. This algebra is known as the **Orlik-Terao algebra** of the Coxeter arrangement of type A_{n-1}. It follows from [PS06, Theorem 4] and [ST09, Proposition 2.7] that the ideal of relations between these generators is generated by $e_{ij} + e_{ji}$ for all i, j and

$e_{ij}e_{jk}+e_{jk}e_{ki}+e_{ki}e_{ij}$ for all distinct triples i, j, k. We regard OT_n as a graded ring with $\deg(e_{ij}) = 2$. Our goal is to understand OT_n as a graded representation of the symmetric group S_n, which acts by permuting the indices.

Let $R_n := \mathbb{C}[z_1, \ldots, z_n]/\langle z_1 + \cdots + z_n\rangle$, with its natural S_n action, and graded by putting $\deg(z_i) = 2$. Consider the S_n-equivariant graded algebra homomorphism $\varphi_n : R_n \to OT_n$ taking z_i to $\sum_{j\neq i} e_{ij}$. This gives OT_n the structure of a graded module over R_n, and it is in fact a free module by [PS06, Proposition 7]. In other words, if we define $M_n := OT_n \otimes_{R_n} \mathbb{C}$ to be the ring obtained by setting $\varphi_n(z_i)$ equal to zero for all i, then there exists an isomorphism of graded R_n-modules

$$OT_n \cong R_n \otimes_{\mathbb{C}} M_n.$$

This isomorphism is not canonical, and is not compatible with the ring structures on the two sides. However, it is compatible with the action of S_n on both sides. Thus, we may reduce the problem of understanding OT_n as a graded representation to the problem of understanding M_n.

Remark 2.1. It is easy to describe M_n as a graded vector space. For any finite dimensional graded vector space V concentrated in even degree, let $H(V, q) := \sum q^i \dim V_{2i}$, where V_{2i} is the degree $2i$ part of V. Then $H(M_n, q)$ is equal to the h-polynomial of the broken circuit complex associated with the Coxeter arrangement of type A_{n-1} [PS06, Proposition 7], which is equal to $(1 + q)(1 + 2q) \cdots (1 + (n - 2)q)$.

The following proposition was proved by the first author [Mos12, Theorem 3.10]; it may also be deduced from [CEF15, Theorem 3.3.3].

Proposition 2.2. *The sequence $\{M_n\}$ of graded representations of symmetric groups is representation stable.*

2.2 Two configuration spaces

Consider the configuration space $\mathrm{Conf}(n, \mathbb{R}^3)$ be the configuration space of n labeled points in $\mathbb{R}^3$, which admits an action of S_n given by permuting the labels. Let

$$C_n := H^*(\mathrm{Conf}(n, \mathbb{R}^3); \mathbb{C}),$$

which is a graded representation of S_n. The ring C_n has a presentation closely related to that of OT_n; it is isomorphic to the quotient of OT_n by the ideal generated by e_{ij}^2 for all i, j [CLM76, Chapter III, Lemma 7.7]. This algebra is also known as the **Artinian Orlik-Terao algebra** of the Coxeter arrangement of type A_{n-1}. The structure of C_n as a graded representation of S_n is complicated but well understood; see Equation (1).

Next, let $G = SU(2) \cong S^3$, and consider the configuration space $\mathrm{Conf}(n, G)/G$ of n labeled points in G up to simultaneous translation by left multiplication. This space admits an action of S_n by permuting the labels; let

$$D_n := H^*(\mathrm{Conf}(n, G)/G; \mathbb{C}),$$

which is a graded representation of S_n.

Proposition 2.3. *There exists an isomorphism*

$$C_{n-1} \cong \mathrm{Res}^{S_n}_{S_{n-1}}(D_n)$$

of graded representations of S_{n-1}.

Proof. We have a diffeomorphism $\mathrm{Conf}(n, G)/G \cong \mathrm{Conf}(n-1, \mathbb{R}^3)$ given by using the action of G to take the n^{th} point to the identity, leaving the remaining n points in $G \setminus \{\mathrm{id}\} \cong \mathbb{R}^3$. This diffeomorphism is equivariant with respect to the action of $S_{n-1} \subset S_n$. $\square$

Remark 2.4. The polynomial $H(D_n, q) = H(C_{n-1}, q)$ is equal to the f-polynomial of the broken circuit complex associated with the Coxeter arrangement of type A_{n-2} [OT94, Theorem 4.3], which is equal to $(1+q)(1+2q)\cdots(1+(n-2)q)$.

Let $W_n := R_n/\langle z_i z_j\rangle$ be the ring obtained by truncating R_n to degree two. As a graded representation of S_n, W_n is isomorphic to the 1-dimensional trivial representation in degree zero plus the irreducible permutation representation of dimension $n-1$ in degree two.

Proposition 2.5. *There exists an isomorphism*

$$C_n \cong D_n \otimes_{\mathbb{C}} W_n$$

of graded representations of S_n.

Proof. Consider the projection $\mathrm{Conf}(n+1, G)/G \to \mathrm{Conf}(n, G)/G$ given by forgetting the $(n+1)^{\text{st}}$ point. This is an S_n-equivariant fiber bundle with fiber diffeomorphic to the complement of n points in G. The base is simply connected and both the base and the fiber have cohomology only in even degree, thus the Leray-Serre spectral sequence degenerates and the cohomology of the total space is isomorphic to the tensor product of the cohomology of the base and the cohomology of the fiber. This yields the desired isomorphism. $\square$

Remark 2.6. Proposition 2.3 tells us that, if we know how to compute D_{n+1}, we know how to compute C_n. Conversely, since W_n is not a zero divisor in the semiring of graded representations of S_n, Proposition 2.5 tells us that we can recover D_n from C_n. This is important because there exist extremely explicit formulas for C_n in the literature; see Equation (1).

Corollary 2.7. *The sequence* $\{D_n\}$ *of graded representations of symmetric groups is representation stable.*

Proof. Representation stability of $\{C_n\}$ (or, more generally, for the cohomology of the configuration space of any manifold) was proved by Church [Chu12, Theorem 1]. For $\{W_n\}$, it is obvious. Representation stability of $\{D_n\}$ then follows from Proposition 2.5. $\square$

Given a graded representation V of S_n, let $\overline{V}$ be the ungraded representation obtained by forgetting the grading. In this section, we describe $\overline{C}_n$ and $\overline{D}_n$. Let $Z_n \subset S_n$ be the cyclic group.

Proposition 2.8. *There exist isomorphisms*

$$\overline{C}_n \cong \mathbb{C}[S_n] \qquad \textit{and} \qquad \overline{D}_n \cong \mathbb{C}[S_n/Z_n] \cong \operatorname{Ind}_{Z_n}^{S_n}(\mathrm{triv})$$

of representations of S_n.

Proof. The first isomorphism is well-known, but we quickly review one proof here because we will use a very similar argument for the second isomorphism. Consider the action of $U(1)$ on $\mathbb{R}^3 \cong \mathbb{R} \oplus \mathbb{C}$ given by rotation on the second factor, which induces an action of $U(1)$ on $\operatorname{Conf}(n, \mathbb{R}^3)$. Since $H^*(\operatorname{Conf}(n, \mathbb{R}^3); \mathbb{C})$ is concentrated in even degree, this action is equivariantly formal, meaning that the $U(1)$-equivariant cohomology of $\operatorname{Conf}(n, \mathbb{R}^3)$ is a free module over the equivariant cohomology of a point. It follows that there is a natural filtration on the cohomology of the fixed point set $\operatorname{Conf}(n, \mathbb{R}^3)^{U(1)}$ whose associated graded is isomorphic to the cohomology of $\operatorname{Conf}(n, \mathbb{R}^3)$ [Mos, Corollary 2.6]. Since the action of $U(1)$ commutes with the action of S_n, this isomorphism is S_n-equivariant [Mos, Proposition 2.8]. We have

$$\operatorname{Conf}(n, \mathbb{R}^3)^{U(1)} \cong \operatorname{Conf}(n, \mathbb{R}) \simeq S_n,$$

so

$$H^*\Big(\operatorname{Conf}(n, \mathbb{R}^3)^{U(1)}; \mathbb{C}\Big) \cong \mathbb{C}[S_n].$$

Passing to the associated graded does not change the isomorphism type of an (ungraded) representation of a finite group, thus $\overline{C}_n \cong \mathbb{C}[S_n]$.

For the second isomorphism, we note that $U(1)$ acts on $\operatorname{Conf}(n, G)/G$ by *right* translation, commuting with the action of S_n, with fixed point set

$$(\operatorname{Conf}(n, G)/G)^{U(1)} \cong \operatorname{Conf}(n, U(1))/U(1) \simeq S_n/Z_n.$$

The second isomorphism follows by the same argument. $\square$

Remark 2.9. The filtration of $H^*(\operatorname{Conf}(n, \mathbb{R})) \cong \mathbb{C}[S_n]$ whose associated graded is isomorphic to C_n can be described very explicitly. First, note that $\operatorname{Conf}(n, \mathbb{R})$ is a disjoint union of contractible pieces, so its cohomology ring is simply the ring of locally constant functions. A **Heaviside function** h_{ij} is a function that takes the value 1 on one side of a given hyperplane $\{x_i = x_j\}$ and 0 on the other side. We define the p^{th} filtered piece $F_p\, \mathbb{C}[S_n]$ to be the vector space of functions that can be expressed as polynomials of degree at most p in the Heaviside functions. This filtration, was first studied by Varchenko and Gelfand [VG87], coincides with the one arising from equivariant cohomology [Mos, Remark 4.9].

Similarly, we may define a **cyclic Heaviside function** h_{ijk} on $\operatorname{Conf}(n, U(1))/U(1)$ by specifying a cyclic ordering of the i^{th}, j^{th} and k^{th} points. This is equal to the pullback of h_{ij} from $\operatorname{Conf}(n-1, \mathbb{R})$

along the isomorphism from $\mathrm{Conf}(n, U(1))/U(1)$ to $\mathrm{Conf}(n-1, \mathbb{R})$ given by using the action of $U(1)$ to move the k^{th} point to the origin. Since we know that the filtration of $H^*(\mathrm{Conf}(n-1, \mathbb{R}); \mathbb{C})$ arising from equivariant cohomology coincides with the one induced by Heaviside functions, we may conclude that, for any fixed index k, the filtration of $H^*(\mathrm{Conf}(n, U(1))/U(1); \mathbb{C})$ arising from equivariant cohomology coincides with the filtration generated by the cyclic Heaviside functions $\{h_{ijk} \mid 0 \leq i < j \leq n\}$. Since the filtration arising from equivariant cohomology is preserved by the action of S_n, it must also coincide with the filtration generated by *all* cyclic Heaviside functions, where all three indices are allowed to vary.

2.3 The main conjecture

Our main conjecture is as follows.

Conjecture 2.10. *There exists an isomorphism of graded S_n representations $M_n \cong D_n$.*

Remark 2.11. Using the computational technique described in Section 3 (specifically Proposition 3.6), we have checked Conjecture 2.10 on a computer up to $n = 10$.

Remark 2.12. Remarks 2.1 and 2.4 tell us that Conjecture 2.10 holds at the level of graded vector spaces.

Remark 2.13. Since W_n is not a zero divisor in the semiring of graded representations of S_n, Conjecture 2.10 is equivalent to the statement that $M_n \otimes_{\mathbb{C}} W_n \cong D_n \otimes_{\mathbb{C}} W_n$. Since $OT_n \cong R_n \otimes_{\mathbb{C}} M_n$, we have

$$M_n \otimes_{\mathbb{C}} W_n \cong OT_n / \langle z_i z_j \rangle.$$

On the other hand, Proposition 2.5 says that

$$D_n \otimes_{\mathbb{C}} W_n \cong C_n \cong OT_n / \langle e_{ij}^2 \rangle.$$

We know that $\mathbb{C}\{z_i z_j\}$ and $\mathbb{C}\{e_{ij}^2\}$ are both isomorphic to the symmetric square of the irreducible permutation representation, thus Conjecture 2.10 holds in degrees zero, two, and four for all values of n.

Remark 2.14. Since $z_i z_j = -e_{ij}^2 + f_{ij}$, where f_{ij} is a certain sum of square-free monomials, it is natural to consider the family of rings

$$A_n(t) := OT_n / \langle (1-t) e_{ij}^2 - t z_i z_j \rangle = OT_n / \langle e_{ij}^2 - t f_{ij} \rangle,$$

where $t \in \mathbb{C}$. By Remark 2.13, $A_n(0) \cong D_n \otimes_{\mathbb{C}} W_n$ and $A_n(1) \cong M_n \otimes_{\mathbb{C}} W_n$. There exists a nonempty Zariski open subset $U \subset \mathbb{C}$ such that the restriction of this family to U is flat, which means that the graded S_n representations $A_n(t)$ are isomorphic for all $t \in U$. If $0, 1 \in U$, this would imply Conjecture 2.10. Unfortunately, this is not the case. For example, when $n = 4$, computations in Macaulay 2 reveal that $U = \mathbb{C} \smallsetminus \{0, 1, -\frac{1}{2}\}$.

Put differently, this means that *most* ideals in OT_4 that are generated by a copy of the symmetric square of the permutation representation in degree four are strictly larger than both $\langle e_{ij}^2\rangle$ and $\langle z_i z_j\rangle$. These two ideals are exceptional, and our conjecture (which is true when $n = 4$) says that they are exceptional in the same way.

2.4 Generalizing to graphs

In this section we generalize some of our results and conjectures to graphs; the cases described above correspond to the complete graph.

Let Γ be a simple connected graph with vertex set $[n]$, and let $\mathrm{Aut}(\Gamma) \subset S_n$ be the group of automorphisms of Γ. Let OT_Γ be the Orlik-Terao algebra of the hyperplane arrangement associated with Γ; this is the subalgebra of rational functions on $\mathbb{C}^n$ generated by $\frac{1}{x_i - x_j}$ whenever i and j are connected by an edge. It is a graded representation of the automorphism group $\mathrm{Aut}(\Gamma) \subset S_n$, with the generators in degree two. We again have a map from R_n to OT_Γ as before, and we let

$$M_\Gamma := OT_\Gamma \otimes_{R_n} \mathbb{C}.$$

Then there exists a graded $\mathrm{Aut}(\Gamma)$-equivariant isomorphism

$$OT_\Gamma \cong R_n \otimes_{\mathbb{C}} M_\Gamma,$$

and $H(M_\Gamma, q) = h_\Gamma(q)$, the h-polynomial of the corresponding broken circuit complex [PS06, Proposition 7].

For any space X, consider the space $\mathrm{Conf}(\Gamma, X)$ of maps from the vertices of Γ to X such that adjacent vertices map to different points. Let

$$C_\Gamma := H^*(\mathrm{Conf}(\Gamma, \mathbb{R}^3) \qquad \text{and} \qquad D_\Gamma := H^*(\mathrm{Conf}(\Gamma, G)/G; \mathbb{C}),$$

both graded representations of $\mathrm{Aut}(\Gamma)$. Let $\hat{\Gamma}$ be the cone over Γ; this is the graph with vertex set $[n+1]$ such that the $(n+1)^{\text{st}}$ vertex is connected to all other vertices and the subgraph spanned by the remaining vertices is equal to Γ. The following proposition is a straightforward generalization of Proposition 2.3.

Proposition 2.15. *There exists an isomorphism*

$$C_\Gamma \cong \mathrm{Res}^{\mathrm{Aut}(\hat{\Gamma})}_{\mathrm{Aut}(\Gamma)} \left(D_{\hat{\Gamma}} \right)$$

of graded representations of $\mathrm{Aut}(\Gamma)$.

The following conjecture is a natural generalization of Conjecture 2.10.

Conjecture 2.16. *For any simple connected graph* Γ*, there exists an isomorphism*

$$M_\Gamma \cong D_\Gamma$$

of graded representations of $\mathrm{Aut}(\Gamma)$*. In particular, there exists an isomorphism*

$$\mathrm{Res}^{\mathrm{Aut}(\hat{\Gamma})}_{\mathrm{Aut}(\Gamma)}\left(M_{\hat{\Gamma}}\right) \cong C_{\Gamma}.$$

Remark 2.17. We have $H(M_{\hat{\Gamma}}, q) = h_{\hat{\Gamma}}(q) = f_{\Gamma}(q) = H(C_{\Gamma}, q)$, thus the second part of Conjecture 2.16 holds at the level of graded vector spaces.

3 Computing M_n via hypertoric geometry

In this section, we explain how to use the geometry of hypertoric varieties to compute M_n.

3.1 Hypertoric varieties

Given any hyperplane arrangement $\mathcal{A}$ defined over the rational numbers, one may define a variety called a **hypertoric variety**. Rather than giving a general construction, we will instead give a direct definition of the hypertoric variety X_n associated with the (doubled) Coxeter arrangement of type A_{n-1}. For a general definition, see [Pro08].

Let K_n be the lattice of rank $n(n-1)$ with basis $\{y_{ij} \mid i \neq j \in [n]\}$. Consider the map $\pi : K_n \to \mathbb{Z}\{x_1, \ldots, x_n\}$ taking y_{ij} to $x_i - x_j$, and let L_n be the image of π. Consider the polynomial ring in $2n(n-1)$ variables

$$Q_n := \mathbb{C}[z_{ij}, w_{ij}]_{i\neq j}.$$

This ring has a grading by K_n^* defined by putting $\deg(z_{ij}) = y_{ij}^* = -\deg(w_{ij})$. Let Q_n^L denote the subring of Q_n spanned by homogeneous elements whose degree lies in the sublattice $L_n^* \subset K^*$. Consider the map

$$\mu_n : \mathrm{Sym}\, K_n^{\mathbb{C}} \to Q_n^L$$

taking y_{ij} to $z_{ij}w_{ij}$, and define

$$P_n := Q_n^L / \langle \mu_n(y) \mid \pi(y) = 0\rangle \qquad \text{and} \qquad X_n := \mathrm{Spec}\, P_n.$$

The variety X_n is the hypertoric variety that will be the main object of our attention. Let

$$T_n := \mathrm{Hom}(L_n^*, \mathbb{C}^\times)$$

be the algebraic torus of dimension $n-1$ with character lattice L_n^*; the grading of P_n by L_n^* induces an action of T_n on X_n. We also have an action of the symmetric group S_n on X_n given by permuting indices. This action does not commute with the action of T_n, but rather defines an action of the semidirect product $T_n \rtimes S_n$ on X_n, where S_n acts on T_n in the obvious way. The variety X_n and its various symmetries are important to us due to the following theorem [BP09, Corollary 4.5] (see also [MP15, Proposition 3.16]).

Theorem 3.1. *There exists a canonical isomorphism*

$$IH^*_{T_n}(X_n; \mathbb{C}) \cong OT_n$$

between the T_n-equivariant intersection cohomology of X_n and OT_n. This isomorphism is compatible with the maps from

$$H^*_{T_n}(*; \mathbb{C}) \cong \mathrm{Sym}(L_n^*)_{\mathbb{C}} \cong R_n.$$

In particular, this implies that

$$IH^*(X_n; \mathbb{C}) \cong M_n.$$

Furthermore, all of these isomorphisms are compatible with the natural actions of the symmetric group S_n.

We next define a stratification of X_n, following the general construction in [PW07, Section 2]. For each partition $B_1 \sqcup \cdots \sqcup B_\ell$ of the set $[n]$, consider the ideal

$$J_n^B := \langle z_{ij}, w_{ij} \mid \text{there exists an } r \text{ such that } i, j \in B_r \rangle \subset Q_n.$$

This ideal descends to an ideal in P_n, which cuts out a subvariety $X_n^B \subset X_n$. We have $X_n^{B'} \subset X_n^B$ if and only if B refines B', and we define

$$\mathring{X}_n^B := X_n^B \smallsetminus \bigcup_{B \text{ refines } B'} X_n^{B'}.$$

Then

$$X_n = \bigsqcup_B \mathring{X}_n^B$$

is a T_n-equivariant stratification of X_n. For each partition B, consider the subtorus

$$T_n^B := T_{|B_1|} \times \cdots \times T_{|B_\ell|} \subset T_n,$$

embedded in the natural way. Then T_n^B is the stabilizer of every point in $\mathring{X}_n^B$ [PW07, Remark 2.3], thus the torus T_n/T_n^B acts freely on $\mathring{X}_n^B$. The quotient space is not Hausdorff, but if we take the quotient of $\mathring{X}_n^B$ by the maximal compact subtorus of T_n/T_n^B, we obtain a manifold homeomorphic to $\mathrm{Conf}(\ell, \mathbb{R}^3)$ [PW07, Proposition 5.2]. Finally, the stratum $\mathring{X}_n^B$ has a normal slice that is T_n^B-equivariantly isomorphic to $X_{|B_1|} \times \cdots \times X_{|B_\ell|}$ [PW07, Lemma 2.4].

3.2 A geometric recursion

Given any partition B of $[n]$, let S_B be the stabilizer of B. Letting m_i be the number of parts of B of size i, we may express S_B as a product of wreath products:

$$S_B \;\cong\; \prod_{i=1}^{n} S_i \wr S_{m_i}.$$

Given any partition λ of n, let $B(\lambda)$ be the partition of $[n]$ given by putting $B_1 = \{1, \ldots, \lambda_1\}$, $B_2 = \{\lambda_1 + 1, \ldots, \lambda_1 + \lambda_2\}$, and so on. Let $S_\lambda := S_{B(\lambda)} \subset S_n$ be the stabilizer of the partition $B(\lambda)$, and let $W_\lambda := \prod S_{m_i} \subset S_\lambda$.

We define a graded representation M_n^c of S_n by putting $(M_n^c)_i := (M_n)_{4(n-1)-i}$. Theorem 3.1 says that $M_n \cong IH^*(X_n; \mathbb{C})$, and $4(n-1) = 2\dim_{\mathbb{C}} X_n = \dim_{\mathbb{R}} X_n$, thus we have $M_n^c \cong IH_c^*(X_n; \mathbb{C})$ by Poincaré duality. We will use the geometry of the hypertoric variety X_n to prove the following result.

Theorem 3.2. *For any positive integer n, there exists an isomorphism of graded S_n representations*

$$OT_n \;\cong\; \bigoplus_{\lambda \vdash n} \operatorname{Ind}_{S_\lambda}^{S_n} \left(C_{\ell(\lambda)} \otimes (M_{\lambda_1}^c \otimes R_{\lambda_1}) \otimes \cdots \otimes (M_{\lambda_{\ell(\lambda)}}^c \otimes R_{\lambda_{\ell(\lambda)}}) \right).$$

Here the subgroup $W_\lambda \subset S_\lambda$ acts on $C_{\ell(\lambda)}$ via the embedding $W_\lambda \hookrightarrow S_{\sum m_i} = S_{\ell(\lambda)}$, and it also permutes the remaining tensor factors of the same size. In addition, each factor of the form $M_{\lambda_j}^c \otimes R_{\lambda_j}$ is acted on by a separate subgroup $S_{\lambda_j} \subset S_\lambda$.

Remark 3.3. We claim that Theorem 3.2 provides a recursive means of computing M_n for all $n \geq 2$. To see this, we first observe that, since $OT_n \cong R_n \otimes_{\mathbb{C}} M_n$, it is possible to recover M_n from OT_n. Moreover, since M_n vanishes in degrees greater than $2(n-2)$, it is possible to recover M_n from the truncation of OT_n to degree $2(n-2)$. If we try to use Theorem 3.2 to compute OT_n and M_n in terms of M_k for $k < n$, we run into the problem that M_n^c appears on the right-hand side of the isomorphism. However, M_n^c vanishes in degrees less than $4(n-1) - 2(n-2) = 2n$, therefore we can compute the truncation of OT_n to degree $2(n-2)$ without knowing M_n, and we avoid any circularity.

Remark 3.4. Theorem 3.2 can be generalized to a recursive expression for $OT_{\mathcal{A}}$ in terms $M_{\mathcal{A}'}^c$ for various restrictions $\mathcal{A}'$ of $\mathcal{A}$ and $C_{\mathcal{A}''}$ for various localizations $\mathcal{A}''$ of $\mathcal{A}$. Taking $\mathcal{A}$ to be a graphical arrangement, this means we may compute OT_Γ in terms of $M_{\Gamma'}^c$ for various contractions Γ' of Γ and $C_{\Gamma''}$ for various subgraphs Γ'' of Γ.

Let IC_{X_n} be the T_n-equivariant intersection cohomology sheaf on X_n. For each partition $B = B_1 \sqcup \cdots \sqcup B_\ell$ of $[n]$, let $\iota_B : \mathring{X}_n^B \hookrightarrow X_n$ be the inclusion. To prove Theorem 3.2, we first establish the following lemma.

Lemma 3.5. *There exists an S_B-equivariant isomorphism of graded vector spaces*

$$\mathbb{H}_{T_n}^*(\mathring{X}_n^B; \iota_B^! IC_{X_n}) \cong C_\ell \otimes (M_{|B_1|}^c \otimes R_{|B_1|}) \otimes \cdots \otimes (M_{|B_\ell|}^c \otimes R_{|B_\ell|}).$$

Proof. The cohomology of the complex $\iota_B^! IC_{X_n}$ is a T_n-equivariant local system on $\mathring{X}_n^B$ whose fiber at a point is the compactly supported cohomology of the stalk of IC_{X_n} at that point. This is the same as the compactly supported intersection cohomology of the normal slice $X_{|B_1|} \times \cdots \times X_{|B_\ell|}$ to $\mathring{X}_n^B \subset X_n$. Since the quotient of $\mathring{X}_n^B$ by the maximal compact subtorus of T_n is homeomorphic to the simply connected space $\mathrm{Conf}(\ell,\mathbb{R}^3)$, this local system is trivial. We therefore have a spectral sequence E with

$$E_2^{p,q} = H^p_{T_n}(\mathring{X}^B_n;\mathbb{C}) \otimes IH^q_c(X_{|B_1|}\times\cdots\times X_{|B_\ell|};\mathbb{C})$$

that converges to $\mathbb{H}^*_{T_n}(\mathring{X}^B_n;\iota^!_B IC_{X_n})$. Since these cohomology groups are concentrated in even degree, all differentials are zero, therefore

$$\begin{aligned}
E_\infty = E_2 &= H^*_{T_n}(\mathring{X}^B_n;\mathbb{C}) \otimes IH^*_c(X_{|B_1|}\times\cdots\times X_{|B_\ell|};\mathbb{C})\\
&\cong H^*(\mathrm{Conf}(\ell,\mathbb{R}^3);\mathbb{C}) \otimes H^*_{T^B_n}(*;\mathbb{C}) \otimes IH^*_c(X_{|B_1|}\times\cdots\times X_{|B_\ell|};\mathbb{C})\\
&\cong C_\ell \otimes R_{|B_1|}\otimes\cdots\otimes R_{|B_\ell|}\otimes M^c_{|B_1|}\otimes\cdots\otimes M^c_{|B_\ell|}\\
&\cong C_\ell \otimes (M^c_{|B_1|}\otimes R_{|B_1|})\otimes\cdots\otimes(M^c_{|B_\ell|}\otimes R_{|B_\ell|}).
\end{aligned}$$

Since the category of graded representations of S_B is semisimple, we have a (noncanonical) S_B-equivariant isomorphism of graded vector spaces $\mathbb{H}^*_{T_n}(\mathring{X}^B_n;\iota^!_B IC_{X_n}) \cong E_\infty$. □

Proof of Theorem 3.2: There is a spectral sequence E with

$$E_1^{p,q} = \bigoplus_{\substack{B_1\sqcup\cdots\sqcup B_\ell=[n]\\ \ell=n-p}} \mathbb{H}^{p+q}_{T_n}(\mathring{X}^B_n;\iota^!_B IC_{X_n})$$

that converges to $IH^*_{T_n}(X_n;\mathbb{C})$ [BGS96, Section 3.4]. By Lemma 3.5, $E_1^{p,q}=0$ unless $p+q$ is even, thus

$$\begin{aligned}
E_\infty = E_1 &\cong \bigoplus_B \mathbb{H}^*_{T_n}(\mathring{X}^B_n;\iota^!_B IC_{X_n})\\
&\cong \bigoplus_B C_\ell \otimes (M^c_{|B_1|}\otimes R_{|B_1|})\otimes\cdots\otimes(M^c_{|B_\ell|}\otimes R_{|B_\ell|}).
\end{aligned}$$

As a representation of S_n, this is isomorphic to

$$\bigoplus_{\lambda\vdash n} \mathrm{Ind}^{S_n}_{S_\lambda}\left(C_{\ell(\lambda)}\otimes(M^c_{\lambda_1}\otimes R_{\lambda_1})\otimes\cdots\otimes(M^c_{\lambda_{\ell(\lambda)}}\otimes R_{\lambda_{\ell(\lambda)}})\right).$$

Since the category of graded representations of S_n is semisimple, we have a (noncanonical) S_n-equivariant isomorphism of graded vector spaces $IH^*_{T_n}(X_n;\mathbb{C})\cong E_\infty$. The result now follows from Theorem 3.1. □

3.3 Symmetric functions

In order to implement the recursive formula in Theorem 3.2 in SAGE, it is convenient to convert everything to the language of symmetric functions. Let Λ be the ring of symmetric functions in infinitely many variables with coefficients in the formal power series ring $\mathbb{Z}[[q]]$. If V is a graded representation of S_n, concentrated in even degree, with finite dimensional graded parts, then its

graded Frobenius characteristic $\operatorname{ch} V$ is an element of Λ of symmetric degree n; the coefficient of q^i is equal to the usual Frobenius characteristic of V_{2i}. The Frobenius characteristic map is an isomorphism of vector spaces, thus it is sufficient to compute $\operatorname{ch} OT_n$ and $\operatorname{ch} M_n$ for each n. More concretely, expressing M_n as an $\mathbb{N}[q]$-linear combination of irreducible representations is equivalent to expressing $\operatorname{ch} M_n$ as an $\mathbb{N}[q]$-linear combination of Schur functions.

We begin by analyzing a single summand from Theorem 3.2. The first piece that we need to understand better is $C_{\ell(\lambda)}$, which is acted on by the subgroup $W_\lambda \subset S_\lambda$. We want to decompose $C_{\ell(\lambda)}$ into irreducible representations for this subgroup:

$$C_{\ell(\lambda)} \cong \bigoplus_{\substack{(\nu_1,\ldots,\nu_n) \\ \nu_i \vdash m_i}} V_{\nu_1} \otimes \cdots \otimes V_{\nu_n} \otimes U(\nu_1, \ldots, \nu_n),$$

where

$$\begin{aligned} U(\nu_1, \ldots, \nu_n) &:= \operatorname{Hom}_{W_\lambda}\left(V_{\nu_1} \otimes \cdots \otimes V_{\nu_n}, C_{\ell(\lambda)}\right) \\ &\cong \operatorname{Hom}_{S_n}\left(\operatorname{Ind}_{W_\lambda}^{S_n}\left(V_{\nu_1} \otimes \cdots \otimes V_{\nu_n}\right), C_{\ell(\lambda)}\right) \end{aligned}$$

is the graded vector space that records the graded multiplicity of $V_{\nu_1} \otimes \cdots \otimes V_{\nu_n}$ in $C_{\ell(\lambda)}$.

Let Y_λ denote the Young subgroup $\prod_{i=1}^n S_{im_i}$, so that we have $S_\lambda \subset Y_\lambda \subset S_n$. We will break up our induction into two steps, first from S_λ to Y_λ and then from Y_λ to S_n. We have

$$\begin{aligned} &\operatorname{Ind}_{S_\lambda}^{S_n}\left(C_{\ell(\lambda)} \otimes (M^c_{\lambda_1} \otimes R_{\lambda_1}) \otimes \cdots \otimes (M^c_{\lambda_{\ell(\lambda)}} \otimes R_{\lambda_{\ell(\lambda)}})\right) \\ \cong\ &\operatorname{Ind}_{Y_\lambda}^{S_n} \operatorname{Ind}_{S_\lambda}^{Y_\lambda}\left(C_{\ell(\lambda)} \otimes (M^c_{\lambda_1} \otimes R_{\lambda_1}) \otimes \cdots \otimes (M^c_{\lambda_{\ell(\lambda)}} \otimes R_{\lambda_{\ell(\lambda)}})\right) \\ \cong\ &\bigoplus_{\substack{(\nu_1,\ldots,\nu_n) \\ \nu_i \vdash m_i}} U(\nu_1, \ldots, \nu_n) \otimes \operatorname{Ind}_{Y_\lambda}^{S_n}\left(\bigotimes_{i=1}^n \operatorname{Ind}_{S_i \wr S_{m_i}}^{S_{im_i}}\left(V_{\nu_i} \otimes (M^c_i \otimes R_i)^{\otimes m_i}\right)\right). \end{aligned}$$

The graded Frobenius characteristic map has the following properties [Mac95, Sections I.7-8]:

- $\operatorname{ch} V_\nu = s_\nu$ (irreducibles go to Schur functions)
- if $S_n \curvearrowright V$ and $S_n \curvearrowright V'$, then $\operatorname{ch}(V \oplus V') = \operatorname{ch} V + \operatorname{ch} V'$
- if $S_n \curvearrowright V$ and $S_n \curvearrowright V'$, then $\operatorname{ch}(V \otimes V') = \operatorname{ch} V * \operatorname{ch} V'$ (internal or "Kroneker" product)
- if $S_n \curvearrowright V$ and $S_n \curvearrowright V'$, then $H(\operatorname{Hom}_{S_n}(V, V'), q) = \langle \operatorname{ch} V, \operatorname{ch} V' \rangle$ (inner product)
- if $S_i \curvearrowright V$ and $S_j \curvearrowright V'$, then $\operatorname{ch} \operatorname{Ind}_{S_i \times S_j}^{S_{i+j}}(V \otimes V') = \operatorname{ch} V \cdot \operatorname{ch} V'$ (ordinary product)
- if $S_i \curvearrowright V$ and $S_j \curvearrowright V'$, then $\operatorname{ch} \operatorname{Ind}_{S_i \wr S_j}^{S_{ij}}\left(V' \otimes V^{\otimes j}\right) = \operatorname{ch} V'[\operatorname{ch} V]$ (plethysm).

The analysis that we have done in this section, combined with Theorem 3.2, gives us the following result.

Proposition 3.6. *We have*

$$\operatorname{ch} OT_n \;=\; \sum_{\substack{(\nu_1,\ldots,\nu_n)\\ \sum i|\nu_i|=n}} \left\langle s_{\nu_1}\cdots s_{\nu_n}, \operatorname{ch} C_{\sum|\nu_i|}\right\rangle \;\prod_{i=1}^{n} s_{\nu_i}[\operatorname{ch} M_i^c * \operatorname{ch} R_i].$$

Recall that M_i^c is just M_i "backward", so $\operatorname{ch} M_i^c$ is obtained from $\operatorname{ch} M_i$ by replacing q with q^{-1} and multiplying by $q^{2(i-1)}$. Thus, in order to use Proposition 3.6 to compute $\operatorname{ch} OT_n$ and $\operatorname{ch} M_n$ recursively, it remains only to find explicit formulas for $\operatorname{ch} C_n$ and $\operatorname{ch} R_n$. A formula for C_n is given by Hersh and Reiner [HR, Theorem 2.7], based on the work of Sundaram and Welker [SW97, Theorem 4.4(iii)]. Let ζ_n be an irreducible 1-dimensional representation of the cyclic group $Z_n \subset S_n$ whose character takes a generator of Z_n to a primitive n^{th} root of unity, and let $\ell_n := \operatorname{ch}\operatorname{Ind}_{Z_n}^{S_n}(\zeta_n)$. Let h_n denote the complete homogeneous symmetric function of degree n. Then

$$\operatorname{ch} C_n = \sum_{\lambda\vdash n} q^{\sum(i-1)m_i} \prod_{i=1}^{n} h_{m_i}[\ell_i]. \tag{1}$$

The description of $\operatorname{ch} R_n$ can be found in [Pro03, Section 5.6]:

$$\operatorname{ch} R_n = (1-q)\sum_{\lambda\vdash n} s_\lambda(1,q,q^2,\ldots)s_\lambda.$$

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