Nicholas Proudfoot: Research Statement

Most of my current research involves the application of geometric techniques to representation theory. We can learn a lot about a geometric object $\mathfrak{M}$ by studying its ring of functions $\mathrm{Fun}(\mathfrak{M})$, and conversely we can express many properties of the ring $\mathrm{Fun}(\mathfrak{M})$ in terms of the geometry of $\mathfrak{M}$. This correspondence is a powerful tool, and it is the main idea behind the modern foundations of algebraic geometry. It has, however, a severe limitation from the point of view of algebra: all of the rings that arise in this manner are commutative. Thus most rings do not arise as the ring of functions on a geometric object, and therefore cannot be studied from this perspective—at least not directly.

My work revolves around the study of geometric objects that I will call, for the purpose of this statement, **quantum algebraic varieties**. In this context, the word "quantum" means that these objects are endowed with a new, twisted notion of what constitutes a function on them. The resulting ring of quantum functions, which I'll call $\mathrm{QFun}(\mathfrak{M})$, is closely related to but not equal to $\mathrm{Fun}(\mathfrak{M})$. Most important, $\mathrm{QFun}(\mathfrak{M})$ is not commutative. In many cases, this ring was discovered first by purely algebraic means, and only later was it found to be equal to the ring of quantum functions on a quantum algebraic variety. The best known example is when $\mathfrak{M}$ is the cotangent bundle to a flag variety. In this case, $\mathrm{QFun}(\mathfrak{M})$ is closely related to the universal enveloping algebra of a Lie algebra, which is one of the fundamental objects of study in representation theory. Many statements about representations of Lie algebras may be proven using the geometry of $\mathfrak{M}$.

My work focuses on building a theory of which the above example is a special case. That is, I consider more general quantum algebraic varieties, and prove theorems about their representation categories that generalize theorems about representations of Lie algebras. The most exciting aspect of this program has been the notion of **symplectic duality**, which posits that every sufficiently nice quantum algebraic variety is intimately related to a second, dual variety. In the case of hypertoric varieties, symplectic duality has provided us with a new way to understand a combinatorial theory known as Gale duality for matroids. In the case of Slodowy slices, it reflects a phenomenon in representation theory known as parabloic-singular duality. Though I do not yet have a general theory of how dual pairs arise, there is evidence to suggest that it is related to a topic of current interest in string theory known as mirror duality.

I will now give a more technical account of my research program, including a detailed discussion of the examples mentioned above. Everything described here is joint work with Tom Braden, Anthony Licata, and Ben Webster.

Fix a smooth, symplectic, complex algebraic variety $(\mathfrak{M}, \omega)$, along with an action of $\mathbb{S} := \mathbb{C}^\times$ on $\mathfrak{M}$. We assume that $\mathfrak{M}$ is a resolution of its affinization $\mathfrak{M}_0 := \operatorname{Spec} \mathbb{C}[\mathfrak{M}]$, that the action of $\mathbb{S}$ contracts $\mathfrak{M}_0$ to a single point, and that there exists a positive integer n such that $s^*\omega = s^n\omega$ for all $s \in \mathbb{S}$. We refer to this collection of data as a **conical symplectic resolution**. Examples of conical symplectic resolutions include Hilbert schemes of points on ALE spaces, cotangent bundles of partial flag varieties, hypertoric varieties, and Nakajima quiver varieties.

Next, fix an $\mathbb{S}$-equivariant quantization $\mathcal{D}$ of $(\mathfrak{M}, \omega)$. Roughly speaking, $\mathcal{D}$ is a non-commutative deformation of the structure sheaf of $\mathfrak{M}$, with the property that the first order term of the deformation is given by the Poisson bracket induced by ω. The main thing to know about $\mathcal{D}$ is that the ring $A := \Gamma_{\mathbb{S}}(\mathcal{D})$ of equivariant sections of $\mathcal{D}$ is a non-commutative filtered ring whose associated graded is canonically isomorphic to the ring $\mathbb{C}[\mathfrak{M}]$ of functions on $\mathfrak{M}$. (This is the ring that I previously called $\mathrm{QFun}(\mathfrak{M})$.) Many of the above examples of conical symplectic resolutions admit quantizations for which the ring A is of independent interest.

- The Hilbert scheme of n points on a crepant resolution of $\mathbb{C}^2/\Gamma$ admits quantizations for which A is isomorphic to any central quotient of the spherical rational Cherednik algebra of $S_n \wr \Gamma$.

- Let G be a reductive algebraic group and $B \subset G$ a Borel subgroup. Then $T^*(G/B)$ admits quantizations for which A is isomorphic to any central quotient of the universal enveloping algebra $U(\mathfrak{g})$.

- The resolution of a Slodowy slice to a nilpotent orbit in $\mathfrak{g}$ admits quantizations for which A is isomorphic to any central quotient of a finite W-algebra.

- A hypertoric variety admits quantizations for which A is isomorphic to any central quotient of the hypertoric enveloping algebra, originally introduced by Musson and van den Bergh.

Consider the category A-mod of finitely generated A-modules, along with the category $\mathcal{D}$-mod of good, coherent, $\mathbb{S}$-equivariant sheaves of $\mathcal{D}$-modules in the sense defined by Kashiwara and Rouquier. In the special case where $\mathfrak{M}$ is the cotangent bundle of a projective variety, $\mathcal{D}$-mod is equivalent to the category of twisted D-modules on the base.

One can define a pair of adjoint functors

$$\Gamma_{\mathbb{S}} : \mathcal{D}\text{-mod} \to A\text{-mod} \qquad \text{and} \qquad \mathrm{Loc} : A\text{-mod} \to \mathcal{D}\text{-mod}\,.$$

The functor $\Gamma_{\mathbb{S}}$ is given simply by taking $\mathbb{S}$-equivariant global sections, while the functor Loc is defined by the formula $\mathrm{Loc}(N) = \mathcal{D} \otimes_A N$. It is sometimes (but not always) the case that $\Gamma_{\mathbb{S}}$ and Loc are quasi-inverse equivalences of categories; when they are, we say that **localization holds** for $\mathcal{D}$. Philosophically, this means that the pair $(\mathfrak{M}, \mathcal{D})$ is "affine", even though $\mathfrak{M}$ itself is not. Localization is known to hold in many special cases, including quantizations of the Hilbert scheme of points in the plane, the cotangent bundle of a partial flag variety, resolved Slodowy slices of simply laced type, and hypertoric varieties. We conjecture that every conical symplectic resolution admits many quantizations for which localization holds. We can prove this conjecture for a wide class of examples, including all of those discussed above.

The last piece of data that I would like to introduce is a second action of the multiplicative group on $\mathfrak{M}$. More precisely, let $\mathbb{T} := \mathbb{C}^\times$, and suppose that $\mathfrak{M}$ is equipped with an action of $\mathbb{T}$ that commutes with the action of $\mathbb{S}$. We assume that the action of $\mathbb{T}$ lifts to a hamiltonian action on $\mathcal{D}$; in particular, this induces an integer grading of A such that the corresponding grading of $\operatorname{gr} A \cong \mathbb{C}[\mathfrak{M}]$ corresponds with the one induced by the $\mathbb{T}$-action. Consider the subring $A^+ \subset A$ spanned by all elements of non-negative degree. We define $\mathcal{O}$ to be the full subcategory of A-mod consisting of modules on which A^+ acts locally finitely. More precisely, a finitely generated A-module N lives in $\mathcal{O}$ if and only if $A^+ \cdot n$ is a finite dimensional vector space for all $n \in N$.

Example 1 Let G be a reductive algebraic group, $B \subset G$ a Borel subgroup, and $H \subset B$ a Cartan subgroup. A block of the classical BGG category $\mathcal{O}$ is by definition the full subcategory of finitely generated $U(\mathfrak{g})$ modules for which $U(\mathfrak{b})$ acts locally finitely, $U(\mathfrak{h})$ acts semisimply, and the center of $U(\mathfrak{g})$ acts with a fixed generalized character. It is a theorem of Soergel that, if the central character is sufficiently general, this is equivalent to the category obtained by dropping the condition that $U(\mathfrak{h})$ acts semisimply but adding the condition that the center of $U(\mathfrak{g})$ acts with a fixed honest character.

In our setup, A is the analogue of a simple quotient of $U(\mathfrak{g})$, A^+ is the analogue of $U(\mathfrak{b})$, and we have no analogue of $U(\mathfrak{h})$. When $\mathfrak{M} = T^*(G/B)$ and A is isomorphic to a generic simple quotient of $U(\mathfrak{g})$, our category $\mathcal{O}$ is equivalent to a regular block of BGG category $\mathcal{O}$ by Soergel's theorem. Many of the deepest results about this category are obtained by applying the localization functor and working with D-modules (or perverse sheaves) on G/B.

For the remainder of this document, we take $\mathcal{D}$ to be a quantization of $\mathfrak{M}$ with integral period (in the sense defined by Bezrukavnikov and Kaledin) and we assume that localization holds. Assume also that $\mathbb{T}$ acts on $\mathfrak{M}$ with isolated fixed points indexed by a finite set $\mathcal{I}$. We conjecture that the category $\mathcal{O}$ has many beautiful properties, which we collect below.

1. The category $\mathcal{O}$ has finite length, and has simple objects indexed by $\mathcal{I}$. It is Koszul, as well as highest weight with respect to a geometrically defined order on $\mathcal{I}$.
2. The complexified Grothendieck group $K(\mathcal{O})_{\mathbb{C}}$ is isomorphic to via a geometrically defined cycle map to the group $H^{\dim \mathfrak{M}}_{\mathbb{T}}(\mathfrak{M};\mathbb{C})$. The Beilinson-Bernstein-Deligne filtration of $H^{\dim \mathfrak{M}}_{\mathbb{T}}(\mathfrak{M};\mathbb{C})$ coincides with a filtration of $K(\mathcal{O})_{\mathbb{C}}$ defined using an analogue of Kazhdan-Lusztig cells in $\mathcal{I}$.
3. Let E be the Yoneda algebra of $\mathcal{O}$, that is, the Ext algebra of the direct sum of all of the simple objects of $\mathcal{O}$ (up to isomorphism). Thus E-mod is the Koszul dual of $\mathcal{O}$. The cohomology ring $H^*(\mathfrak{M};\mathbb{C})$ is isomorphic (as a graded $\mathbb{C}$-algebra) to the center of E.
4. Let T be a maximal torus in the hamiltonian symplectomorphism group of $\mathfrak{M}$ that contains $\mathbb{T}$. (If $\mathbb{T}$ is generic, then there will be a unique such T.) The ring $H^*_T(\mathfrak{M};\mathbb{C})$ is isomorphic to the center of the universal deformation of E, as defined in [5].
5. There are two collections of geometrically defined derived auto-equivalences of $\mathcal{O}$, generalizing Arkhipov's twisting functors and Irving's shuffling functors on a block of BGG category $\mathcal{O}$.

Remark 2 When $\mathfrak{M} = T^*(G/P)$, Properties (1)-(3) and (5) are well known, and we have proven Property (4) in [5]. When $\mathfrak{M}$ is a hypertoric variety, we have proven everything above [1, 2, 5]. The first part of Property (2) is proven for arbitrary quiver varieties by Zheng, who uses the category $\mathcal{O}$ to categorify tensor products of irreducible representations of quantum groups. In this setting, we conjecture that the twisting functors categorify the braiding action by R-matrices [4].

Perhaps the most interesting aspect of this program is the notion of symplectic duality. Given any $\mathbb{S}\times\mathbb{T}$-equivariant quantized conical symplectic resolution $\mathfrak{M}$ as above, we conjecture that there exists a **dual** $\mathbb{S}\times\mathbb{T}$-equivariant quantized conical symplectic resolution $\mathfrak{M}^!$ with an associated category $\mathcal{O}^!$ having the following properties.

- The fixed point set of $\mathfrak{M}^!$ is indexed by the same set $\mathcal{I}$. The left cells in $\mathcal{I}$ for $\mathcal{O}^!$ coincide with the right cells for $\mathcal{O}$, and vice-versa.
- The categories $\mathcal{O}$ and $\mathcal{O}^!$ are Koszul dual. In particular, the center of $\mathcal{O}^!$ is isomorphic to the cohomology ring of $\mathfrak{M}$, and vice-versa. Furthermore, the derived categories of their graded lifts are canonically identified.
- Under the identification of the derived categories, the (graded) twisting functors for $\mathcal{O}$ coincide with the (graded) shuffling functors for $\mathcal{O}^!$, and vice-versa.

The following is a list of some examples, both proven and conjectural, of dual pairs of symplectic varieties.

- We conjecture that $T^*(G/B)$ is dual to $T^*(G^!/B^!)$, where $G^!$ is the Langlands dual of G.

- We prove in [4] that $T^*(\mathrm{GL}(n)/P)$ is dual to a resolution of the Slodowy slice to a $\mathfrak{gl}(n)$ nilpotent orbit determined by the parabolic P. We have used this duality, along with Properties (3) and (4) of category $\mathcal{O}$, to explain a relationship between the equivariant cohomology rings of these spaces that was originally observed by Goresky and MacPherson [5]. We hope to use this example to relate the knot invariants of of Mazorchuk, Stroppel, and Sussan, defined using twisting functors $T^*(\mathrm{GL}(n)/P)$, to those of Seidel and Smith, defined using shuffling functors on the resolved Slodowy slice.

- We prove that hypertoric variety is dual to the Gale dual hypertoric variety [1, 2]. This gives us a geometric interpretation of the relationship between the Tutte polynomial of a matroid and that of its dual matroid.

- We conjecture that the Hilbert scheme of points on a crepant resolution of $\mathbb{C}^2/\Gamma$ is dual to the moduli space of certain torsion-free sheaves on $\mathbb{P}^2$, framed at infinity. More generally, the moduli space of G-instantons on a crepant resolution of $\mathbb{C}^2/\Gamma$ is dual to the moduli space of G'-instantons on a crepant resolution of $\mathbb{C}^2/\Gamma'$, where G is matched to Γ' and G' is matched to Γ via the MacKay correspondence. We hope to use this example, along with Property (2) of category $\mathcal{O}$, to provide a geometric interpretation of level-rank duality in the representation theory of affine Lie algebras.

- We conjecture that a quiver variety of finite type ADE is dual to a normal slice to one Schubert variety inside another one in the affine grassmannian for the Langlands dual group. Through this example we hope to relate Zheng's categorification of the Nakajima action to the work of Cautis and Kamnitzer, which uses the affine grassmannian.

- We conjecture that the Higgs branch of the moduli space of vacua for an $N = 4, d = 3$ supersymmetric quantum field theory is dual to the Higgs branch of the moduli space of the mirror dual theory, or (equivalently) the Coulomb branch of the moduli space for the same theory. The fact that cotangent bundles of Langlands dual flag varieties arise in this way appears in the recent work of Gaiotto and Witten, while the fact that Gale dual hypertoric varieties appear was shown by Kapustin and Strassler.

The meaning of the relationship between symplectic duality in mathematics and mirror duality in physics remains mysterious; understanding it is one of my main goals in the coming years.

References

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