Stability phenomena for resonance arrangements

Nicholas Proudfoot[1] and Eric Ramos[2]

Department of Mathematics, University of Oregon, Eugene, OR 97403

Department of Mathematics, Bowdoin College, Brunswick, ME 04011

Abstract. We prove that the i^{th} graded pieces of the Orlik–Solomon algebras or Cordovil algebras of resonance arrangements form a finitely generated $\mathrm{FS}^{\mathrm{op}}$-module, thus obtaining information about the growth of their dimensions and restrictions on the irreducible representations of symmetric groups that they contain.

1 Introduction

Let $\mathcal{A}(n)$ be the collection of all hyperplanes in $\mathbb{R}^n$ that are perpendicular to some nonzero vector with entries in the set $\{0,1\}$. This hyperplane arrangement is called the **resonance arrangement** of rank n. The resonance arrangement has connections to algebraic geometry, representation theory, geometric topology, mathematical physics, and economics; for a survey of these connections, see [Küh, Section 1]. Of particular interest is the set of chambers of $\mathcal{A}(n)$. Amazingly, despite the simplicity of the definition, no formula for the number of chambers as a function of n is known. A more refined invariant of $\mathcal{A}(n)$ is its characteristic polynomial, whose coefficients (after taking absolute values) have sum equal to the number of chambers. Kühne has made some progress toward understanding the coefficient of t^{n-i} in the characteristic polynomial as a function of n with i fixed. Our purpose is to shed a new light on Kühne's result, to generalize it to a wider class of arrangements, and to study the action of the symmetric group Σ_n on various algebraic invariants of these arrangements.

Let $S \subset \mathbb{R}$ be any finite set, and let $\mathcal{A}_S(n)$ be the collection of hyperplanes that are perpendicular to a nonzero vector with entries in S. If $S = \{0,1\}$, $\mathcal{A}_S(n)$ is the resonance arrangement. If $S = \{\pm 1\}$, it is the **threshold arrangement**, which is studied in [GMP]. For each positive integer d, let $M_S(n,d)$ denote the set of n-tuples of vectors in $\mathbb{R}^d$ such that no nontrivial[3] linear combination of all n vectors with coefficients in S is equal to zero. The cohomology ring of $M_S(n,d)$ is generated in degree $d-1$ [dS01, Corollary 5.6]. If d is even, the presentation of this ring in [dS01] coincides with that of the **Orlik–Solomon algebra** of $\mathcal{A}_S(n)$ (with all degrees multiplied by $d-1$) [OS80]. If d is odd and greater than 1, then it coincides with that of the **Cordovil algebra** of $\mathcal{A}_S(n)$ (with all degrees multiplied by $d-1$) [Cor02]; see also [Mos17, Example 5.8].[4] In particular, for any $n \geq 1$, $d \geq 2$, and $i \geq 0$, the dimension $b^i_S(n) = \dim H^{(d-1)i}\big(M_S(n,d);\mathbb{Q}\big)$ is equal to $(-1)^i$ times the coefficient of t^{n-i} in the characteristic polynomial of $\mathcal{A}_S(n)$.

These vector spaces carry more information than just their dimension; they also carry actions of the symmetric group Σ_n, which acts by permuting the n vectors. These representations are

[1]Supported by NSF grants DMS-1565036 and DMS-1954050.

[2]Supported by NSF grant DMS-1704811.

[3]Nontrivial means that, if $0 \in S$, we do not allow all coefficients to be 0.

[4]For d odd, the presentation in [dS01] incorrectly omits the relations that each of the generators squares to zero.

isomorphic for all even $d \geq 2$ and for all odd $d \geq 3$, but the $d = 2$ and $d = 3$ cases are genuinely different. The total cohomology $H^*\big(M_S(n,3);\mathbb{Q}\big)$ with all degrees combined is isomorphic as a representation of Σ_n to $H^0\big(M_S(n,1);\mathbb{Q}\big)$, which is the permutation representation with basis indexed by the chambers of $\mathcal{A}_S(n)$ [Mos17, Theorem 1.4(b)].

For fixed $S \subset \mathbb{R}$, $d \geq 2$, and $i \geq 0$, we will define in the next section a contravariant module $B_S^{i,d}$ over the category of finite sets with surjections that takes the set $[n]$ to $H^{(d-1)i}\big(M_S(n,d);\mathbb{Q}\big)$.

Theorem 1.1. *The module $B_S^{i,d}$ is finitely generated in degrees $\leq |S|^i$.*

Combining Theorem 1.1 with [PY17, Theorem 4.1], we obtain the following numerical results:[5]

Corollary 1.2. *Fix a finite set $S \subset \mathbb{R}$ and a pair of integers $d \geq 2$ and $i \geq 0$.*

1. *The generating function*
$$G_S^i(t) := \sum_{n=1}^{\infty} b_S^i(n) t^n$$
is a rational function with poles contained in the set $\{1/j \mid 1 \leq j \leq |S|^i\}$, with at worst a simple pole at $|S|^{-i}$. Equivalently, there exist polynomials $\{c_S^{i,j}(n) \mid 1 \leq j \leq |S|^i\}$ such that, for n sufficiently large,
$$b_S^i(n) = \sum_{j=1}^{|S|^i} c_S^{i,j}(n) j^n,$$
and the last polynomial $c_S^{i,|S|^i}(n)$ is a constant polynomial.

2. *For any partition λ of n, let V_λ denote the irreducible representation of Σ_n indexed by λ. If $\operatorname{Hom}_{\Sigma_n}\Big(V_\lambda, H^{(d-1)i}\big(M_S(n,d);\mathbb{Q}\big)\Big) \neq 0$, then λ has at most $|S|^i$ rows.*

3. *For any partition λ with $n \geq |\lambda| + \lambda_1$, let $\lambda(n)$ be the* **padded partition** *of n obtained from λ by adding a row of length $n - |\lambda|$. For any λ, the function*
$$n \mapsto \dim \operatorname{Hom}_{\Sigma_n}\Big(V_{\lambda(n)}, H^{(d-1)i}\big(M_S(n,d);\mathbb{Q}\big)\Big)$$
is bounded above by a polynomial in n. In particular, if λ is the empty partition, this says that the multiplicity of the trivial representation in $H^{(d-1)i}\big(M_S(n,d);\mathbb{Q}\big)$ is bounded above by a polynomial in n.

Remark 1.3. A stronger version of item (1) above for the resonance arrangement appears in [Küh, Theorem 1.4]. Kühne proves that the polynomials $c_{\{0,1\}}^{i,j}(n)$ are *all* constant (i.e. that all poles of $G_{\{0,1\}}^i(t)$ are simple), obtains bounds on their sizes, and shows that the equality holds for all n, not just sufficiently large n (i.e. that the limit as t goes to ∞ of $G_{\{0,1\}}^i(t)$ is zero). It should be possible to categorify Kühne's theorem by proving that the restriction of $B_{\{0,1\}}^{i,d}$ to the category of **ordered surjections** [SS17] is isomorphic to a direct sum of shifts of principal projectives, with summands

[5] The deepest of these statements, namely the fact that the dimension generating function for a finitely generated $\mathrm{FS}^{\mathrm{op}}$-module is rational with prescribed poles, is due to Sam and Snowden [SS17, Corollary 8.1.4].

indexed by Kühne's **functional prototypes**. The cost of working with ordered surjections would be that we would lose all information about the action of the symmetric group.

Acknowledgments: The authors are grateful to Lou Billera for telling them about the arrangement $\mathcal{A}(n)$ and about Kühne's work.

2 The proof

Let FS denote the category whose objects are nonempty finite sets and whose morphisms are surjective maps. An **FS$^{\mathbf{op}}$-module** over $\mathbb{Q}$ is a contravariant functor from FS to the category of rational vector spaces. For each finite set F, we have the **principal projective** module P_F, which sends a finite set E to the vector space with basis $\operatorname{Hom}_{\mathrm{FS}}(E,F)$, with morphisms defined on basis elements by composition. An FS$^{\mathrm{op}}$-module N is said to be **finitely generated** if it is a quotient of a finite sum $\oplus_i P_{F_i}$ of principal projectives, and it is said to be **finitely generated in degrees** $\boldsymbol{\leq m}$ if the sets F_i can all be taken to have cardinality less than or equal to m. This is equivalent to saying that, for all E, the vector space $N(E)$ finite dimensional and is spanned by the images of the pullbacks along various maps $\varphi : E \to F$, where F has cardinality less than or equal to m.

Lemma 2.1. *Suppose that N_1 is finitely generated in degrees $\leq m_1$ and N_2 is finitely generated in degrees $\leq m_2$. Then the pointwise tensor product $N_1 \otimes N_2$ is finitely generated in degrees $\leq m_1m_2$.*

Proof. We immediately reduce to the case where $N_1 = P_{[m_1]}$ and $N_2 = P_{[m_2]}$. For any $\varphi : E \to [m]$, let e_φ denote the corresponding basis element of $P_{[m]}(E)$. Then $N_1 \otimes N_2$ has basis

$$\{e_{\varphi_1} \otimes e_{\varphi_2} \mid \varphi_1 : E \to [m_1], \varphi_2 : E \to [m_2]\}\,.$$

Given the pair of surjections (φ_1, φ_2), let $F \subset [m_1] \times [m_2]$ denote the image of $\varphi_1 \times \varphi_2$, let $\varphi = \varphi_1 \times \varphi_2 \in \operatorname{Hom}_{\mathrm{FS}}(E,F)$, and let $\psi_1 : F \to [m_1]$ and $\psi_2 : F \to [m_2]$ denote the coordinate projections. It is clear that we have $e_{\varphi_1} \otimes e_{\varphi_2} = \varphi^*(e_{\psi_1} \otimes e_{\psi_2})$. Since the cardinality of F is at most m_1m_2, this completes the proof. $\square$

Fix a positive integer d and a finite set $S \subset \mathbb{R}$. To any finite set E, we associated the space $M_S(E,d)$ of E-tuples of vectors in $\mathbb{R}^d$ such that any nontrivial linear combination of the vectors with coefficients in S is nonzero. Given a surjection $\varphi : E \to F$, we obtain a map

$$\varphi_* : M_S(E,d) \to M_S(F,d)$$

by adding the vectors in each fiber of φ. These maps define a functor from FS to the category of topological spaces. By taking rational cohomology in degree $(d-1)i$, we obtain an FS$^{\mathrm{op}}$-module $B_S^{i,d}$. We prove the following theorem, which implies the three statements in the introduction.

Proof of Theorem 1.1. As noted above, the cohomology of $M_S(E,d)$ is generated as an algebra in degree $d-1$, hence $B_S^{i,d}$ is a quotient of $(B_S^{1,d})^{\otimes i}$. By Lemma 2.1, this means that it is sufficient to prove that $B^{1,d}_S$ is finitely generated in degrees $\leq |S|$. For any finite set F, the vector space $B^{1,d}_S(F)$ has a generating set indexed by nonzero elements of S^F [dS01, Corollary 5.6] (these generators form a basis unless two nonzero elements of S^F are proportional, in which case the corresponding generators are equal). For any nonzero $v \in S^F$, let $x_v \in B^{1,d}_S(F)$ be the corresponding generator. Concretely, if we take $x \in H^{d-1}(\mathbb{R}^d \smallsetminus \{0\}; \mathbb{Q})$ to be the standard generator, then x_v is equal to the pullback of x along the map

$$f_v : M_S(F, d) \to \mathbb{R}^d \smallsetminus \{0\}$$

that sends an F-tuple of vectors to its linear combination with coefficients determined by v. Given a surjection $\varphi : E \to F$, we have $f_v \circ \varphi_* = f_{\varphi^* v}$, and therefore

$$\varphi^*(x_v) = \varphi^* \circ f_v^*(x) = f^*_{\varphi^* v}(x) = x_{\varphi^* v} \in B^{1,d}_S(E).$$

Since every element of S^E may be pulled back from a subset of cardinality at most $|S|$, $B^{1,d}_S$ is generated in degrees $\leq |S|$. □

Remark 2.2. Our construction also works if we replace $\mathbb{R}$ with an arbitrary field k and we take S to be a finite subset of k. We define the arrangement $\mathcal{A}_{k,S}(n)$ in k^n as above, we denote its complement by $M_{k,S}(E, 1)$, and we take $B^{i,1}_{k,S}(E)$ to be the étale cohomology group $H^i_{\text{ét}}\big(M_{k,S}(E, 1) \otimes_k \bar{k}; \mathbb{Q}_l\big)$ for some prime l not equal to the characteristic of k, which is isomorphic to the degree i part of the Orlik–Solomon algebra of $\mathcal{A}_{k,S}(n)$. This is an $\mathrm{FS}^{\mathrm{op}}$-module over $\mathbb{Q}_l$, and the same argument shows that it is finitely generated in degrees $\leq |S|^i$.

An interesting special case is where $k = \mathbb{F}_q$ is a finite field and $S = k$, so that our arrangement $\mathcal{A}_{\mathbb{F}_q,\mathbb{F}_q}(n)$ is the collection of all hyperplanes in $\mathbb{F}_q^n$. This arrangement has characteristic polynomial $(t-1)(t-q)\cdots(t-q^{n-1})$, and therefore the i^{th} Betti number is equal to the evaluation of the i^{th} elementary symmetric polynomial at the values $1, q, \ldots, q^{n-1}$. This implies that the Hilbert series of our module is

$$q^{\binom{i}{2}} t^i \prod_{j=0}^{i} \frac{1}{1 - q^j t},$$

which has simple poles at q^{-j} for $j = 0, 1, \ldots, i$. The projectivization of $M_{\mathbb{F}_q,\mathbb{F}_q}(n, 1) \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ is a Deligne–Lusztig variety for the group $\mathrm{GL}_n(\mathbb{F}_q)$.

References

[Cor02] R. Cordovil, *A commutative algebra for oriented matroids*, Discrete Comput. Geom. **27** (2002), no. 1, 73–84, Geometric combinatorics (San Francisco, CA/Davis, CA, 2000).

[dS01] Mark de Longueville and Carsten A. Schultz, *The cohomology rings of complements of subspace arrangements*, Math. Ann. **319** (2001), no. 4, 625–646.

[GMP21] Samuel C. Gutekunst, Karola Mészáros, and T. Kyle Petersen, *Root cones and the resonance arrangement*, Electron. J. Combin. **28** (2021), no. 1, Paper No. 1.12, 39.

[Küh] Lucas Kühne, *The universality of the resonance arrangement and its Betti numbers*, arXiv:2008.10553.

[Mos17] Daniel Moseley, *Equivariant cohomology and the Varchenko-Gelfand filtration*, J. Algebra **472** (2017), 95–114.

[OS80] Peter Orlik and Louis Solomon, *Combinatorics and topology of complements of hyperplanes*, Invent. Math. **56** (1980), no. 2, 167–189.

[PY17] Nicholas Proudfoot and Ben Young, *Configuration spaces,* $\mathrm{FS}^{\mathrm{op}}$*-modules, and Kazhdan-Lusztig polynomials of braid matroids*, New York J. Math. **23** (2017), 813–832.

[SS17] Steven V Sam and Andrew Snowden, *Gröbner methods for representations of combinatorial categories*, J. Amer. Math. Soc. **30** (2017), no. 1, 159–203.