Let $G$ be a reductive algebraic group over the complex numbers and $V$ a finite dimensional linear representation of $G$. Let $X$ be the Coulomb space obtained via the construction of Braverman, Finkelberg, and Nakajima [BFN] and let $X^!$ be the Higgs space obtained as a symplectic quotient of $T^*V$ by $G$. We assume that $X$ and $X^!$ are both conical symplectic singularities, and that there exists a cocharacter of $G$ that induces a Hamiltonian $G_m$-action on $X^!$ with a unique fixed point along with a conical symplectic resolution $\tilde{X}^!$ of $X^!$. We refer to $X^!$ as the symplectic dual of $X^!$; see [BLPW16] and [Web] for more on this notion.

We first focus on a purely on the Coulomb side. Let $\mathcal{A}$ be the canonical quantization of the universal filtered Poisson deformation of a $\mathbb{Q}$-factorial terminalization of $X$; roughly speaking, this is an algebra with a large center whose central quotients give all possible quantizations of $\mathbb{C}[X]$. Let $T$ be a maximal torus of the automorphism group of $X$. Using the algebra $\mathcal{A}$, we construct in [KMP] a D-module on an affine $T$-toric variety which we call the D-module of traces because it serves as a universal source for certain types of graded trace maps. The fiber over the identity of $T$ is isomorphic to the degree zero Hochschild homology of $\mathcal{A}$, while the fiber over the $T$-fixed point is isomorphic to the degree zero Hochschild homology of the $B$-algebra of $\mathcal{A}$, a gadget which is useful for studying category $\mathcal{O}$.

We now move over to the Higgs side. Assuming some conjectures of Okounkov [Oko §2.3.4], we define the quantum D-module for $\tilde{X}^!$, which is a D-module over an open subset of the same affine toric variety that appeared on the Coulomb side. We then pass to the Calabi-Yau specialization by setting the Rees parameter equal to the $G_m$-equivariant parameter. This allows us to formulate the quantum Hikita conjecture [KMP].

Conjecture 1. The D-module of graded traces for $X$, after restricting to the appropriate open subset, becomes isomorphic to the specialized quantum D-module for $\tilde{X}^!$.

Remark 2. By taking fibers over the $T$-fixed point, Conjecture 1 specializes to a version of Nakajima’s extension of the Hikita conjecture [KTW] 8.9, which relates the $B$-algebra of $\mathcal{A}$ to the equivariant cohomology of $\tilde{X}^!$. If we further kill the equivariant parameters, we obtain the original conjecture of Hikita [Hik17], which relates the $B$-algebra of $\mathbb{C}[X]$ to the ordinary cohomology of $\tilde{X}^!$.

Remark 3. Conjecture 1 is proved for hypertoric varieties (which are dual to other hypertoric varieties) and for the Springer resolution (which is self-dual) [KMP]. Note that the Springer resolution only arises via the Coulomb/Higgs construction described above in type A, so we are actually using a slightly more flexible notion of symplectic duality here.

References


